A simplicial monotone-light factorization theorem

by

J. R. Walker (Mansfield, Penn.)

Abstract. The Eilenberg–Whyburn monotone-light factorization theorem states that under certain very general conditions a continuous function may be factored into the composition of two continuous functions, the first of which is monotone and the second light. This paper establishes that if the original function is simplicial (resp., piecewise linear), then the factors may also be chosen to be simplicial (resp. piecewise linear).

1. Introduction. A fiber of a mapping \( f \) from a topological space \( X \) to a topological space \( Y \) is any non-empty set of the form \( f^{-1}(y) \), where \( y \in Y \). Such a mapping is semi-monotone if each fiber of \( f \) is connected; it is monotone if, in addition, each fiber of \( f \) is compact. The mapping \( f \) is light if each fiber of \( f \) is totally disconnected. The following theorem is a generalization of the classical Monotone-light factorization theorem of Eilenberg [2] and Whyburn [4].

Theorem (Bauer [1]). Let \( X \) and \( Y \) be locally compact topological spaces. If \( f \) is a mapping from \( X \) to \( Y \) such that each component of each fiber of \( f \) is compact then there is a (unique) factorization \( f = gh \), where \( g \) is monotone and \( h \) is light.

The current paper contains a piecewise linear (PL) analog of the theorem above. This states that for a PL mapping the factorization can take place within the PL category. It appears as a corollary to the main theorem, below.

The notation used here follows Spanier [3]. In particular, if \( Y \) is a simplicial complex, then \( |Y| \) denotes the space of \( Y \) and if \( \sigma \) is a simplex in \( Y \), then \( <\sigma> \) denotes the open simplex in \( |Y| \).

Main Theorem. Let \( f: X \to Y \) be a simplicial mapping of simplicial complexes. There exist subdivisions \( X' \) of \( X \) and \( Y' \) of \( Y \), an induced simplicial mapping \( f': X' \to Y' \), a simplicial complex \( W \) and simplicial mappings \( g: X' \to W \) and \( h: W \to Y' \) such that:

a. \( f' = hg \),

b. \( f' = f \) as mappings of \( |X| \) to \( |Y| \),

c. \( g: |X| \to |W| \) is semi-monotone,

d. \( h: |W| \to |Y| \) has discrete fibers.
This theorem may be converted to a monotone-light factorization theorem by the addition of any hypothesis which implies that the fibers of $g$ are compact. For example, it suffices to require that $X$ be a finite complex or that $f$ be a proper mapping.

**Corollary.** Let $X$ and $Y$ be locally finite complexes, and let $f : X \to Y$ be a proper, PL mapping. Then there is a (unique) factorization $f = h g$, where $g$ is PL and monotone and $h$ is PL with discrete fibers.

The proof of the corollary consists of noting that, since $f$ is PL, $X$ and $Y$ can be subdivided in such a way that $f$ is simplicial. We then apply the theorem to obtain the desired factorization.

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2. Example. This example will illustrate the necessity of the subdivision in the theorem and should aid in understanding the proof which will follow. Let $X$ consist of the proper faces of a single 2-simplex $(v_1, v_2, v_3)$ and let $Y$ consist of a single 1-simplex $(u_1, u_2)$ together with its faces. Define $f : X \to Y$ by $f(v_1) = u_2$ and $f(v_3) = f(v_2) = u_1$. The middle space in the monotone-light factorization of $f$ contains two 1-simplices: one only and two "vertices". Thus it is not a simplicial complex. Furthermore, any attempt to subdivide this middle space so that it becomes a simplicial complex will prevent the monotone and light factors of $f$ from being simplicial mappings.

3. Lemma. If $f : X \to Y$ is a simplicial mapping of simplicial complexes and if $\beta_1, \beta_2 \in \langle \sigma \rangle$, where $\sigma$ is a simplex of $Y$, then $f^{-1}(\beta_1)$ is homeomorphic to $\langle \beta_1 \rangle$.

Proof. $f^{-1}(\beta_1)$ consists exactly of those points $x \in |X|$ such that

$$\sum_{x \in |X|} \sigma(x) = \beta_1$$

for each vertex $x$ of $X$. Define $\theta : f^{-1}(\beta_1) \to f^{-1}(\beta_1)$ by

$$\theta(x) = \begin{cases} \frac{\beta_1 f(x)}{\beta_1 f(x)} & \text{if } x \text{ is a vertex of } \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

It will be seen that $\theta$ is the required homeomorphism.

