Some properties of fundamental dimension

by

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Abstract. By a fundamental dimension of compactum $X$ we understand the number $\text{Fd}(X) = \min(\dim Y : \text{Sh}(X) \leq \text{Sh}(Y))$.

The main purpose of this paper is to give characterizations of compacta with fundamental dimension $\leq n$, where $n$ is a positive integer or 0, and to apply these characterizations to the study of some problems.

By a fundamental dimension of compactum $X$ we understand the number $\text{Fd}(X) = \min(\dim Y : \text{Sh}(X) \leq \text{Sh}(Y))$.

This notion has been introduced by K. Borsuk in [3] (see also [5], p. 31). In the theory of shape it has a similar role to that of dimension in topology. Therefore it is one of the most important invariants of the theory of shape.

The main purpose of this paper is to give characterizations of compacta with fundamental dimension $\leq n$ (where $n$ is a positive integer or 0) and to apply these characterizations to the study of the following problems:

(1) Suppose that all components of the compactum $X$ have fundamental dimension $\leq n$. Is it true that $\text{Fd}(X) \leq n$?

(2) Suppose that $X = \lim (X_k, F_{k+1})$ and $\text{Fd}(X_k) \leq n$ for every $k = 1, 2, \ldots$ Is it true that $\text{Fd}(X) \leq n$?

(3) Suppose that $X, Y$ are compacta. Is it true that $\text{Fd}(X \cup Y) \leq \max(\text{Fd}(X), \text{Fd}(Y), 1 + \text{Fd}(X \cap Y))$?

(4) Suppose that $X, A$ are compacta and $X \supset A$. Is it true that $\text{Fd}(X \setminus A) \leq \max(\text{Fd}(X), \text{Fd}(A) + 1)$?

(5) Suppose that $M$ is a connected $n$-manifold and let $X \subset M$ be a compactum. Is it true that $\text{Fd}(X) \leq n - 1$?

Theorems (2.1), (3.2), (4.1) and (4.18) give positive answers to problems (1), (3), (3), and (4). Theorem (5.6) gives a partial answer to problem (3). The above-mentioned characterizations are given in Theorems (1.3), (1.7) and (1.9). The first section contains also Theorem (1.4), which implies that $\text{Fd}(X) = \min(\dim Y : \text{Sh}(X) = \text{Sh}(Y))$. This result was obtained by W. Holstżyński as early as 1968.
By \( Q \) we denote the Hilbert cube \([0, 1] \times [0, 1] \times \ldots \) and for every \( k = 1, 2, \ldots \) we denote by \( r_k \) the natural projection

\[ r_k: [0, 1] \times [0, 1] \times \ldots = Q \to [0, 1]^k \]

given by the formula

\[ r_k(x_1, x_2, \ldots, x_k, x_{k+1}, \ldots) = (x_1, x_2, \ldots, x_k) \]

for \((x_1, x_2, \ldots, x_k, x_{k+1}, \ldots) \in Q\).

The expression \( A \cong B \) means that the sets \( A \) and \( B \) are homeomorphic.

By a map we always understand a continuous function.

We now give the definition of a strong deformation retract. Let \( X, Y \) be topological spaces and \( A \subseteq X \) and suppose that the maps \( f_0, f_1: X \to Y \) agree on \( A \). Then \( f_0 \) is homotopic to \( f_1 \) relative to \( A \) (denoted by \( f_0 \simeq f_1 |_{rel A} \)) if there exists a map \( \varphi: X \times [0, 1] \to Y \) such that \( \varphi(x, 0) = f_0(x) \) and \( \varphi(x, 1) = f_1(x) \) for \( x \in X \) and \( \varphi(x, t) = f_t(x) \) for \( x \in A \) and \( t \in [0, 1] \). The subspace \( A \) of the topological space \( X \) is a strong deformation retract of \( X \) if there is a retraction \( r \) of \( X \) to \( A \) such that \( fr \simeq f |_{rel A} \), where \( r: X \to X \) is the identity and \( j: A \to X \) is the inclusion map.

I would like to express my sincere gratitude to Professor K. Borsuk for his guidance and valuable remarks.

§ 1. Characterizations of compacta with fundamental dimension \( \leq n \). If \( X, Y \) are compacta and \( f: X \to Y \) is a map, we denote by \( \omega(f) \) the smallest integer \( n \) such that there exists a map \( g: X \to Y \) homotopic to \( f \) and satisfying the condition: \( \dim g(X) \leq n \) or \( \infty \) when there is no such number does not exist. Let \( Y \) be a polyhedron and let \( f: X \to Y \) be a map. Then \( \omega(f) \leq n \) iff \( f \) is homotopic to a map \( g: X \to Y \) such that \( g(X) \) lies in the combinatorial \( n \)-skeleton of \( Y \) (the combinatorial \( n \)-skeleton of \( Y \) is a homotopy \( n \)-skeleton of \( Y \), see [1], p. 612).

S. Gedanken and W. Holsztynski have shown (see [8], p. 376) the following:

(1.1) Theorem. Suppose that \( \text{dim}(X) \leq n \) and \( Y \) is a polyhedron. Then \( \omega(f) \leq n \) for every map \( f: X \to Y \).

