Reflective functors via nearness

by

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Abstract. Using the concept of nearness for arbitrary families of sets, which is a generalization of proximity and constancy, the well-known theorem of Taimanov on extensions of continuous functions from dense subspaces is generalized. This result is then used to show that all $T_1$ extensions are reflections.

1. Introduction. The concept of proximity or nearness between pairs of subsets was first introduced by F. Riesz [22] in 1908 but was ignored by mathematicians until the fifties when Efremović [6] and Smirnov [25] systematically developed the new classical theory (for a compact account see Naimpally and Warrack [21]). Lodato [20] considered a generalized proximity which has proved to be of great value in solving many topological problems; in particular, it was used by Gagnat and Naimpally [8] to obtain a generalization of the well-known theorem of Taimanov [27] concerning extensions of continuous functions from dense subspaces. Another type of generalization was discovered by Herrlich [12] who used topological bases for closed sets.

In considering $T_1$-extensions Ivanova and Ivanov [16] generalized the concept of proximity to constancy in which nearness of finitely many sets is postulated. In this paper we further generalize the concept to nearness of families of sets of arbitrary cardinality. This concept includes as special cases topologies, proximities, uniformities etc. and an account is written by Herrlich [13]. His axioms are different from ours. Our purpose here is to prove, in a certain sense, an ultimate generalization of Taimanov's theorem which yields a general theory of reflective functors.

In our work here, a near structure is generated in either of the two ways:
(i) that induced on $X$ by a super space $aX$ in which $X$ is dense and
(ii) that induced on $X$ by separating bases of Steiner [26]. From our general result it will follow that for strict Hausdorff extensions the best we can do is to get $\delta$-continuous extensions (Fomin [7], Hunsaker and Naimpally [14], Rudolf [23]). In case of simple Hausdorff extensions and all $T_1$ extensions we get continuous extensions which shows that they are all reflections.
We then show that from our results we can handle even non-compact extensions such as Banaaschewski $T_1$-minimal extension [3], Kadetov extension [19], [17], Liu’s $aX$ [18], Liu and Strecker’s $aX$ [19] etc. We also generalize a recent result of Hunsaker and Sharma [15] concerning the Harris regular-closed extension [11]. This should be compared with EF-proximities and contiguities which are useful in dealing with Hausdorff compactifications and $T_1$-compactifications respectively. Also we note that the concept of nearness is related to the theory of structures of Harris [19].

We have given a fairly representative bibliography on reflective functors and the interested reader will find further references in the items included here.

2. Near structure. It is well known that every EF-proximity (see 2.3 (v)) on a Tychonoff space $X$ is induced by the EF-proximity $E_0$ on a suitable compact Hausdorff space $aX$, where $A \approx B$ iff their closures in $aX$ intersect. The most general situation that we can think of is: we are given a topological space $X$ which is dense in a topological space $aX$; in this case we may say that a family $A$ of subsets of $X$ is near iff the closures of the members of $A$ in $aX$ have a common point. This then provides a motivation for defining a near structure on $X$; indeed with certain additional assumptions every near structure on $X$ is obtained in the above manner from some superspace $aX$; see Thron [28] for a similar result concerning LO-proximities and Herrlich [13].

We now axiomatize the concept of nearness and the reader will find the axioms natural if the above example is kept in view. For notation we write “$A$ is near” by “$\eta(A)$” and “$A$ is not near” by “$\eta(A)$”. We write $B(\eta)$ for $\eta(A \cup B)$ and $C(\eta, A) = \{ x \in X : (x, \eta) \}$. $P = X$ denotes the power set $P(X)$ for $\eta: \eta, \eta \setminus \Xi, \eta \setminus \Xi = \{ \eta = A : A \in B, B \setminus \Xi \}.$

2.1. Definition. Let $X$ be a non-empty set and $\eta \in \mathcal{P}(X)$. Then $\eta$ is called a Čech near structure or Čech nearness on $X$ iff

(a) $\bigcap \{ A : A \in \eta \} \neq \emptyset$ implies $\eta(A)$.
(b) $\eta(A)$ and $\eta(B)$ implies $\eta(A \cap B)$.
(c) $\eta(A)$ and for all $B \in B$ there is an $A \in A$ such that $A \cup B$ implies $\eta(B)$.
(d) $\emptyset \notin \eta$ implies $\eta(B)$.