Straightforward calculations show that $\theta(\sigma) = f^{-1}(\beta_1)$. To see that $\theta$ is continuous, let $M = \max_{u \in \sigma} |u_\sigma|/|u_{\beta_1}|$. Then

$$|\theta(\sigma_1) - \theta(\sigma_2)| < \left\{ \begin{array}{ll} |\beta_1 f(x)| & \text{if } f(x) \in \sigma, \\ 0 & \text{otherwise.} \end{array} \right.$$
Call the resulting space $W$. Define $h: W \to Y'$ by letting $h|_{A'}$ map $A'$ onto $\sigma$ isomorphically. Thus if $\sigma = \{w_1, w_2, \ldots, w_n\}$, then $h$ sends the vertex $w_i$ of $\sigma$ to the vertex $u_i$ of $\sigma$. The method of defining the identifications, given above, insures that $h$ will be well-defined.

The key point in the proof is to show that $W$ is a simplicial complex. The only problem is that a single set of vertices may "define" more than one simplex. The example given in section 2 shows how this could happen if we fail to subdivide $X$ and $Y$ before forming $W$. Let $\omega$ be an "$n$-simplex" of $W$. By this we mean that $\omega$ was an $n$-simplex in the pre-identification space from which $W$ was obtained. Now $h(\omega)$ is an $n$-simplex of $Y$. Hence there exists a simplex $\lambda$ of $Y$ such that $\langle h(\omega) \rangle \subset \langle \lambda \rangle$ and such that the barycenter $b$ of $\lambda$ is a vertex of $h(\omega)$.

Let $\beta \in \langle h(\omega) \rangle$. If we apply the lemma and its corollary we see that $f^{-1}(\langle h(\omega) \rangle)$ is homeomorphic to $f^{-1}(\beta) \times \langle h(\omega) \rangle$ which is, in turn, homeomorphic to $f^{-1}(\beta) \times \langle h(\omega) \rangle$.

Thus there is a natural one-to-one correspondence between the components of $f^{-1}(\langle h(\omega) \rangle)$ and the components of $f^{-1}(\beta)$. Hence there is a one-to-one correspondence between the vertices of $W$ which are mapped to $b$ and the components of $f^{-1}(\langle h(\omega) \rangle)$.

Thus any "$n$-simplex" $\omega'$ of $W$ which is distinct from $\omega$ has a vertex which is distinct from the vertices of $\omega$. Hence $W$ is a simplicial complex.

It is clear that $h$ is a simplicial mapping. Define $g: X' \to W$ by letting $g|_{A'}$ map $A'$ onto $f(\tau)$, where $\tau \in A'$ for some $\gamma \in \tau_{\gamma}$, in the "natural way". Thus $f$ maps the vertex $\nu$ of $X'$ to the vertex $v$ of $Y'$, then $g$ maps $v$ to the vertex $\omega'$ of $W$. Again the method of defining the identifications guarantees that $g$ is well-defined. It is clear that $g$ is a simplicial mapping and that $f' = hg$.

The mapping $h$ does not collapse simplices; that is, the image of each simplex of $W$ is a simplex of the same dimension in $Y'$. Hence $h$ is light and has, in fact, discrete fibers.

For each simplex $\sigma_0$ of $W$, $g^{-1}(\langle \sigma_0 \rangle) = A_\sigma$. By the corollary $g^{-1}(\langle \sigma_0 \rangle)$ is homeomorphic to $g^{-1}(\beta) \times \langle \sigma_0 \rangle$ for each $\beta \in \langle \sigma_0 \rangle$. Thus the connectedness of $A_\sigma$ implies the connectedness of $g^{-1}(\beta)$ for each $\beta$. Therefore $g$ is semi-monotone. This completes the proof.

6. Remark. The proof just completed contains a characterization of semi-monotonicity for simplicial mappings which may be stated as follows: if $f: X \to Y$ is any simplicial mapping, then in order for $f$ to be semi-monotone it is necessary and sufficient that for each vertex $v$ of the barycentric subdivision of $Y$, $f^{-1}(v)$ is connected.