Let us prove the following elementary lemmas:

(1.2) Lemma. Suppose that \( Q \supset Z_1 \supset Z_2 \supset \ldots \supset Z_k \) is a compactum for each \( i = 1, 2, \ldots \) and \( Z_k \) is a compactum for each \( k = 1, 2, \ldots \). Let \( Y \) be a compactum. Let \( X, X_k \) be two ANR-sets such that \( Q \subseteq Y \subseteq Z_k \) and let \( \varphi: X \to [0, 1] \to Y \) be a homotopy satisfying the following condition:

\[ \varphi(x, 0) = x \quad \text{and} \quad \varphi(x, 1) \in Y_k \quad \text{for every} \quad x \in X. \]

Then there exist a positive integer \( k' \) and a map \( \tilde{\varphi}: Z_k \to [0, 1] \to Y \) such that

\[ \tilde{\varphi}(x, 0) = x \quad \text{and} \quad \tilde{\varphi}(x, 1) \in Y_k' \quad \text{for every} \quad x \in Z_k. \]

Proof. Since \( Y_k \) is ANR, there exist a neighbourhood \( U \) of \( X \) and a map \( \lambda: U \to Y_k \) such that \( \lambda(x) = \varphi(x, 1) \) for every \( x \in U \).

Let \( k' > 1 \) be a positive integer such that \( Z_k \subseteq U \). Consider now a map \( \tilde{\varphi}: Z_k \to [0, 1] \to X \times [0, 1] \to Y \) given by the following formula:

\[ \tilde{\varphi}(x, t) = \begin{cases} \varphi(x, t) & \text{for every} \quad (x, t) \in X \times [0, 1], \\ \lambda(x) & \text{for every} \quad x \in Z_k, \quad \text{and} \quad t = 1. \end{cases} \]

Since \( \lambda \in \text{ANR} \), there exist a neighbourhood \( U' \) of \( Z_k \subseteq U \subseteq [0, 1] \times X \times [0, 1] \) in \( \text{dim}(U') \leq \text{dim}(X) \times [0, 1] \) and a map \( \tilde{\varphi}: \text{dim}(X) \times [0, 1] \to Y \) such that \( \tilde{\varphi}(x, 0) = x \quad \text{for every} \quad x \in Z_k \) and \( \lambda(x) = \tilde{\varphi}(x, 1) \quad \text{for every} \quad x \in Z_k \).

Moreover, there exists a neighbourhood \( U'' \) of \( X \) in \( Z_k \subseteq U' \subseteq U'' \subseteq [0, 1] \times X \times [0, 1] \) such that \( U'' \subseteq U' \subseteq [0, 1] \times X \times [0, 1] \).

Now let us observe that \( \tilde{\varphi} \simeq \lambda \simeq \varphi \simeq \tilde{\varphi} \).

(1.3) Lemma. Let \( X = \lim(X_k, p_k^{+1}) \), where \( X_k \) is ANR and \( p_k^{+1}: X_k \to X_{k+1} \) is a map. If \( \omega(p_k^{+1}) \leq n \) for each \( k = 1, 2, \ldots \), then \( \text{dim}(X) \leq n \).

Proof. Let \( q_k^{+1}: X_k \to X_k \) be a map homotopic to \( p_k^{+1} \) and such that \( \text{dim}(q_k^{+1}(X_k)) \leq n \). It is known (see [9], p. 1107, Theorem (3.6)) that

\[ \text{Sh}(X) = \text{Sh}(\lim(X_k, p_k^{+1})) \leq \text{Sh}(\lim(X_k, q_k^{+1})). \]

Let \( \lim(X_k, q_k^{+1}) = \lim(X_k, q_k^{+1}) \subseteq \text{dim}(X_k) \).

This means that

\[ q_k^{+1}(X_k) = a_k \in \text{dim}(X_k, q_k^{+1}) \subseteq \text{dim}(X_k). \]

We infer that \( \text{dim}(X_k, q_k^{+1}) = \lambda(Y_k, q_k^{+1}(Y_k)) \subseteq Y_k \). Hence \( \text{dim}(Y_k) \leq n \) for every \( k = 1, 2, \ldots \). This follows that \( \text{dim}(X) \leq n \). This completes the proof.

(1.4) Remark. From this proof it follows that if \( X = \lim(X_k, p_k^{+1}) \) (where \( X_k \) is ANR for \( k = 1, 2, \ldots \)) and \( \omega(p_k^{+1}) \leq n \) for every \( k = 1, 2, \ldots \), then there exists a compactum \( Y \) such that \( \text{dim}(Y) \leq n \) and \( \text{Sh}(X) = \text{Sh}(Y) \).
Let us observe that
\[ r_k \circ \alpha(x) = f_k \circ p_{\alpha}(x) \quad \text{for every } x \in X. \]

It follows hence by an elementary argument that
\[ i_k = f_k \circ p_k. \]

Since \( p_k \) is homotopic to a map \( p_k' : X_k \to X_k \) such that \( p_k'(X) \subset Y^k \), we infer that \( f_k \circ p_k'(X) \subset Y^k \) and \( i_k' = f_k \circ p_k' = i_k \).