The pair $(X, \eta)$ is called a Čech near space. If $X$ is a topological space and $C(\eta, A \cup A)$ (the topological closure) for each $A \in \mathcal{A}$, then $\eta$ is said to be associated with the space $X$. If further $A = C(\eta, A)$ then $\eta$ is said to be compatible with the space $X$.

2.2. Remarks. The term “Čech nearness” is an obvious analogue of “Čech proximity” studied by Thron and Warren [29]. The most important Čech near structure $\eta$ associated with a topological space $X$ is defined as follows:

$$\eta \eta \text{ iff } \bigcap \{ A : A \in \eta \} \neq \emptyset .$$

Further if $X$ is an $R_0$-space (i.e., $x \in (y)^-$ implies $y \in (x)^-$ for all $x, y$ in $X$), then $\eta$ is compatible with $X$.

Notation $\eta \eta = \{ A : A \in \eta \}$. 2.3. Definitions. A Čech near structure $\eta$ on $X$ is called:

(i) LO iff $\eta \eta \eta = \eta \eta$.
(ii) $T_1$ iff $\eta \eta$ is LO and $x \neq y : (x, \eta) \neq (y, \eta)$.
(iii) $H$ iff $\eta \eta$ implies the existence of an $E \subset X$ such that $(x, \eta) E \eta$ and $(x - E) \eta \eta$.
(iv) $R$ iff $\eta \eta = \eta$ and $x \in \eta \eta$ implies the existence of an $E \subset X$ such that $(x, \eta) E \eta$ and $(x - E) \eta \eta$. (Note that this is patterned after Harris’ $R$-proximity [11]; however, an $R$-proximity need not be a LO-proximity.)
(v) $E$ iff $\eta \eta = \eta$ and $A \in \eta \eta$ implies the existence of an $E \subset X$ such that $A \in E \eta$ and $(x - E) \eta \eta$.

Obviously $E O = R H = T_1 O = \text{LO} \Rightarrow \text{Čech}$.

2.4. Definition. Let $b$ be an arbitrary infinite cardinal. A Čech near structure $\eta$ is called a $b$-Čech near structure iff $\eta \eta \eta$ there exists $\exists C \subset X$ such that $|X| < b$ and $\eta \eta \eta$, and $b$ is the smallest cardinal having this property. The contiguity of Ivanova and Ivanov [18] is an $\eta \eta \eta$, near structure.

2.5. Example. Another important method of constructing compatible near structures on $T_1$-spaces is through the separating bases of Steiner [26]. A separating or $\eta_0$-base $\xi$ on a $T_1$-space $X$ is a ring of closed subsets of $X$ such that whenever $x \in A$, a closed subset of $X$, there exist $L_1, L_2 \in \xi$ such that $x \in L_1, A \subset L_1, L_2 \cap L_2 = \emptyset$. An $\eta_0$-base is (i) $\eta_0$ iff $x \in A$ implies the existence of $L_1, L_2 \in \xi$ such that $x \in L_1, L_2 \cap L_2 = \emptyset$ and $L_2 \cap L_2 = X$ (ii) $\eta_0$ iff $x \in A$ a closed subset of $X$ implies the existence of $L_1, L_2$ such that $x \in L_1, A \subset L_1, L_2 \cap L_2 = \emptyset$. A (or normal) iff $L_1, L_2 \in \xi$ and $L_1 \cap L_2 = \emptyset$ implies the existence of $L_1, L_2 \in \xi$ such that $L_1 \cap L_2 = \emptyset, L_1 \cap L_2 = \emptyset$ and $L_1 \cap L_2 = X$.
2.6. Definition. Let \((X, \eta_X)\) and \((Y, \eta_Y)\) be two Čech near spaces. A function \(f: X \to Y\) is called a near map if and only if \(\forall A \subseteq X, \eta_X A \implies \eta_Y f(A)\), where \(f(A) := \{f(a): a \in A\}\).

2.7. Theorem. Let \(X\) and \(Y\) be topological spaces, \(\eta_X\) the associated Čech near structure on \(X\) (see 2.3) and \(\eta_Y\) be any LO near structure on \(Y\). Then \(f: X \to Y\) is a near map if \(f\) is continuous and the converse holds if \(\eta_Y\) is compatible with \(Y\).

