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Some properties of fundamental dimension

by

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Abstract. By a fundamental dimension of compactum X we understand the number $\mathrm{Fd}(X) = \min(\dim Y\colon \mathrm{Sh}(X) \leqslant \mathrm{Sh}\, Y)$.

The main purpose of this paper is to give characterizations of compacta with fundamental dimension $\leqslant n$, where n is a positive integer or 0, and to apply these characterizations to the study of some problems.

By a fundamental dimension of compactum X we understand the number $\mathrm{Fd}(X) = \min(\dim Y\colon \mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y))$.

This notion has been introduced by K. Borsuk in [3] (see also [5], p. 31). In the theory of shape it has a similar role to that of dimension in topology. Therefore it is one of the most important invariants of the theory of shape.

The main purpose of this paper is to give characterizations of compacta with fundamental dimension $\leqslant n$ (where n is a positive integer or 0) and to apply these characterizations to the study of the following problems:

(1) Suppose that all components of the compactum X have fundamental dimension $\leqslant n$. Is it true that $\mathrm{Fd}(X) \leqslant n$?

(2) Suppose that $X = \varprojlim \{X_k, p_k^{k+1}\}$ and $\mathrm{Fd}(X_k) \leqslant n$ for every $k = 1, 2, \ldots$ Is it true that $\mathrm{Fd}(X) \leqslant n$?

(3) Suppose that X, Y are compacta. Is it true that $\mathrm{Fd}(X \cup Y) \leqslant \max(\mathrm{Fd}(X), \mathrm{Fd}(Y), 1 + \mathrm{Fd}(X \cap Y))$?

(4) Suppose that X, A are compacta and $X \supset A$. Is it true that $\mathrm{Fd}(X/A) \leqslant \max(\mathrm{Fd}(X), \mathrm{Fd}(A) + 1)$?

(5) Suppose that M is a connected n-manifold and let $X \subsetneq M$ be a compactum. Is it true that $\mathrm{Fd}(X) \leqslant n-1$?

Theorems (2.1), (3.2), (4.1) and (4.18) give positive answers to problems (1), (2), (3), and (4). Theorem (3.6) gives a partial answer to problem (5). The above-mentioned characterizations are given in Theorems (1.5), (1.7) and (1.9). The first section contains also Theorem (1.6), which implies that $\mathrm{Fd}(X) = \min(\dim Y\colon \mathrm{Sh}(X) = \mathrm{Sh}(Y))$. This result was obtained by W. Holsztyński as early as 1968.

By Q we denote the Hilbert cube $[0,1]\times[0,1]\times\ldots$ and for every $k=1,2,\ldots$ we denote by τ_k the natural projection

$$\tau_k\colon [0,1]\times[0,1]\times\ldots = Q\to[0,1]^k$$

given by the formula

$$\tau_k(x_1, x_2, \ldots, x_k, x_{k+1}, \ldots) = (x_1, x_2, \ldots, x_k)$$

$$\text{for} \quad (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots)\in Q\,.$$

The expression $A\cong B$ means that the sets A and B are homeomorphic.

By a *map* we always understand a continuous function.

We now give the definition of a strong deformation retract. Let X, Y be topological spaces and $A\subset X$ and suppose that the maps $f_0, f_1\colon X\to Y$ agree on A. Then f_0 *is homotopic to* f_1 *relative to* A (denoted by $f_0\simeq f_1\operatorname{rel}A$) if there exists a map $\varphi\colon X\times[0,1]\to Y$ such that $\varphi(x,0) = f_0(x)$ and $\varphi(x,1) = f_1(x)$ for $x\in X$ and $\varphi(x,t) = f_0(x)$ for $x\in A$ and $t\in[0,1]$. The subspace A of the topological space X is a *strong deformation retract* of X if there is a retraction r of X to A such that $i_X\simeq jr\operatorname{rel}A$, where $i_X\colon X\to X$ is the identity and $j\colon A\to X$ is the inclusion map.

I would like to express my sincere gratitude to Professor K. Borsuk for his guidance and valuable remarks.