It is clear that
\[ i_k' = f_k \circ p_k'(X) \subset Y^k \subset Y^k \]

Hence, using Lemma (1.2) in the case where \( Y = Y_k \), \( Y_k = Y^k \), \( Y_k = Y_k ', Z_k = Y_{k+2}, \ldots \), we conclude that there exist a \( k' > k \) and a homotopy \( \psi : \Gamma_k' \times [0, 1] \to X_k \) such that
\[ \psi(\Gamma_k' \times [1]) \subset Y^k \]

and
\[ \psi(y_0, 0) = y \quad \text{for every } y \in \Gamma_k'. \]

Then setting
\[ \lambda(x, t) = g_k \psi(f_k'(x), t) \quad \text{for every } (x, t) \in X_k \times [0, 1], \]

we get a homotopy \( \lambda : X_k \times [0, 1] \to X_k \) such that
\[ \lambda(x, 0) = g_k \psi(f_k'(x), 0) = g_k(f_k'(x), 1) = p_k'(x) \quad \text{for every } x \in X_k \]

and
\[ \lambda(x, 1) = g_k \psi(f_k'(x), 1) \quad \text{for every } x \in X_k. \]

Since \( g_k \psi(f_k'(x), 1) \in Y^k \), we infer that
\[ g_k \psi(f_k'(x), 1) \in g_k(Y^k) \subset X_k, \]

i.e.
\[ \omega(\psi_k) < n. \]

Suppose that \( (X_k, p_k^{(1)}) \) satisfies condition (ii). Then there exists a sequence \( k_i < k_i < \ldots \) of indices such that
\[ \omega(\psi_i) < n \quad \text{for every } i = 1, 2, \ldots \]

It is evident that \( X \equiv \lim X_{k_n} p_{k_n}^{(1)} \). Therefore (by Lemma (1.3)) \( F_k(X) \subset n \). Thus the proof of Theorem (1.5) is completed.

Remark. Since the case where \( F_k(X) \subset n \) every inverse sequence \( (X_k, p_k^{(1)}) \) (where \( X_k \) is a polyhedron for every \( k = 1, 2, \ldots \)) satisfies condition (ii) of (1.5), we get the following:

(1.6) Theorem. If \( F_k(X) \subset n \), then there exists a compactum \( Y \) such that \( \dim(Y) \subset n \) and \( Sh(X) = Sh(Y) \).
The first proof (unpublished) of this theorem was presented by W. Holsztynski at Borsuk’s seminar, December 1968 — January 1969.

From Theorems (1.5) and (1.1) follows:

(1.7) COROLLARY. For every compactum \( X \) the following conditions are equivalent:

(a) \( \text{Fd}(X) \leq n \),

(b) \( \omega(f) \leq n \) for every map \( f : X \to W \), where \( W \) is a polyhedron.

Let us prove the following:

(1.8) LEMMA. Let \( U \) be an open neighbourhood of a compactum \( X \subset Q \) and \( \dim X < n \). Then \( X \) is contractible in \( U \) to a polyhedron \( X' \) of dimension \( < n \).

Proof. It is evident that \( X \) is contractible in \( U \) to a subset of \( P = (0,1) \times (0,1) \times \ldots \subset Q \) whose dimension is \( < n \). Hence we can assume that \( X \subset P \).

It is known (see [10], p. 111) that for every compactum \( X \subset P \) of dimension \( < n \) and every \( \varepsilon > 0 \) there exist a polyhedron \( X_\varepsilon \) of dimension \( < n \) and a map \( p_\varepsilon : X \to X_\varepsilon \), such that

\[
\varphi(x, p_\varepsilon(x)) < \varepsilon,
\]

Since \( U \in \text{ANR}(\mathbb{R}) \), there exist an \( t_\varepsilon > 0 \) and a homotopy \( \chi : X \times [0,1] \to U \) such that \( \chi(x,0) = x \) and \( \chi(x,1) = p_\varepsilon(x) \) for every \( x \in X \). Hence the polyhedron \( X' = X_\varepsilon \) satisfies the required condition and the proof of Lemma (1.8) is finished.

The compacta lying in the Hilbert cube with fundamental dimension \( < n \) are characterized in the following way:

(1.9) THEOREM. Let \( X \) be a compactum lying in the Hilbert cube \( Q \).

Then the following conditions are equivalent:

(a) \( \text{Fd}(X) \leq n \),

(b) For every neighbourhood \( V \subset U \) of \( X \) in \( Q \) there exists a homotopy \( \varphi : X \times [0,1] \to U \) such that

\[
\varphi(x,0) = x \quad \text{for each } x \in X \quad \text{and} \quad \dim \varphi(X \times \{1\}) < n.
\]

(b) For every neighbourhood \( U \) of \( X \) in \( \mathcal{C} \) there exists a neighbourhood \( \mathcal{C} \subset U \) of \( X \) in \( \mathcal{C} \) and a homotopy \( \varphi : X \times [0,1] \to U \) such that

\[
\varphi(x,0) = x \quad \text{for each } x \in V \quad \text{and} \quad \dim \varphi(X \times \{1\}) < n.
\]

Proof. Let \( X_1, X_2, \ldots \) be a decreasing sequence of closed neighbourhoods of \( X \) such that

\[
X = \bigcap_{k=1}^{\infty} X_k,
\]

\[
X_{k+1} \subset \text{Int} X_k \quad \text{for every } k = 1, 2, \ldots
\]

and

\[
X_k \text{ is a prism in } Q \quad \text{for every } k = 1, 2, \ldots,
\]

i.e. there exist a positive integer \( k_0 \) and a polyhedron \( W_k \subset [0,1]^k \) such that \( X_k = n(W_k) \).

It is clear that \( W_k \times [0,1] \times [0,1] \times \ldots \) is a strong deformation retract of \( X_k \) and the inclusion map \( X_k \to X_k \) is homotopic (in \( X_k \)) to some map with values belonging to \( W_k \times [0,1] \times [0,1] \times \ldots \).

