A simplicial monotone-light factorization theorem

by

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Abstract. The Eilenberg–Whyburn monotone-light factorization theorem states that under certain very general conditions a continuous function may be factored into the composition of two continuous functions, the first of which is monotone and the second light. This paper establishes that if the original function is simplicial (resp., piecewise linear), then the factors may also be chosen to be simplicial (resp. piecewise linear).

1. Introduction. A fiber of a mapping f from a topological space X to a topological space Y is any non-empty set of the form $f^{-1}(y)$, where $y \in Y$. Such a mapping is *semi-monotone* if each fiber of f is connected; it is *monotone* if, in addition, each fiber of f is compact. The mapping f is *light* if each fiber of f is totally disconnected. The following theorem is a generalization of the classical Monotone-light factorization theorem of Eilenberg [2] and Whyburn [4].

THEOREM (Bauer [1]). *Let X and Y be locally compact topological spaces. If f is a mapping from X to Y such that each component of each fiber of f is compact then there is a (unique) factorization $f = hg$, where g is monotone and h is light.*

The current paper contains a piecewise linear (PL) analog of the theorem above. This states that for a PL mapping the factorization can take place within the PL category. It appears as a corollary to the main theorem, below.

The notation used here follows Spanier [3]. In particular, if Y is a simplicial complex, then $|Y|$ denotes the space of Y and if σ is a simplex in Y, then $\langle\sigma\rangle$ denotes the open simplex in $|Y|$.

MAIN THEOREM. *Let $f\colon X \to Y$ be a simplicial mapping of simplicial complexes. There exist subdivisions X' of X and Y' of Y, an induced simplicial mapping $f'\colon X' \to Y'$, a simplicial complex W and simplicial mappings $g\colon X' \to W$ and $h\colon W \to Y'$ such that:*

a. $f' = hg$,
b. $f' = f$ *as mappings of* $|X|$ *to* $|Y|$,
c. $g\colon |X| \to |W|$ *is semi-monotone,*
d. $h\colon |W| \to |Y|$ *has discrete fibers.*

This theorem may be converted to a monotone-light factorization theorem by the addition of any hypothesis which implies that the fibers of g are compact. For example it suffices to require that X be a finite complex or that f be a proper mapping.

COROLLARY. *Let K and L be locally finite complexes and let $f\colon K\to L$ be a proper,* PL *mapping. Then there is a (unique) factorization $f = hg$, where g is* PL *and monotone and h is* PL *with discrete fibers.*

The proof of the corollary consists of noting that, since f is PL, X and Y can be subdivided in such a way that f is simplicial. We then apply the theorem to obtain the desired factorization.

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2. EXAMPLE. This example will illustrate the necessity of the subdivision in the theorem and should also aid in understanding the proof which will follow. Let X consist of the proper faces of a single 2-simplex $\{v_0, v_1, v_2\}$ and let Y consist of a single 1-simplex $\{u_0, u_1\}$ together with its faces. Define $f\colon X\to Y$ by $f(v_0) = u_0$ and $f(v_1) = f(v_2) = u_1$. The middle space in the monotone-light factorization of f contains two "1-simplices" and only two "vertices". Thus it is not a simplicial complex. Furthermore, any attempt to subdivide this middle space so that it becomes a simplicial complex will prevent the monotone and light factors of f from being simplicial mappings.

3. LEMMA. *If $f\colon X\to Y$ is a simplicial mapping of simplicial complexes and if $\beta_1, \beta_2 \in \langle\sigma\rangle$, where σ is a simplex of Y, then $f^{-1}(\beta_1)$ is homeomorphic to $f^{-1}(\beta_2)$.*

Proof. $f^{-1}(\beta_i)$ consists exactly of those points $\alpha \in |X|$ such that

$$\sum_{f(v)=u} \alpha(v) = \beta_i(u)$$

for each vertex u of Y, $i = 1, 2$.

Define $\theta\colon f^{-1}(\beta_1)\to f^{-1}(\beta_2)$ by

$$\theta(\alpha)(v) = \begin{cases} \dfrac{\beta_2\big(f(v)\big)}{\beta_1\big(f(v)\big)}\,\alpha(v) & \text{if } f(v) \text{ is a vertex of } \sigma\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

It will be seen that θ is the required homeomorphism.