Proof. Suppose \(f\) is continuous; then
\[
\eta_Y f(A) = \bigcap \{A' : A' \subseteq A \wedge f(A') \neq \emptyset \} = \bigcap \{f(A') : A' \subseteq A \wedge f(A') \neq \emptyset \} = \bigcap \{f(A') : A' \subseteq A \wedge f(A') \neq \emptyset \}
\]

since \(f\) is continuous
\[
\eta_Y f(A') = \eta_Y f(A)
\]
Thus \(f\) is continuous implies \(f\) is a near map.

Conversely, if \(f\) is a near map, then
\[
x \in A \iff (x) \eta_X A = \eta_Y f(A) = f(x) \in f(A)
\]

and \(f\) is continuous.

2.8. Corollary. (Necessity of the generalized Taimanov Theorem.)
Let \(X\) be dense in a topological space \(aX\) and let \(X\) be assigned the Čech near structure \(\eta_X\) induced by \(\eta_X\) on \(aX\). Let \(\eta_Y\) be a compatible LO near structure on \(Y\). Then a necessary condition that \(f: (X, \eta_X) \to (Y, \eta_Y)\) has a continuous extension \(f: aX \to Y\) is that \(f\) is a near map.

Our motivation for the introduction of the concept of a near clan is again the situation with which we began this section. Suppose \(X\) is dense in a topological space \(aX\) and that we have information only about \(X\). In order to explore \(aX\) we must express each \(x \in aX\) in terms of certain objects formed from subsets of \(X\) and the most natural one is \(x^\sigma = \{B \subseteq X: x \in Cl_x(B)\}\). Two simple properties satisfied by \(x^\sigma\) in terms of the Čech near structure \(\eta_X\) induced on \(X\) by \(\eta_X\) on \(aX\) results in the following definition.

2.9. Definition. A near clan \(\sigma\) in a Čech near space \((X, \eta_X)\) is a subset of \(PX\) satisfying:

(a) \(\eta_X\)
(b) \((A \cup B) \in \sigma \iff A \in \sigma \lor B \in \sigma\).

A near clan \(\sigma\) such that \(A \in \sigma \iff A^{-} \in \sigma\) is a near clan.

A near clan \(\sigma\) is a near clan in which \(A \in \sigma\) implies the existence of \(B \in \sigma\) such that \(A \subseteq B\).

2.10. Example. The most important example of a near clan is \(\sigma^\eta\) defined in the paragraph just preceding 2.9. If \(aX\) is compact Hausdorff, then \(\sigma^\eta\) is a near clan.

We state below several results which are either known or follow easily from our definitions:

2.11. Lemma. Let \(\eta\) be a Čech near structure associated with a topological space \(X\). In case \(X^\sigma = aX\) we assume that \(\eta\) is induced by \(\eta_X\) on \(aX\).

(i) If \(F\) is an ultrafilter and \(\eta\) is a LO-contiguity, then \(b(F) = (A \subseteq X: A^{-} \in \eta)\) is a near bunch called contiguity bunch.

(ii) If \(\xi\) is a separating base on \(X, \eta = \eta_(\xi)\) is the contiguity induced by \(\xi\), and \(F\) is an \(\xi\)-ultrafilter on \(X\), then \(\sigma(F) = (E \subseteq X: E \in \eta)\) is a near clan called contiguity clan in \(\xi\).

(iii) Every contiguity clan is contained in a maximal contiguity clan. If \(\eta\) is EF, then each contiguity bunch is contained in a unique maximal contiguity bunch which is a contiguity clan (cf. (8)).

(iv) For each \(x \in X\), \(\sigma_x = (E \subseteq X: x \in Cl_x(E))\) is a near bunch called point near bunch. If \(\eta\) is compatible with \(X\), then \(\sigma_x\) is a near clan called point near clan.

(v) If \(\eta\) is compatible with \(X, \sigma\) is a near bunch and \((x) \in \sigma\), then \(\sigma = \sigma_x\).

(vi) If \(\sigma\) is a near bunch, \(A \in \sigma\) and \(A \subseteq B\), then \(B \in \sigma\). In particular, \(x \in \sigma\).

(vii) If \(\eta\) in \(H\) and \(\sigma\) is a near bunch, then there is at most one point \(x \in X\) such that \((x) \in \sigma\).

(viii) If \(aX\) is a T_1, then for each \(x \in aX, \sigma_x^\eta\) is a near clan.

(ix) If \(aX\) is compact, then \(\sigma\) is a contiguity.