Suppose that \( \text{Fd}(X) < n \). Let \( U \) be a neighbourhood of \( X \) in \( Q \) and let \( U \) be such that \( X_k \subset U \). Since the inclusion map \( X_k \to X_k \) is homotopic (in \( X_k \)) to some map with values belonging to \( W_k \times [0,1] \times [0,1] \times \ldots \), we infer from (1.1) that there is a homotopy joining in \( X_k \subset U \) the inclusion with some map \( g : X \to X_k \subset U \) such that \( \dim g(X) < n \). Hence (i) implies (a).

Suppose that \( X \) satisfies (i). By Lemma (1.8) it follows that there exist an \( m \)-dimensional polyhedron \( X' \subset X_k \), where \( m < n \), and a homotopy \( \varphi : X \times [0,1] \to X' \), such that

\[
\varphi(x,0) = x \quad \text{for every } x \in X,
\]

\[
\varphi(X \times \{1\}) \subset X' _{m+1},
\]

Hence, using Lemma (1.2) in the case where \( X = X_k, X_1 = X', Z_1 = X_{k+1}, \ldots \), we conclude that \( X \) satisfies (a).

Suppose that \( X \) satisfies (a). Then there exists a sequence \( k_1 < k_2 < \ldots \) of indices such that \( \dim(X_{k_i}) < n \) for every \( i = 1, 2, \ldots \), where by \( \dim X \) we denote the inclusion map \( j_{k+1} : X_{k+1} \to X_k \). Using (1.3) we infer that \( \text{Fd}(X) < n \). This completes the proof.

§ 2. Fundamental dimension of components of compacta. Let us prove the following

(2.1) THEOREM. A compactum \( X \) has fundamental dimension \( < n \) if and only if all its components have fundamental dimension \( < n \).

Proof. Suppose that \( \text{Fd}(X) < n \). Then there exists a compactum \( Y \) such that \( \dim Y < n \) and \( S_{X_k}(X) \leq S_{Y_k}(Y) \). Moreover (see [6], p. 29)

for every component \( X_k \) of \( X \) there exists a component \( A(X_k) \) of \( Y \) such that \( S_{X_k}(X) \leq S_{Y_k}(A(X_k)) \). This implies that all components of \( X \) have fundamental dimension \( < n \).

Suppose that every component of \( X \) has fundamental dimension \( < n \).

We can assume that \( X \subset Q \).

Consider a neighbourhood \( U \) of \( X \). Then by Theorem (1.9) for every component \( X_k \) of \( X \) there are an open neighbourhood \( Y_k \) of \( X_k \) such that its boundary is disjoint with \( X \) and a homotopy \( \varphi_k : Y_k \times [0,1] \to U \)
such that
\[ q_0(x, 0) = x \quad \text{for every } x \in \hat{V}_0 \]
and
\[ \dim q_0(\hat{V}_0 \times \{1\}) \leq n. \]

Since \( X \) is compact, there is a finite system of indices \( \mu_1, \mu_2, \ldots, \mu_k \) such that
\[ \hat{V} = \hat{V}_{\mu_1} \cup \hat{V}_{\mu_2} \cup \ldots \cup \hat{V}_{\mu_k} \]
is a neighbourhood of \( X \). Setting
\[ V_i = \hat{V}_{\mu_i} \]
for \( i = 1, 2, \ldots, k \),
we get a system of open and disjoint sets \( V_1, V_2, \ldots, V_k \), such that the set \( V = \bigcup_{i=1}^k V_i \) is a neighbourhood of \( X \).

Setting
\[ \varphi(x, t) = q_0(x, t) \quad \text{for every } (x, t) \in \hat{V}_0 \times [0, 1], \]
we get a homotopy \( \varphi' : V \times [0, 1] \to U \).

Let \( Y \) be a closed neighbourhood of \( X \) such that \( V \subseteq V' \) and let \( \varphi = \varphi'(V \times [0, 1]) \).

It follows that \( V \cap V_i \) is a closed subset \( Q \). Hence \( \dim \varphi(V \times \{1\}) \leq n \)
and \( \varphi(x, 0) = x \) for every \( x \in V \).

Using Theorem (1.9), we infer that \( \text{Fd}(X) \leq n \) and the proof of Theorem (2.1) is finished.

### § 3. Fundamental dimension of the inverse limit of compacta and of subsets of manifolds.

Let us prove the following:

#### (3.1) Theorem. Suppose that \( X_1 \supset X_2 \supset \ldots \) are compacta such that
\[ \text{Fd}(X_k) \leq n \quad \text{for every } k = 1, 2, \ldots, \]
Then \( \text{Fd}(\lim_{k \to \infty} X_k) \leq n. \)

**Proof.** We can assume that \( X_1 \subset Q \). Take an arbitrary neighbourhood \( U \) of \( X = \lim_{k \to \infty} X_k \) in \( Q \). Let \( k \) be such that \( X \subset U \). From Theorem (1.9), we infer that \( \omega(i) \leq n \), where by \( i_k \) we denote the inclusion \( i_k : X_k \to U \). It follows at once that \( \omega(i) \leq n \), where by \( i_0 \) we denote the inclusion \( i_0 : X_0 \to U \). From Theorem (1.9), we infer that \( \text{Fd}(X) \leq n \).