Straightforward calculations show that $\theta(\alpha) \in f^{-1}(\beta_2)$. To see that θ is continuous let $M = \max\limits_{u\in\sigma}\left|\dfrac{\beta_2(u)}{\beta_1(u)}\right|$.

Then

$$|\theta(\alpha_1)(v) - \theta(\alpha_2)(v)| \leqslant \begin{cases} \left|\dfrac{\beta_2\big(f(v)\big)}{\beta_1\big(f(v)\big)}\big(\alpha_1(v)-\alpha_2(v)\big)\right| \leqslant M|\alpha_1(v)-\alpha_2(v)| & \text{if } f(v)\in\sigma\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

Thus θ is continuous.

Define $\eta\colon f^{-1}(\beta_2)\to f^{-1}(\beta_1)$ in a manner similar to the definition of θ. Exactly as was seen for θ, η is well defined and continuous. Furthermore it is clear that η is an inverse for θ. Thus θ is a homeomorphism.

4. COROLLARY. *Let $\beta_0 \in \langle\sigma\rangle$. Then there exists a homeomorphism $\varphi\colon f^{-1}(\langle\sigma\rangle)\to f^{-1}(\beta_0)\times\langle\sigma\rangle$ such that f restricted to $f^{-1}(\langle\sigma\rangle)$ is equal to φ followed by the projection $\pi\colon f^{-1}(\beta_0)\times\langle\sigma\rangle\to\langle\sigma\rangle$.*

Proof. For each $\beta\in\langle\sigma\rangle$ let $\theta_\beta\colon f^{-1}(\beta)\to f^{-1}(\beta_0)$ and $\eta_\beta\colon f^{-1}(\beta_0)\to f^{-1}(\beta)$ be as in the lemma.

Define $\varphi\colon f^{-1}(\langle\sigma\rangle)\to f^{-1}(\beta_0)\times\langle\sigma\rangle$ by $\varphi(\alpha) = \big(\theta_{f(\alpha)}(\alpha), f(\alpha)\big)$ and $\psi\colon f^{-1}(\beta_0)\times\langle\sigma\rangle\to f^{-1}(\langle\sigma\rangle)$ by $\psi(\gamma,\beta) = \eta_\beta(\gamma)$. It is clear that φ and ψ are well-defined and continuous. Routine calculation shows that ψ is an inverse for φ. Thus φ is a homeomorphism.

5. Proof of Main theorem. Let Y' be the first barycentric subdivision of Y. Let X' be the subdivision of X constructed in a manner similar to the barycentric subdivision with the exception that the barycenter, $b(\tau)$, of a simplex τ will be replaced by another point of $\langle\tau\rangle$ chosen as follows.

Let $f(\tau) = \{u_0, u_1, \dots, u_k\}$. For each $j = 0, 1, \dots, k$ let τ_j be the largest face of τ such that $f(\tau_j) = u_j$ and let n_j be the number of vertices of τ_j. Now define $c(\tau)$ (which will replace $b(\tau)$ in the subdivision of X) by

$$c(\tau)(v) = \frac{1}{n_j}\cdot\frac{1}{k+1}\,,\qquad \text{where}\qquad v\in\tau_j\,.$$

Note that with this definition $f\big(c(\tau)\big) = b\big(f(\tau)\big)$. Define $f'\colon X'\to Y'$ by $f'\big(c(\tau)\big) = b\big(f(\tau)\big)$. It may be seen that as mappings of $|X| = |X'|$ into $|Y| = |Y'|$, f and f' are identical. Furthermore f' is simplicial.

For each simplex σ in Y' let $\{A_\gamma\colon \gamma\in\Gamma_\sigma\}$ be the set of all components of $f^{-1}(\langle\sigma\rangle)$. For each A_γ, $\gamma\in\Gamma_\sigma$, let σ_γ be a simplex of the same dimension as σ, together with all its faces. Thus if $\sigma = \{u_0, u_1, \dots, u_n\}$, then σ_γ consists of $\{u_0^\gamma, u_1^\gamma, \dots, u_n^\gamma\}$ together with its faces.