§ 1. Characterizations of compacta with fundamental dimension $\leqslant n$. If X, Y are compacta and $f\colon X\to Y$ is a map, we denote by $\omega(f)$ the smallest integer n such that there exists a map $g\colon X\to Y$ homotopic to f and satisfying the condition: $\dim g(X)\leqslant n$ or ∞ when there this number does not exist. Let Y be a polyhedron and let $f\colon X\to Y$ be a map. Then $\omega(f)\leqslant n$ iff f is homotopic to a map $g\colon X\to Y$ such that $g(X)$ lies in the combinatorial n-skeleton of Y (the combinatorial n-skeleton of Y is a homotopic n-skeleton of Y, see [1], p. 612).

S. Godlewski and W. Holsztyński have shown (see [8], p. 376) the following:

(1.1) Theorem. *Suppose that* $\operatorname{Fd}(X)\leqslant n$ *and* Y *is a polyhedron. Then* $\omega(f)\leqslant n$ *for every map* $f\colon X\to Y$.

Let us prove the following elementary lemmas:

(1.2) Lemma. *Suppose that* $Q\supset Z_1\supset Z_2\supset\ldots$, *and* Z_i *is a compactum for each* $i=1,2,\ldots$ *Let* $X=\bigcap_{k=1}^{\infty}Z_k$. *Let* Y, Y_0 *be two* ANR-*sets such that* $Q\supset Y\supset Z_1$ *and* $Y\supset Y_0$ *and let* $\varphi\colon X\times[0,1]\to Y$ *be a homotopy satisfying the following condition*:

(a) $$\varphi(x,0) = x \quad \textit{and} \quad \varphi(x,1)\in Y_0 \quad \textit{for each } x\in X\,.$$

Then there exist a positive integer k', *and a map* $\tilde\varphi\colon Z_k\times[0,1]\to Y$ *such that*

(b) $$\tilde\varphi|X\times[0,1] = \varphi\,,$$

(c) $$\tilde\varphi(x,0) = x \quad \textit{and} \quad \tilde\varphi(x,1)\in Y_0 \quad \textit{for each } x\in Z_{k'}\,.$$

Proof. Since $Y_0\in\mathrm{ANR}$, there exist a neighbourhood U of X and a map $\lambda\colon U\to Y_0$ such that $\lambda(x) = \varphi(x,1)$ for every $x\in U$.

Let $k''>1$ be a positive integer such that $Z_{k''}\subset U$. Consider now a map $\chi\colon Z_{k''}\times\{0,1\}\cup X\times[0,1]\to Y$ given by the following formula:

$$\chi(x,t) = \begin{cases} x & \text{for every } x\in Z_{k''} \text{ and } t=0\,,\\ \varphi(x,t) & \text{for every } (x,t)\in X\times[0,1]\,,\\ \lambda(x) & \text{for every } x\in Z_{k''} \text{ and } t=1\,.\end{cases}$$

Since $Y\in\mathrm{ANR}$, there exist a neighbourhood U' of $Z_{k''}\times\{0,1\}\cup X\times\times[0,1]$ in $Z_{k''}\times[0,1]$ and a map $\tilde\chi\colon U'\to Y$ such that

$$\tilde\chi|Z_{k''}\times\{0,1\}\cup X\times[0,1] = \chi\,.$$

Moreover, there exists a neighbourhood U'' of X in $Z_{k''}$ such that $U\times\times[0,1]\subset U'$. Let $k'>k''$ be a positive integer such that $Z_{k'}\subset U''$.

Now let us observe that $\tilde\varphi = \tilde\chi|Z_{k'}\times[0,1]$ satisfies (b) and (c).

(1.3) Lemma. *Let* $X=\varprojlim\{X_k, p_k^{k+1}\}$, *where* $X_k\in\mathrm{ANR}$ *and* $p_k^{k+1}\colon X_{k+1}\to X_k$ *is a map. If* $\omega(p_k^{k+1})\leqslant n$ *for each* $k=1,2,\ldots$, *then* $\operatorname{Fd}(X)\leqslant n$.