#### (3.2) Corollary. Suppose that \( X = \lim_{k \to \infty} (X_k, P_k^{k+1}) \), where \( X_1, X_2, \ldots \)
are compacta and \( \text{Fd}(X_k) \leq n \) for every \( k = 1, 2, \ldots \). Then \( \text{Fd}(X) \leq n. \)

**Proof.** Using an analogous argument to that used in the proof of Theorem (1.6), one infers that there exists a sequence \( X'_1 \supset X'_2 \supset \ldots \)
of compacta such that \( X'_{k+1} \) is homotopically equivalent to \( X'_k \) for every \( k = 1, 2, \ldots \), and \( X = \bigcup_{k=1}^{\infty} X'_k \). Hence
\[ \text{Fd}(X'_k) \leq \text{Fd}(X'_k) \leq n \quad \text{and} \quad \text{Fd}(X) = \text{Fd}(\bigcup_{k=1}^{\infty} X'_k) \leq n. \]

A connected \( n \)-manifold \( M \) is said to be **regular** provided for every compactum \( X \subset M \) there is a sequence \( X_n = M \cap N_1 \cap N_2 \cap \ldots \) such that
\[ (3.3) \quad X_n \text{ is an } n \text{-manifold with non-empty boundary for every } k = 1, 2, \ldots \]

### (3.4)
\[ X = \bigcup_{k=1}^{\infty} N_k. \]

M. Brown and B. Cassler have shown (see [13], p. 94) that if \( N \) is a compact and connected \( n \)-manifold with boundary \( B \neq \emptyset \), then there is a map \( g : B \to R \), \( \dim R \leq n-1 \), such that the mapping cylinder \( C_g \) is homeomorphic to \( M \). Therefore we have the following

#### (3.5) Lemma. Let \( N \) be a connected and compact \( n \)-manifold with boundary \( B \neq \emptyset \). Then \( \text{Fd}(N) \leq n-1 \).

Let \( M \) be a regular \( n \)-manifold and let \( X \subseteq M \) be a compactum. Suppose that \( X_n = M \cap N_1 \cap N_2 \cap \ldots \) is a sequence which satisfies (3.3) and (3.4). Lemma (3.5) implies that \( \text{Fd}(N_k) \leq n-1 \) for each \( k = 1, 2, \ldots \). By conditions (3.3) and (3.4) and Theorem (3.1) imply that \( \text{Fd}(X) \leq n-1 \).

Thus we get the following

#### (3.6) Theorem. If \( M \) is a regular \( n \)-manifold and compactum \( X \subseteq M \),
then \( \text{Fd}(X) \leq n-1 \).

A PL \( n \)-ball is a polyhedron that is piecewise linearly homeomorphic to an \( n \)-simplex.

A PL \( n \)-manifold is a polyhedron \( M \) such that for every \( x \in M \) there is a subpolyhedron \( N \) of \( M \) which is a neighbourhood of \( x \) in \( M \) and such that it is a PL \( n \)-ball.

It is known (see [15], Chapter III) that for every subpolyhedron \( W \) of \( PL \) \( n \)-manifold \( M \) and for every neighbourhood \( U \) of \( W \) in \( M \) there is a PL \( n \)-manifold \( P \subseteq U \) such that

#### (3.7) \( P \) is a closed polyhedral neighbourhood of \( W \) in \( M \).

#### (3.8) \( P \) collapses to \( W \).

We call \( P \) a regular neighbourhood of \( W \).

It is known (see [15], Chapter III) that \( P \) is an \( n \)-manifold with a non-empty boundary if \( W \neq M \).

If \( M \) is a PL connected \( n \)-manifold and \( X \subseteq M \) is a compactum, then for every open neighbourhood \( U \) of \( X \), \( U \neq M \), there are a polyhedral \( W \subseteq U \) and a regular neighbourhood \( P \subseteq U \) of \( W \) and we infer that \( M \) is a regular manifold.
(3.9) Corollary. Let $M$ be a PL $n$-manifold and let $X \subseteq M$ be a compactum. Then $\text{Fd}(X) \leq n-1$.

(3.10) Corollary. Let the compactum $X$ be a subset of $E^p$. Then $\text{Fd}(X) \leq n-1$.

Remark. It is known (see [14], p. 70) that for $n > 6$ every $n$-manifold without a boundary is a handlebody. Using this theorem M. Stan’ko has shown (unpublished) that every closed and connected $n$-manifold is a regular manifold for every $n > 6$.

(3.11) Problem. Suppose that $M$ is a connected $n$-manifold and let $X \subseteq M$ be a compactum and $M \neq X \subseteq M$. Is it true that $\text{Fd}(X) \leq n-1$?

§ 4. Fundamental dimension of the union of compacta. Let us prove the following:

(4.1) Theorem. Let $X$, $Y$ be compacta. Then

$$\text{Fd}(X \cup Y) \leq \max\{\text{Fd}(X), \text{Fd}(Y), \text{Fd}(X \cap Y) + 1\}.$$  

Proof. Let $T$ be a fixed triangulation of $X$ such that $T$ induces triangulations of $X_0$, $X_1$, and $X_0$. It is evident that for every geometric simplex $\sigma$ the sets $\sigma \cap [-1, 1]$ and $\sigma \cap [0, 1]$ are strong deformation retracts of $\sigma \cap [-1, 1]$, where $\sigma$ is the boundary of $\sigma$. Let $X^0$ be the $n$-skeleton of $X$ and let

$$A_0 = [X \cap [-1, 1] \cup X \cap [0, 1], X \cap [-1, 1], X \cap [-1, 1]]$$

for $n > 1$.