For every simplex τ of X' and every face ϱ of τ, $\langle\tau\rangle\subset A_\gamma$ for some $\gamma\in\Gamma_{f(\tau)}$ and $\langle\varrho\rangle\subset A_\delta$ for some $\delta\in\Gamma_{f(\varrho)}$. For each such τ and ϱ we identify $f(\varrho)_\delta$ with the "corresponding face" of $f(\tau)_\gamma$ in the "natural way". Thus if $f(\tau) = \{u_0, u_1, \dots, u_n\}$ and $f(\varrho) = \{u_{i_0}, u_{i_1}, \dots, u_{i_m}\}$ $(m<n)$, we identify $f(\varrho)_\delta = \{u_{i_0}^\delta, u_{i_1}^\delta, \dots, u_{i_m}^\delta\}$ with the face $\{u_{i_0}^\gamma, u_{i_1}^\gamma, \dots, u_{i_m}^\gamma\}$ by matching $u_{i_k}^\delta$ with $u_{i_k}^\gamma$ for each $k = 0, 1, \dots, m$.

Call the resulting space W.

Define $h\colon W \to Y'$ by letting $h|\sigma_\gamma$ map σ_γ onto σ isomorphically. Thus if $\sigma = \{u_0, u_1, \ldots, u_n\}$, then h sends the vertex u_i^γ of σ_γ to the vertex u_i of σ. The method of defining the identifications, given above, insures that h will be well-defined.

The key point in the proof is to show that W is a simplicial complex. The only problem is that a single set of vertices may "define" more than one simplex. The example given in section 2 shows how this could happen if we fail to subdivide X and Y before forming W. Let ω be an "n-simplex" of W. By this we mean that ω was an n-simplex in the pre-identification space from which W was obtained. Now $h(\omega)$ is an n-simplex of Y'. Hence there exists a simplex λ of Y such that $\langle h(\omega)\rangle \subset \langle\lambda\rangle$ and such that the barycenter b of λ is a vertex of $h(\omega)$.

Let $\beta \in \langle h(\omega)\rangle$. If we apply the lemma and its corollary we see that $f^{-1}(\langle h(\omega)\rangle)$ is homeomorphic to $f^{-1}(\beta) \times \langle h(\omega)\rangle$ which is, in turn, homeomorphic to $f^{-1}(b) \times \langle h(\omega)\rangle$.

Thus there is a natural one-to-one correspondence between the components of $f^{-1}(\langle h(\omega)\rangle)$ and the components of $f^{-1}(b)$. Hence there is a one-to-one correspondence between the vertices of W which are mapped to b and the components of $f^{-1}(\langle h(\omega)\rangle)$.

Thus any "n-simplex" ω' of W which is distinct from ω has a vertex which is distinct from the vertices of ω. Hence W is a simplicial complex.

It is clear that h is a simplicial mapping.

Define $g\colon X' \to W$ by letting $g|\tau$ map τ onto $f(\tau)_\gamma$, where $\langle\tau\rangle \subset A_\gamma$ for some $\gamma \in \Gamma_{f(\tau)}$, in the "natural way". Thus if f maps the vertex v of τ to the vertex u of Y', then g maps v to the vertex u^γ of W. Again the method of defining the identifications guarantees that g is well-defined. It is clear that g is a simplicial mapping and that $f' = hg$.

The mapping h does not collapse simplices; that is, the image of each simplex of W is a simplex of the same dimension in Y'. Hence h is light and has, in fact, discrete fibers.

For each simplex σ_γ of $W g^{-1}(\langle\sigma_\gamma\rangle) = A_\gamma$. By the corollary $g^{-1}(\langle\sigma_\gamma\rangle)$ is homeomorphic to $g^{-1}(\beta) \times \langle\sigma_\gamma\rangle$ for each $\beta \in \langle\sigma_\gamma\rangle$. Thus the connectedness of A_γ implies the connectedness of $g^{-1}(\beta)$ for each β. Therefore g is semi-monotone. This completes the proof.

6. Remark. The proof just completed contains a characterization of semi-monotoniety for simplicial mappings which may be stated as follows: if $f\colon X \to Y$ is any simplicial mapping, then in order for f to be semi-monotone it is necessary and sufficient that for each vertex v of the barycentric subdivision of Y, $f^{-1}(v)$ is connected.

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