Proof. Let $q_k^{k+1}\colon X_{k+1}\to X_k$ be a map homotopic to p_k^{k+1} and such that $\dim p_k^{k+1}(X_{k+1})\leqslant n$. It is known (see [9], p. 1107, Theorem (3.6)) that

$$\operatorname{Sh}(X) = \operatorname{Sh}(\varprojlim\{X_k, p_k^{k+1}\}) = \operatorname{Sh}(\varprojlim\{X_k, q_k^{k+1}\}).$$

Let

$$\{x_k\}_{k=1}^{\infty}\in\varprojlim\{X_k, q_k^{k+1}\}\subset\mathop{\mathbf{P}}_{k=1}^{\infty}X_k\,.$$

This means that

$$q_k^{k+1}(x_{k+1}) = x_k\in q_k^{k+1}(X_{k+1}) = Y_k\,.$$

We infer that $\varprojlim\{X_k, q_k^{k+1}\} = \varprojlim\{Y_k, q_k^{k+1}|Y_{k+1}\} = Y$. Hence $\dim Y\leqslant n$ (because $\dim Y_k\leqslant n$ for every $k=1,2,\ldots$). It follows that $\operatorname{Fd}(X)\leqslant n$. This completes the proof.

(1.4) Remark. From this proof it follows that if $X=\varprojlim\{X_k, p_k^{k+1}\}$ (where $X_k\in\mathrm{ANR}$ for $k=1,2,\ldots$) and $\omega(p_k^{k+1})\leqslant n$ for every $k=1,2,\ldots$, then there exists a compactum Y such that $\dim Y\leqslant n$ and $\operatorname{Sh}(X)=\operatorname{Sh}(Y)$.

(1.5) THEOREM. *Let* $X = \varprojlim\{X_k, p_k^{k+1}\}$, *where* X_k *is a polyhedron, and* $p_k^{k+1}\colon X_{k+1}\to X_k$ *is a map for every* $k = 1, 2, \ldots$ *Then the following conditions are equivalent*:

(i) $\mathrm{Fd}(X) \leqslant n$,

(ii) $\omega(p_k) \leqslant n$ *for every* $k = 1, 2, \ldots$, *where* $p_k\colon X\to X_k$ *is the natural projection*,

(iii) *for every* $k = 1, 2, \ldots$ *there exists a* $k' > k$ *such that* $\omega(p_k^{k'}) \leqslant n$.

Proof. Suppose that $\mathrm{Fd}(X) \leqslant n$. Then we conclude from Theorem (1.1) that $\omega(p_k) \leqslant n$ for every $k = 1, 2, \ldots$

Suppose that (ii) is satisfied. Let $X_k^{(n)}$ be the n-dimensional combinatorial skeleton of the polyhedron X_k. From (ii) we infer that there exists a map $\tilde{\varphi}_k\colon X\times[0,1]\to X_k$ such that

$$\tilde{\varphi}_k(x, 0) = p_k(x) \quad \text{and} \quad \tilde{\varphi}_k(x, 1) \in X_k^{(n)} \quad \text{for every } x \in X.$$

Suppose that $X_k \subset [0,1]^{l_k}$ for every $k = 1, 2, \ldots$ Let $Y_k \subset I^{l_1}\times I^{l_2}\times I^{l_3}\times \ldots \cong Q$ be defined as the set of all points $(y_1, y_2, \ldots) \in X_1\times X_2\times\ldots$ with y_k in X_k such that if $j \leqslant k$, then $y_j = p_j^k(y_k)$ and if $j \nleqslant k$, then $y_j \in I^{l_j}$ is arbitrary.

Let $Y_k^{(n)} \subset Y_k$ be the set of all points $(y_1, y_2, \ldots) \in I^{l_1}\times I^{l_2}\times\ldots$ with $y_k \in X_k^{(n)}$ such that if $j \leqslant k$, then $y_j = p_j^k(y_k)$ and if $j \nleqslant k$, then $y_j = 0$.

Since Y_k or $Y_k^{(n)}$ is the Cartesian product of a set homeomorphic to X_k or $X_k^{(n)}$, respectively, and an AR-set, we infer that $Y_k, Y_k^{(n)} \in \mathrm{ANR}$.