We show that for each $n > 0$ the space $A_{n+1}$ is a strong deformation retract of $A_n$. For each $i = 1, 2$, and each geometric $n$-simplex $\sigma \subseteq X \cap X_i$, let $\phi^i_0(\sigma \cap [-1, 1]) \cap [0, 1] \times \sigma \cap [-1, 1]$ be a homotopy satisfying the conditions:

$$\phi^i_0((x, s), \tilde{\sigma}) = (x, s)$$

for $(x, s) \in \sigma \cap [-1, 1] \cap [0, 1] \times \sigma \cap [-1, 1]$, and $\phi^i_0((x, s), \tilde{\sigma}) = \sigma \cap [-1, 1] \

\text{Defining a map}

$$\pi: \text{Fd}(X \cap Y) \leq \max\{\text{Fd}(X), \text{Fd}(Y), \text{Fd}(X \cap Y) + 1\}.$$  

by the conditions

$$\pi_0(\sigma \cap [-1, 1]) \cap [0, 1] = \phi^0_0(\sigma \cap [-1, 1]) \cap [0, 1]$$

for an $n$-simplex $\sigma \subseteq X \cap X_i$, and

$$\pi_0(\sigma \cap [-1, 1]) \cap [0, 1] = \phi^0_0(\sigma \cap [-1, 1]) \cap [0, 1]$$

for an $n$-simplex $\sigma \subseteq X \cap X_i$,

and

$$\pi_0(x, t) = x$$

for every $(x, t) \in A_{n-1} \times [0, 1]$.

Then $\pi_0$ is well-defined and continuous and a map

$$r_0: A_0 \to \pi_0$$

given by the formula

$$r_0(x) = \pi_0(x, 1)$$

for every $x \in A_0$.

is a strong deformation retract of $A_{n+1}$, for every $i = 1, 0, 1, \ldots, n-1$, $A_{n-1}$ is a strong deformation retract $X = A_n$, and this completes the proof.

(4.2) Lemma. Let $X$, $Y$ be polyhedra and let $A$ be a subpolyhedron of $X$. Let $T_1$, $T_2$ be fixed triangulations of $X$ and $Y$, respectively, such that $T_i$ induces the triangulation $T_i[A]$ of $A$. Suppose that $f: X \to Y$ is a map such that $f: A \to Y$ is the simplicial map and $\omega(f) \subset n$. Then there exist a map $\phi: X \cap [0, 1] \to Y$ satisfying the conditions:

(a) $\phi(x, 0) = f(x)$ for $x \in X$,

(b) $\dim_{\phi}(X \cap [1]) \subset n$,

(c) $\dim_{\phi}(A \cap [0, 1]) \subset A + 1$.

Proof. It is known (see [12], pp. 126 and 128) that there exist a positive integer $N_1$ and a simplicial map $g: X \to Y$ with respect to the triangulation $T_1$ of $X$ homotopic to $f$ relative to $A$ (g is a simplicial approximation of f). The assumptions of (4.2) imply that there exists a map $f_1: X \to Y$ such that $f_1(x) \in T_1$ (by $T_1$ we denote the $n$-skeleton of $T_1$). We can assume that $f_1$ is a simplicial map with respect to $T_1$ (where $N_1 \supset N_1$) and we can assume that $g$ is a simplicial map with respect to $T_2$. Since $f_1 \leq g$ and $f_1$, $g$ are simplicial maps, one infers (see [12], pp. 121 and 122) that there exist $N_2 \supset N_2$ and a finite sequence $a_1, a_2, \ldots, a_k$: $X \to Y$ of simplicial maps with respect to $T_2$ such that $a_1 = g$, $a_k = f_1$, and such that $a_1 = g_1$, $a_k = f_1$, and $a_1 = g_1$, $a_k = f_1$, and $a_1 = g_1$, $a_k = f_1$. It is clear that $\dim_{\phi}(A) \subset A$ for every $i = 1, 2, \ldots, k$.
...k and that for every i = 0, 1, ..., k−1 there exists a homotopy
\( \varphi_t : X \times [0, 1+i/k] \to Y \) such that
\[ \varphi_t(x, 0) = f(x) \quad \text{and} \quad \varphi_t(x, 1/k) = g(x) \quad \text{for} \quad x \in X \]
and
\[ \varphi_t(x, i/k) = a(x) \quad \text{and} \quad \varphi_t(x, i+1/k) = a_{i+1}(x) \quad \text{for} \quad x \in X \]
and
\[ \dim \varphi_t(A \times [i, i+1/k]) \leq \dim A + 1 \quad \text{for} \quad i = 1, 2, \ldots, k−1. \]

Setting \( \varphi_t(x, t) = \varphi_t(x) \) for every \( (x, t) \in X \times [i, i+1/k] \), we get a homotopy satisfying conditions (a), (b), (c) of (4.3). Thus the proof of the lemma is completed.