2.12. Definitions. Suppose \(X^\sigma = aX\) and the Čech near structure on \(X\) is induced by \(\eta\) on \(aX\). We say that \(X\) is relatively clan (resp. cluster) complete in \(aX\) iff for every near clan (resp. cluster) in \((X, \eta)\) there corresponds an \(x \in aX\) such that \((x) \in \sigma\). If \(X = aX\) we drop the word "relatively"; note that in this case if \(\sigma\) is a near clan, then \((x) \in \sigma\) implies \((x) \in \sigma\) and \(\sigma = \sigma_x\).

2.13. Remarks. The most important example of a relatively clan complete space is \(X^\sigma = aX\) and \(aX\) has the Čech near structure \(\eta_X\). In case \(aX\) is compact Hausdorff, \(X\) is relatively cluster complete in \(aX\) and
we have the classical EF proximity theory, due to Leader [30], of Efremovich-Smirnov ([6], [30]). Another important example is: suppose $\mathcal{I}$ is a separating base on a $T_1$-space $X$ and $w(X, \mathcal{I})$ is the corresponding Wallman compactification of $X$; then $X$ is relatively cluster complete in $w(X, \mathcal{I})$ when $X$ is assigned the contiguity $\eta(\mathcal{I})$ which is the subspace contiguity induced by $\eta(\mathcal{I})$ on $w(X, \mathcal{I})$ (see [4] for details). Note that $(X, \eta(\mathcal{I}))$ is bunch-complete.

2.14. DEFINITIONS. Let $\eta$ be a Čech near structure associated with a topological space $X$ and let $\Sigma_X$ denote the set of all near classes in $X$. For $P \subseteq \Sigma_X$ and $A \subseteq X$ we say that $A$ absorbs $P$ iff $A \cap \sigma = \emptyset$ for each $\sigma \in P$. It is easy to verify that, "$\sigma \in \text{Cl}_X P$ iff whenever $A$ absorbs $P$, $A \cap \sigma = \emptyset$" defines a Kuratowski closure operator on $\Sigma_X$. The resulting topology is called the abstraction or $A$-topology.

2.15. THEOREM. Let $X' = aX$ and let $\eta$ be the Čech near structure on $X$ induced by $\eta_0$ on $aX$. The map $\Phi: aX \to \Sigma_X$ defined by $\Phi(\sigma) = \sigma^\eta = \{E \subseteq X: \{a \in \eta_0, E \} \text{ is continuous}\}$ is continuous.

Proof. Let $x \in aX$, $x \in aX$ and $\Phi(x) \notin \Phi(\eta)$. Then there is an $A \subseteq X$ which absorbs $\Phi(\eta)$ and $A \notin \Phi(x)$. This implies that $B \subseteq \text{Cl}_X A$ and $x \notin \text{Cl}_X B$, i.e., $x \notin \text{Cl}_X \Sigma_X$, showing thereby that $\Phi$ is continuous.

2.16. COROLLARY. If $aX$ is $T_1$ and $\text{Cl}_X A \subseteq X$ is a base for closed sets of $aX$ (i.e., $aX$ has the strict extension topology, see Banachowski [2]), then $\Phi$ is a homeomorphism.

Proof. $\Phi$ is obviously one-to-one. We show that $\Phi$ is closed. Suppose $x \in \text{Cl}_X A$; then there is an $A \subseteq X$ such that $x \notin \text{Cl}_X A$ and $\Phi(x) \notin \Phi(A)$, i.e., $\Phi(x) \notin \Phi(x)$. Thus showing that $\Phi$ is continuous.

2.17. THEOREM. Let $\eta_0, \eta_1$ be two Čech near structures associated with topological spaces $X$ and $Y$ respectively. If $f: X \to Y$ is a near map, then the function $f_2: \Sigma_X \to \Sigma_Y$ defined by

$$f_2(\sigma) = \{E \subseteq X: f^{-1}(E) \cap \sigma = \emptyset\}$$

is continuous.

Proof. It is easily checked that $f_2(\sigma) \in \Sigma_Y$. Let $\sigma \in \Sigma_X$ and $f_2(\sigma) \notin \text{Cl}_Y(\Sigma_X)$. Then there is an $A \subseteq \Sigma_X$ such that $A \cap \sigma = \emptyset$ and $A \notin \text{Cl}_Y(\Sigma_X)$. Clearly $f^{-1}(A)$ absorbs $P$ and $f^{-1}(A) \cap \sigma = \emptyset$, i.e., $\sigma \notin \text{Cl}_X P$, showing that $f_2$ is continuous.