Let us observe that $Y_k \supset Y_{k+1}$ and let $i_k^{k+1}\colon Y_{k+1}\to Y_k$ be the inclusion map for every $k = 1, 2, \ldots$

It is clear that $X = \bigcap_{k=1}^{\infty} Y_k$. Let $i_k\colon X\to Y_k$ be the inclusion map for every $k = 1, 2, \ldots$

Setting

$$g_k(\{y_j\}_{j=1}^{\infty}) = y_k \quad \text{for every } \{y_j\}_{j=1}^{\infty} \in Y_k$$

and for every $x \in X_k$

$$f_k(x) = \{y_j\}_{j=1}^{\infty},$$

where $y_k = x$, $y_j = p_j^k(x)$ for every $j \leqslant k$ and $y_j = 0$ for every $j > k$, we get maps (see [11], p. 42) $f_k\colon X_k\to Y_k$ and $g_k\colon Y_k\to X_k$ such that $f = (\mathrm{id}_N, f_n)$ and $g = (\mathrm{id}_N, g_n)$ are maps of the ANR-sequences $\{X_k, p_k^{k+1}\}$ and $\{Y_k, q_k^{k+1}\}$ (id_N is the identity of the set N of all natural numbers).

Let Y_k' be the set of all sequences $(y_1, y_2, \ldots) \in Y_k$ such that $y_j = 0$ for each $j > k$. Let us observe that Y_k' is a strong deformation retract of Y_k and a strong deformation retractions is a map $r_k\colon Y_k\to Y_k'$ given for every $\{y_j\}_{j=1}^{\infty} \in Y_k$ by the formula

$$r_k(\{y_j\}_{j=1}^{\infty}) = \{y_j'\}_{j=1}^{\infty},$$

where $y_j' = y_j$ for $j \leqslant k$ and $y_j = 0$ for $j > k$.

Let us observe that

$$r_k i_k(x) = f_k p_k(x) \quad \text{for every } x \in X.$$

It follows hence by an elementary argument that

$$i_k \simeq f_k p_k.$$

Since p_k is homotopic to a map $p_k'\colon X\to X_k$ such that $p_k'(X) \subset X_k^{(n)}$, we infer that $f_k p_k'(X) \subset Y_k^{(n)}$ and $i_k' = f_k p_k \simeq i_k$.

It is clear that

$$i_k'(X) = f_k p_k'(X) \subset f_k(X_k^{(n)}) \subset Y_k^{(n)}.$$

Hence, using Lemma (1.2) in the case where $Y = Y_k$, $Y_0 = Y_k^{(n)}$, $Z_1 = Y_{k+1}$, $Z_2 = Y_{k+2}, \ldots$, we conclude that there exist a $k' > k$ and a homotopy $\psi\colon Y_{k'}\times[0,1]\to Y_k$ such that

$$\psi(Y_{k'}\times\{1\}) \subset Y_k^{(n)}$$

and

$$\psi(y, 0) = y \quad \text{for every } y \in Y_{k'}.$$

Then setting

$$\lambda(x, t) = g_k\psi(f_{k'}(x), t) \quad \text{for every } (x, t) \in Y_{k'}\times[0,1],$$

we get a homotopy $\lambda\colon X_{k'}\times[0,1]\to X_k$ such that

$$\lambda(x, 0) = g_k\psi(f_{k'}(x), 0) = g_k(f_{k'}(x)) = p_k^{k'}(x) \quad \text{for every } x \in X_{k'}$$

and

$$\lambda(x, 1) = g_k\psi(f_{k'}(x), 1) \quad \text{for every } x \in X_{k'}.$$

Since $\psi(f_{k'}(x), 1) \in Y_k^{(n)}$, we infer that

$$g_k\psi(f_{k'}(x), 1) \in g_k(Y_k^{(n)}) \subset X_k^{(n)},$$

i.e.

$$\omega(p_k^{k'}) \leqslant n.$$

Suppose that $\{X_k, p_k^{k+1}\}$ satisfies condition (ii). Then there exists a sequence $k_1 < k_2 < \ldots$ of indices such that

$$\omega(p_{k_i}^{k_{i+1}}) \leqslant n \quad \text{for every } i = 1, 2, \ldots$$

It is evident that $X \cong \varprojlim\{X_{k_i}, p_{k_i}^{k_{i+1}}\}$. Therefore (by Lemma (1.3)) $\mathrm{Fd}(X) \leqslant n$. Thus the proof of Theorem (1.5) is completed.

Remark. Since in the case where $\mathrm{Fd}(X) \leqslant n$ every inverse sequence $\{X_k, p_k^{k+1}\}$ (where X_k is a polyhedron for every $k = 1, 2, \ldots$) satisfies condition (ii) of (1.5), we get the following:

(1.6) THEOREM. *If* $\mathrm{Fd}(X) \leqslant n$, *then there exists a compactum* Y *such that* $\dim(Y) \leqslant n$ *and* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

The first proof (unpublished) of this theorem was presented by W. Holsztyński on Borsuk's seminar, December 1968 — January 1969.