(4.4) Lemma. Let \( X = \lim \{Z_k, f_k^{k+1}\} \), and let \( B_k \) be a strong deformation retract of \( Z_k \) for every \( k = i, 2, \ldots \) Suppose that \( Z_{k+1} : B_k \in \text{ANR}_k \) and \( f_k^{k+1}(B_{k+1}) \subset B_k \) for every \( k = 1, 2, \ldots \). Then
\[ \text{Sh}(X) = \text{Sh}(\lim \{B_k, f_k^{k+1}(B_{k+1})\}). \]

The simple proof of this lemma may be read to the reader.

(4.5) Lemma. Let \( X, Y \) be compacta and \( Z = \lim \{Z_i, f_i^{i+1}\} \). Then there exist inverse sequences \( \{Z_i, f_i^{i+1}\}, \{Z_{i+1}, f_{i+1}^{i+2}\}, \{Z_i, f_i^{i+2}\} \) and \( \{Z_{i+1}, f_{i+1}^{i+2}\} \) such that for every \( k = 1, 2, \ldots \) the following conditions are satisfied:

(i) \( Z_k = Z_k \cap Z_{k+1}, \quad Z_{k+1} = Z_{k+1} \cap Z_k \)
(ii) \( Z_{k+1} \cap Z_k = Z_{k+1} \cap Z_k \)
(iii) \( Z_{k+1} \cap Z_k = Z_{k+1} \cap Z_k \)
(iv) \( \dim Z_{k+1} \cap Z_k = \dim Z_{k+1} \cap Z_k \)
(v) \( \omega(f_{k+1}^{i+2}(X) \subset \text{Fd}(X \cap Y) \)
(vi) \( \omega(f_{k+1}^{i+2}(X) \subset \text{Fd}(Z \cap Y) \)
(vii) \( \omega(f_{k+1}^{i+2}(X) \subset \text{Fd}(Z \cap Y) \)
(viii) \( \omega(f_{k+1}^{i+2}(X) \subset \text{Fd}(Z \cap Y) \)
(ix) \( \omega(f_{k+1}^{i+2}(X) \subset \text{Fd}(Z \cap Y) \)

By Theorem (1.8) we can assume that \( \omega(f_{k+1}^{i+2}(X) \subset \text{Fd}(X \cap Y) \)

It follows by the simplicial approximation theorem that there exist a triangulation \( T_k \) of \( Z_k \) which induces triangulations of \( Z_{k+1} \) for \( i = 0, 1, 2 \) and simplicial maps \( f_{k+1}^{i+2} : Z_{k+1} \to Z_k \) with respect to \( T_k \) and \( T_{k+1} \) homotopic to \( f_{k+1}^{i+2} \) and such that
\[ \dim f_{k+1}^{i+2}(Z_{k+1}) \leq \text{Fd}(X \cap Y) \]

Using Borsuk's homotopy extension theorem to the homotopy joining \( f_{k+1}^{i+2} \) with \( f_{k+1}^{i+2} \) in \( Z_{k+1} \subset Z_k \) for \( i = 1, 2 \), we infer that there exist maps \( f_{k+1}^{i+2} : Z_{k+1} \to Z_k \) such that conditions (i)-(vii) of (4.5) are satisfied.
Proof of Theorem (4.1). Suppose that $\text{Fd}(X), \text{Fd}(Y), \text{Fd}(X \cap Y)$ < $\infty$. Let $(Z_k^a, p_k^a), (Z_k^b, q_k^b), (Z_k^c, r_k^c)$ and $(Z_k^d, s_k^d)$ be sequences from Lemma (4.5). Consider the inverse sequence $(V_k^a, p_k^a)$ where $V_k = Z_k \times [-1, 1]$ and $p_k^a: V_k \rightarrow V_k$ is a map given by the formula

$$p_k^a(x, t) = (q_k^b(x), t) \quad \text{for every} \quad (x, t) \in V_k \Rightarrow Z_k \times [-1, 1].$$

It is evident that $\text{Sh}(X \cap Y) = \text{Sh}(\lim (V_k^a, p_k^a))$. From Lemma (4.3) we infer that the set $V_k = Z_k^a \times [-1, 1] \cup Z_k^b \times [1] \cup Z_k^c \times [-1, 1]$ is a strong deformation retract of $V_k$. By Lemma (4.4) we infer that

$$\text{Sh}(X \cap Y) = \text{Sh}(\lim (V_k, p_k^a)).$$

Conditions (i) and (iv) of (4.5) imply that

$$\phi^a_k(Z_k^a) \subset A_k,$$

where by $A_k$ we denote the $\text{Fd}(X \cap Y)$-dimensional skeleton of $Z_k^a$ (with respect to $T_k$). Consider now the sequence $(U_k, u_k^a)$, where $U_k = Z_k \times [-1, 1] \cup Z_k^a \times [-1, 1] \cup U_k \times [-1, 1]$ and $u_k^a: U_k \rightarrow U_k$ is given by the formula

$$u_k^a(x) = \phi^a_k(x) \quad \text{for} \quad x \in U_k.$$

(4.12) follows (compare the proof of Lemma (1.3)) that

$$\lim (V_k, p_k^a) = \lim (U_k, u_k^a).$$

Condition (i)-(vii) of (4.5) and Lemma (4.3) imply that for every $k = 1, 2, \ldots$, there exist homotopies

$$\psi^a_1((Z_k^a \times [-1, 1]) \times [0, 1]) \Rightarrow Z_k^a \times [-1, 1]$$

and

$$\psi^a_2((Z_k^a \times [1]) \times [0, 1]) \Rightarrow Z_k^a \times [1]$$

such that

$$\psi^a_1((x, -1), 0) = u_k^a(x) \quad (\phi^a_k(x), -1) \quad \text{for every} \quad x \in Z_k^a,$$