2.18. Remark. If in the above theorem, $f$ is also continuous, then it is easily checked that

$$f_2(\sigma) = \sigma_{\text{fin}},$$

and we may consider $f_2$ to be a continuous extension of $f$.

We recall that a function $f: X \to Y$ is called $\theta$-continuous (Pomin [7]) iff for each $x \in X$, $f(x) \in V$ open in $Y$ implies the existence of an open set $U$ in $X$ such that $x \in U$ and $f(U) \subseteq V$. An important problem in Topology is to find conditions for $\theta$-continuous extensions of continuous functions (see Rudolf [23]).

2.19. THEOREM. Let $X$ be dense in $aX$ and let $\eta_0$ be a near structure on $X$ induced by an $A$ near structure $\eta$ on $aX$. Let $X$ be relatively clan complete in $aX$. Then the map

$$\psi: \Sigma_X \to aX,$$

where $\psi(\sigma) = x_0$, the unique point such that $(a_0, \eta, \sigma)$ is $\theta$-continuous.

Proof. Suppose $U' = \emptyset$ is an open nhbd. of $x_0$ in $aX$ and let $U = U' \cap X$. Then $(a_0, E) \in \Sigma_X$, where $E = X - U_0$ and $P \subseteq \text{Cl}_X E$ implies $E \notin \Sigma_X - E^\mathfrak{m}$, where $E^\mathfrak{m} = \{\sigma \in \Sigma_X: P \ni \sigma\}$. We must show that $a_0 \in \text{Cl}_X \Sigma_X - E^\mathfrak{m}$. Then $a_0 \in \text{Cl}_X U'$, where $a_0 \in \Sigma_X$, $U_0$ open in $aX$, $U_0 = aU_0 \cap X$ and $F = X - U_0$. $(a_0, E) \in F_0$ implies

$$a_0 \in E^\mathfrak{m} = \{X_0 \subseteq \Sigma_X: P \ni X_0\}$$

is continuous.

2.20. Remark. Herrlich's Example 3 in § 2 [12] shows that we cannot hope to get continuity of $\psi$.

2.21. COROLLARY. If $\eta$ is $\mathcal{I}$, then the map $\psi$ is continuous.

2.22. THEOREM. (Generalized Taimanov Theorem.) Suppose $X$ is dense in a topological space $aX$ and has an associated Čech near structure $\eta_0$ induced by $\eta_0$ on $aX$. Let $Y$ be relatively clan complete in $\lambda$ and $\eta_0$ on $Y$ be induced by an $\eta$ near structure $\eta$ on $\lambda$. Then a necessary and sufficient condition that a continuous function $f: X \to Y$ has a continuous extension $\lambda: aX \to \lambda$ is that $f$ is a near map.

Proof. The result follows from the following:

$$aX \xrightarrow{\eta_0} \Sigma_X \xrightarrow{f_2} \Sigma_Y \xrightarrow{\eta_0} \lambda$$

2.23. Remark. The above result includes, as special cases, several previously known generalizations of Taimanov's theorem; this is what we propose to show below. 4
First suppose that \( \eta = \eta_n \) and \( \lambda Y \) is compact. Then from 2.11 (ix) it follows that \( \eta = \eta_n \) is a contiguity and we may take \( f \) to be a contiguity map. If further \( \lambda Y \) is Hausdorff, then \( \eta = \eta_n \) is EF and \( \lambda Y \) is the Smirnov compactification of \( (Y, \delta) \), where \( \delta \) is the natural proximity associated with \( \eta_n \).