From Theorems (1.5) and (1.1) follows:

(1.7) COROLLARY. *For every compactum X the following conditions are equivalent*:

(i) $\mathrm{Fd}(X) \leqslant n$,

(a) $\omega(f) \leqslant n$ *for every map* $f\colon X \to W$, *where* W *is a polyhedron.*

Let us prove the following:

(1.8) LEMMA. *Let U be an open neighbourhood of a compactum $X \subset Q$ and $\dim X \leqslant n$. Then X is contractible in U to a polyhedron X' of dimension $\leqslant n$.*

Proof. It is evident that X is contractible in U to a subset of $P = (0,1)\times(0,1)\times \ldots \subset Q$ whose dimension is $\leqslant n$. Hence we can assume that $X \subset P$.

It is known (see [10], p. 111) that for every compactum $X \subset P$ of dimension $\leqslant n$ and every $\varepsilon > 0$ there exist a polyhedron X_ε of dimension $\leqslant n$ and a map $p_\varepsilon\colon X \to X_\varepsilon$ such that

$$\varrho\big(x, p_\varepsilon(x)\big) < \varepsilon .$$

Since $U \in \mathrm{ANR}(\mathfrak{M})$, there exist an $\varepsilon_0 > 0$ and a homotopy $\chi\colon X \times [0,1] \to U$ such that $\chi(x, 0) = x$ and $\chi(x,1) = p_{\varepsilon_0}(x)$ for every $x \in X$. Hence the polyhedron $X' = X_{\varepsilon_0}$ satisfies the required condition and the proof of Lemma (1.8) is finished.

The compacta lying in the Hilbert cube with fundamental dimension $\leqslant n$ are characterized in the following way:

(1.9) THEOREM. *Let X be a compactum lying in the Hilbert cube Q. Then the following conditions are equivalent*:

(i) $\mathrm{Fd}(X) \leqslant n$.

(α) *For every neighbourhood U of X in Q there exists a homotopy $\varphi\colon X \times [0,1] \to U$ such that*

$$\varphi(x, 0) = x \quad \textit{for each } x \in X \quad \textit{and} \quad \dim \varphi(X \times \{1\}) \leqslant n.$$

(β) *For every neighbourhood U of X in Q there exist a neighbourhood $V \subset U$ of X in Q and a homotopy $\varphi\colon V \times [0,1] \to U$ such that $\varphi(x, 0) = x$ for each $x \in V$ and $\dim \varphi(V \times \{1\}) \leqslant n$.*

Proof. Let $X_1, X_2, \ldots$ be a decreasing sequence of closed neighbourhoods of X such that

$$X = \bigcap_{k=1}^{\infty} X_k ,$$

$$X_{k+1} \subset \mathrm{Int}\, X_k \quad \text{for every } k = 1, 2, \ldots$$

and

$$X_k \text{ is a prism in } Q \quad \text{for every } k = 1, 2, \ldots ,$$

i.e. there exist a positive integer l_k and a polyhedron $W_k \subset [0,1]^{l_k}$ such that $X_k = \tau_{l_k}^{-1}(W_k)$.

It is clear that $W_k \times \{0\} \times \{0\} \times \ldots$ is a strong deformation retract of X_k and the inclusion map X into X_k is homotopic (in X_k) to some map with values belonging to $W_k \times \{0\} \times \{0\} \times \ldots$

Suppose that $\mathrm{Fd}(X) \leqslant n$. Let U be a neighbourhood of X in Q and let k be such that $X_k \subset U$. Since the inclusion map X into X_k is homotopic (in X_k) to some map with values belonging to $W_k \times \{0\} \times \{0\} \times \ldots$, we infer from (1.1) that there is a homotopy joining in $X_k \subset U$ the inclusion with some map $g\colon X \to X_k \subset U$ such that $\dim_g(X) \leqslant n$. Hence (i) implies (α).