$$\psi^a_2((x, 1), 0) = u_k^a(x) \quad (\phi^a_k(x), 1) \quad \text{for every} \quad x \in Z_k^a,$$

$$\dim \psi^a_1((Z_k^a \times [-1, 1]) \times [0, 1]) \leq \text{max}(\text{Fd}(X), \text{Fd}(Y)),$$

$$\dim \psi^a_2((Z_k^a \times [1]) \times [0, 1]) \leq \text{max}(\text{Fd}(X), \text{Fd}(Y)) + 1,$$

$$\dim \psi^a_1((A_k \times [-1, 1]) \times [0, 1]) \leq \text{dim}A_k + 1 \leq \text{Fd}(X \cap Y) + 1,$$

$$\dim \psi^a_2((A_k \times [1]) \times [0, 1]) \leq \text{Fd}(X \cap Y) + 1.$$
Using Theorem (4.1) one can obtain easily some theorems:

(4.13) Theorem. Let $X, A$ be compacta and $A \subseteq X$. Then $\text{Fd}(X/A) \leq \max(\text{Fd}(X), \text{Fd}(A) + 1)$.

Proof. T. A. Chapman (see [7, p. 653]) has proved that if $A_1, A_2, B$ and $Y$ are compacta such that $A_1, A_2 \subseteq B \subseteq Y$, $\text{Sh}(A_1) = \text{Sh}(A_2)$ and $\text{Sh}(B) = Sh((\text{point}))$, then $\text{Sh}(Y/A_1) = \text{Sh}(Y/A_2)$. Using this theorem in the case where $Y = (X \times [0, 1])/(A \times \{1\})$ and $\sim = R = \pi(A \times [0, 1], \{0\})$, where $\pi_0 \in A_1$ and $\pi: X \times [0, 1] \to Y$ is the natural projection, we get

$$\text{Sh}(Y/A) = \text{Sh}(Y/\pi(A \times [0, 1])) = \text{Sh}(Y/A_1) = \text{Sh}(Y)$$

(because $Y/B \cong X/A$). It is evident that

$$\text{F}(\pi(A \times [0, 1])) \leq \text{F}(A_1) = 0 \quad \text{and} \quad \text{F}(\pi(A) \cap A_1) = \text{F}(A_1).$$

From Theorem (4.1) we infer

$$\text{F}(X/A) = \text{F}(X) \leq \max(\text{F}(\pi(A)), \text{F}(\pi(A) \cap A_1) + 1) = \max(\text{F}(X), \text{F}(A) + 1).$$

This proves (4.13).

Remark. Easy examples show that the fundamental dimension of the space $X/A$, where $X$ and $A$ are compacta and $X \supseteq A$, may be equal to $\text{F}(X)$ and also to $\text{F}(A) + 1$, and it can be less than either of these numbers, as well.

Suppose that $(X, x_0)$ and $(Y, y_0)$ are two pointed compacta. It is clear that there exists a pointed compactum $(Z, z_0)$ such that $Z \simeq Z'$ and there exist two homeomorphisms:

$$k: (X, x_0) \to (Z, z_0) \quad \text{and} \quad k': (Y, y_0) \to (Z', z_0).$$

It is evident that the topological properties of $(Z, z_0)$ depend only on the topological properties of $(X, x_0)$ and of $(Y, y_0)$. Thus we can say (see [4], p. 234) that the topological type of $(Z, z_0)$ is the sum of the topological types of $(X, x_0)$ and of $(Y, y_0)$ and we write shortly $(Z, z_0) = (X, x_0) + (Y, y_0)$.

(4.14) Theorem. Suppose that $X, Y$ are compacta and $X \cap Y$ is of trivial shape. Then $\text{F}(X \cup Y) = \text{F}(X, Y)$.

The above-mentioned result of Chapman implies that if $A$ is a compactum of trivial shape and $A \subseteq X$, then $\text{Sh}(X) = \text{Sh}(X/A)$. Using this theorem in the case where $X$ and $Y$ are compacta and $X \cap Y$ is a compactum of trivial shape, we conclude that

$$\text{Sh}(X \cup Y) = \text{Sh}(X \cup Y/X \cap Y),$$

$$\text{Sh}(X) = \text{Sh}(X/X \cap Y),$$

$$\text{Sh}(Y) = \text{Sh}(Y/X \cap Y).$$

Hence, Theorem (5.1) is a consequence of the following

(4.20) Theorem. Suppose that $X, Y$ are non-empty compacta and $(Z, z_0) = (X, x_0) + (Y, y_0)$. Then $\text{F}(Z) = \text{F}(X, Y) + \text{F}(X, Y)$. top

This theorem gives a positive answer to the question raised by Professor K. Borsuk (see [1], p. 240).

Proof. Theorem (4.20), Theorem (4.1) implies that $\text{F}(Z) \leq \max(\text{F}(X), \text{F}(Y), 1)$. Since the sets $X, Y$ are retracts of $Z$, we infer that $\text{F}(Z) \geq \max(\text{F}(X), \text{F}(Y))$. Suppose that $\text{F}(X) = \text{F}(Y) = 0$. Theorem (2.1) implies that we may assume that $X, Y$ are continua and it is evident that the one-point union of compacta of trivial shape is a compactum of trivial shape.

References