We now show that in this case we may further reduce the map \( f \) to a proximally continuous map. We need only show that if \( f : X \to Y \) is proximally continuous, then \( f \) is a contiguity map. Suppose \( \eta \) (A1: \( 1 \leq i \leq n \)), \( A_i \subset Y \). Then \( \bigcap_{p \in \lambda Y} \bigcup_{A_i} \Lambda = \emptyset \) and so for each \( p \in \lambda Y \), there corresponds a \( \epsilon_p \in (1, \ldots, n) \) such that \( p \in \bigcup_{A_{\epsilon_p}} \Lambda_p \). Since \( \eta = \eta_n \) is EF and hence EF, there is a nbhd. \( \mathcal{U}_p \) of \( \eta \) such that \( \eta_p \cap \Lambda_{\epsilon_p} \subseteq \mathcal{U}_p \). Since \( \lambda Y \) is compact, there are \( p_i \in \lambda Y \), \( 1 \leq i \leq n \), such that \( \lambda Y = \bigcup_{i=1}^n A_{\epsilon_i} \). If \( V_i = V_{p_i} \cap Y \), then \( Y = \bigcup_{i=1}^n A_{\epsilon_i} \cap V_i \). Since \( X \) is dense in \( \lambda X \), it follows that \( \eta \cap \left( \bigcup_{A_{\epsilon_i}} \mathcal{V}_i \right) \) and \( \left( \bigcup_{A_{\epsilon_i}} \mathcal{V}_i \right) \cap \bigcup_{A_{\epsilon_i}} \mathcal{V}_i \). Since \( \lambda Y \) is compact, we have shown:

2.24. Lemma. If in Theorem 2.22, \( \lambda Y \) is compact Hausdorff and \( \eta = \eta_n \) on \( \lambda Y \), then \( f \) is a near map if and only if \( f \) is proximally continuous.

The main result of [8] is Theorem 2.1. This now follows from 2.23 and 2.24; in fact, we get a slightly more general result since we do not assume \( \lambda X \) to be EF. It was shown in [8] how the above mentioned result includes as special cases several of the known results on extensions of continuous functions from dense subspaces. However, as proximities as they deal with only two sets at a time, were found to be rather awkward in dealing with Lindelöf or real compact extensions, where countably many sets occur (see e.g. Theorem (6.1) of [3]). The concept of nearness is, on the other hand, capable of handling such cases, as we show below.

Suppose \( \lambda Y \) is \( T_1 \) Lindelöf (and hence normal) and \( \eta = \eta_n \) on \( \lambda Y \). Clearly in this case \( \lambda Y \) is EF and \( \eta_n \), there exist \( \delta = \bigcup_{A_n} \subset \mathcal{N} \) and \( f : X \to Y \). Suppose \( \eta = \eta_n \) is a contiguity map and \( \lambda Y \) is a near map. Proceeding in a similar manner, we may take a strong delta normal base \( \gamma \) on \( Y \) and take \( \lambda Y \) to be the \( \gamma \)-compactification of \( Y \) (see Alô and Shapiro [13]). In this case, again \( \eta_n = \eta_{n(n)} \) and \( f \) is a near map if \( \gamma \) is a near map.

3. Reflective functors. In this section we show how, in contrast to the main result (5.3) of [8], our Theorem 2.22 enables us to handle non-compact extensions with ease. In particular, some of the results of [14] follow easily.

In this section we suppose that all topological spaces are Hausdorff and that each space \( X \) is dense in some extension \( \lambda X \). Further each \( \lambda X \) is assigned the \( \delta \)-near structure \( \eta_n \) and each \( X \) is assigned the near structure induced by \( \eta_n \). An important class of problems is to find necessary and sufficient conditions on \( f : X \to Y \) to have a continuous extension \( \bar{f} : \lambda X \to Y \); in other words, to find a class of maps for which \( \lambda X \) is a reflection of \( X \).

3.1. Strict extensions. If each \( \lambda X \) is a strict extension of \( X \), then Theorems 3.10, 3.17, 3.19 show that each near map \( f : X \to Y \) has a \( \delta \)-continuous extension \( \bar{f} : \lambda X \to Y \). Examples of this type are the Banachewski \( T_0 \)-minimal extension [3] and the Poincaré extension [7]. In this connection we note that every \( \lambda \)-map of [14] is a near map (the proof of this is similar to that of "\( p \)-map implies near map" proved below) and so our present theory includes some of the results of [14].

3.2. Simple extensions. If each extension \( \lambda X \) is simple, then Theorems 3.1, 3.17, 3.19 show that a necessary and sufficient condition that \( f : X \to Y \) has a continuous extension \( \bar{f} : \lambda X \to Y \) is that \( f \) is a near map.

We now discuss the case of Kaschewski extension [17] and relate our results to those of Harris [9]; we do not consider here \( \lambda X \) of Linus, \( \delta \)- extension of Lin and Strecker [19] which are discussed in [14].