Suppose that X satisfies (α). By Lemma (1.8) it follows that there exist an m-dimensional polyhedron $X_k' \subset X_k$, where $m \leqslant n$, and a homotopy $\varphi\colon X \times [0,1] \to X_k$, such that

$$\varphi(x, 0) = x \quad \text{for every } x \in X ,$$

$$\varphi(X \times \{1\}) \subset X_k' .$$

Hence, using Lemma (1.2) in the case where $Y = X_k$, $Y_0 = X_k'$, $Z_1 = X_{k+1}, \ldots$, we conclude that X satisfies (β).

Suppose that X satisfies (β). Then there exists a sequence $k_1 < k_2 < \ldots$ of indices such that $\omega(i_{k_i}^{k_{i+1}}) \leqslant n$ for every $i = 1, 2, \ldots$, where by $i_{k_i}^{k_{i+1}}$ we denote the inclusion map $i_{k_i}^{k_{i+1}}\colon X_{k_{i+1}} \to X_{k_i}$. Using (1.3) we infer that $\mathrm{Fd}(X) \leqslant n$. This completes the proof.

§ 2. Fundamental dimension of components of compacta.

Let us prove the following

(2.1) THEOREM. *A compactum X has fundamental dimension $\leqslant n$ if and only if all its components have fundamental dimension $\leqslant n$.*

Proof. Suppose that $\mathrm{Fd}(X) \leqslant n$. Then there exists a compactum Y such that $\dim Y \leqslant n$ and $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$. Moreover (see [6], p. 28), for every component X_μ of X there exists a component $\Lambda(X_\mu)$ of Y such that $\mathrm{Sh}(X_\mu) \leqslant \mathrm{Sh}\big(\Lambda(X_\mu)\big)$. This implies that all components of X have fundamental dimension $\leqslant n$.

Suppose that every component of X has fundamental dimension $\leqslant n$. We can assume that $X \subset Q$.

Consider a neighbourhood U of X. Then by Theorem (1.9) for every component X_μ of X there are an open neighbourhood $\hat{V}_\mu$ of X_μ such that its boundary is disjoint with X and a homotopy $\varphi_\mu\colon \hat{V}_\mu \times [0,1] \to U$

such that
$$\varphi_\mu(x, 0) = x \quad \text{for every } x \in \hat{V}_\mu$$
and
$$\dim \varphi_\mu(\hat{V}_\mu \times \{1\}) \leqslant n .$$

Since X is compact, there is a finite system of indices $\mu_1, \mu_2, \ldots, \mu_k$ such that
$$\hat{V} = \hat{V}_{\mu_1} \cup \hat{V}_{\mu_2} \cup \ldots \cup \hat{V}_{\mu_k}$$
is a neighbourhood of X. Setting
$$V_i = \hat{V}_{\mu_i} \setminus \overline{\bigcup_{j<i} \hat{V}_{\mu_j}} \quad \text{for} \quad i = 1, 2, \ldots, k,$$
we get a system of open and disjoint sets $V_1, V_2, \ldots, V_k$, such that the set $V' = \bigcup_{i=1}^{k} V_i$ is a neighbourhood of X.

Setting
$$\varphi'(x, t) = \varphi_{\mu_i}(x, t) \quad \text{for every } (x, t) \in V_i \times [0, 1],$$
we get a homotopy $\varphi'\colon V' \times [0, 1] \to U$.

Let V be a closed neighbourhood of X such that $V \subset V'$ and let $\varphi = \varphi' | V \times [0, 1]$.

It follows that $V \cap V_i$ is a closed subset Q. Hence $\dim \varphi(V \times \{1\}) \leqslant n$ and $\varphi(x, 0) = x$ for every $x \in V$.

Using Theorem (1.9), we infer that $\mathrm{Fd}(X) \leqslant n$ and the proof of Theorem (2.1) is finished.

§ 3. Fundamental dimension of the inverse limit of compacta and of subsets of manifolds. Let us prove the following:

(3.1) THEOREM. *Suppose that $X_1 \supset X_2 \supset \ldots$ are compacta such that*
$$\mathrm{Fd}(X_k) \leqslant n \quad \textit{for every } k = 1, 2, \ldots$$
Then $\mathrm{Fd}(\bigcap_{k=1}^{\infty} X_k) \leqslant n$.