Let \( \lambda X = \lambda T \) the Kaschewski extension of \( X \) and let \( \eta_n \) be the near structure induced on \( X \) by \( \eta_n \) on \( \lambda X \). The following is obvious:

3.3. Theorem. For \( AC \), \( \eta_n \) iff there is a \( \eta \)-filter \( \mathcal{F} \) in \( X \) such that \( A \cap \mathcal{F} \neq \emptyset \) for each \( A \in \mathcal{A} \) and each \( \mathcal{F} \in \mathcal{F} \) iff for every \( \eta \)-filter \( \mathcal{F} \) there is an \( A \in \mathcal{A} \) such that \( A \cap \mathcal{F} \neq \emptyset \) for each \( A \in \mathcal{A} \). Further \( \eta_n \) implies the existence of a finite subset \( \{A_i \subset \mathcal{A} : 1 \leq i \leq n\} \) of \( \mathcal{A} \) such that \( \lambda \mathcal{A} = \emptyset \).

3.4. Theorem. A function \( f : (X, \eta_n) \to (Y, \eta_n) \) is a near map if and only if \( f \) is a \( p \)-map. (Here, \( \eta_n \) are induced by \( \eta_n \) on \( \lambda X \).)

Proof. Suppose \( f \) is a \( p \)-map and \( \lambda X \) for \( AC \). Then there exists a \( \eta \)-filter \( \mathcal{F} \) in \( X \) such that \( A \cap \mathcal{F} = \emptyset \) (i.e. \( A \cap \mathcal{F} = \emptyset \) for all \( A \in \mathcal{A} \). \( F \in \mathcal{F} \). Clearly \( f(A) \cap f(\mathcal{F}) \neq \emptyset \) and so \( f(A) \cap f(\mathcal{F}) \neq \emptyset \), where \( f(\mathcal{F}) = \{f(E) : E \cap \mathcal{F} = \emptyset \} \). Since \( f \) is the \( \eta \)-filter in \( Y \) (see Harris [9]).

Conversely if \( f \) is a near map and \( \mathcal{F} \) is a \( p \)-filter in \( X \), then \( \mathcal{F} \) and so \( f(\mathcal{F}) \). Hence there is a \( \eta \)-filter \( \mathcal{F} \) in \( Y \) such that \( f(\mathcal{F}) \neq \emptyset \). Hence \( f(\mathcal{F}) \) is a \( \eta \)-filter in \( Y \) and \( f \) is a \( p \)-map.

3.5. \( T_0 \) extensions. If each \( \lambda X \) is \( T_0 \), then Theorem 2.22 shows that, irrespective of whether \( \lambda X \) is simple or strict, \( f : X \to Y \) has a continuous extension \( \bar{f} : \lambda X \to Y \); if only if \( f \) is a near map. Thus the category of objects \( \lambda X \) and continuous functions as morphisms is the largest epireflective subcategory of \( T_0 \) spaces and near maps. We will now show...
how this can be used to obtain a recent solution to Problem II of Harris [11] by Hunsaker and Sharma [15] (also see [24]).

First consider a slightly general situation. Let \( aX \) consist of some distinguished or \( d \)-clusters in \( X \) with the absorption topology. Then an argument similar to Theorem 2 of [4] shows that \( f: aX \to aY \) is a continuous extension of \( f: X \to Y \) iff for each \( \sigma \in aX \), \( f(\sigma) \subseteq f(\sigma') \), i.e.

\[ (*) \text{ for each } \sigma \in aX \text{ there is a unique } \sigma' \in aX \text{ such that } f(\sigma) \subseteq \sigma' \].

Let a proximally continuous map satisfying \( (*) \) be called strongly \( p \)-continuous. Obviously, every near map is strongly \( p \)-continuous. Conversely, every strongly \( p \)-continuous map is a near map: \( \eta A \) for \( A \subseteq PX \) implies there is a \( \sigma \in aX \) such that \( A \subseteq \sigma \). Since \( f \) is strongly \( p \)-continuous it satisfies \( (*) \) and so \( f(A) \subseteq f(\sigma) \subseteq \sigma' \) and \( \eta f(\sigma) \). (We note \( \eta f(\sigma) \) is a \( \sigma \in aX \) such that \( A \subseteq \sigma \).) In particular, if \( f \) is the Harris regular-closed extension of an \( RO \)-regular space \( X \), then \( aX \) is the space of all contractive clusters (Hunsaker and Sharma [15]) and hence \( aX \) is an epireflection.

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References