Proof. We can assume that $X_1 \subset Q$. Take an arbitrary neighbourhood U of $X = \bigcap_{k=1}^{\infty} X_k$ in Q. Let k be such that $X \subset U$. From Theorem (1.9) we infer that $\omega(i_k) \leqslant n$, where by i_k we denote the inclusion $i_k\colon X_k \to U$. It follows at once that $\omega(i) \leqslant n$, where by i we denote the inclusion $i\colon X \to U$. From Theorem (1.9) we infer that $\mathrm{Fd}(X) \leqslant n$.

(3.2) COROLLARY. *Suppose that $X = \varprojlim \{X_k, p_k^{k+1}\}$, where $X_1, X_2, \ldots$ are compacta and $\mathrm{Fd}(X_k) \leqslant n$ for every $k = 1, 2, \ldots$ Then $\mathrm{Fd}(X) \leqslant n$.*

Proof. Using an analogous argument to that used in the proof of Theorem (1.5) one infers that there exists a sequence $Y_1 \supset Y_2 \supset Y_3 \supset \ldots$ of compacta such that Y_k is homotopically equivalent to X_k for every $k = 1, 2, \ldots$ and $X = \bigcap_{k=1}^{\infty} Y_k$. Hence
$$\mathrm{Fd}(X_k) = \mathrm{Fd}(Y_k) \leqslant n \quad \text{and} \quad \mathrm{Fd}(X) = \mathrm{Fd}(\bigcap_{k=1}^{\infty} Y_k) \leqslant n .$$

A connected n-manifold M is said to be *regular* provided for every compactum $X \subsetneqq M$ there is a sequence $N_0 = M \supset N_1 \supset N_2 \supset \ldots$ such that

(3.3) *N_k is an n-manifold with non-empty boundary for every $k = 1, 2, \ldots$*

(3.4) $$X = \bigcap_{k=1}^{\infty} N_k .$$

M. Brown and B. Cassler have shown (see [13], p. 94) that if N is a compact and connected n-manifold with boundary B, then there is a map $g\colon B \to R$, $\dim R \leqslant n-1$, such that the mapping cylinder C_g is homeomorphic to M. Therefore we have the following

(3.5) LEMMA. *Let N be a connected and compact n-manifold with boundary $B \neq \emptyset$. Then* $\mathrm{Fd}(N) \leqslant n-1$.

Let M be a regular n-manifold and let $X \subsetneqq M$ be a compactum. Suppose that $N_0 = M \supset N_1 \supset N_2 \supset \ldots$ is a sequence which satisfies (3.3) and (3.4). Lemma (3.5) implies that $\mathrm{Fd}(N_k) \leqslant n-1$ for every $k = 1, 2, \ldots$ and conditions (3.3), (3.4) and Theorem (3.1) imply that $\mathrm{Fd}(X) \leqslant n-1$. Thus we get the following

(3.6) THEOREM. *If M is a regular n-manifold and compactum $X \subsetneqq M$, then* $\mathrm{Fd}(X) \leqslant n-1$.

A PL *n-ball* is a polyhedron that is piecewise linearly homeomorphic to an n-simplex.

A PL *n-manifold* is a polyhedron M such that for every $x \in M$ there is a subpolyhedrom N of M which is a neighbourhood of x in M and such that it is a PL n-ball.

It is known (see [15], Chapter III) that for every subpolyhedron W of PL n-manifold M and for every neighbourhood U of W in M there is a PL n-manifold $P \subset U$ such that

(3.7) *P is a closed polyhedral neighbourhood of W in M,*

(3.8) *P collapses to W.*

We call P a *regular neighbourhood* of N.

It is known (see [15], Chapter III) that P is an n-manifold with a non-empty boundary if $W \neq M$.

If M is a PL connected n-manifold and $X \subsetneqq M$ is a compactum, then for every open neighbourhood U of X, $U \neq M$, there are a polyhedral neighbourhood $W \subset U$ of X and a regular neighbourhood $P \subset U$ of W and we infer that M is a regular manifold.

(3.9) COROLLARY. *Let M be a PL n-manifold and let $X \subsetneq M$ be a compactum. Then* $\mathrm{Fd}(X) \leqslant n-1$.

(3.10) COROLLARY. *Let the compactum X be a subset of E^n. Then* $\mathrm{Fd}(X) \leqslant n-1$.

Remark. It is known (see [14], p. 70) that for $n \geqslant 6$ every n-manifold without a boundary is a handlebody. Using this theorem M. Štan'ko has shown (unpublished) that every closed and connected n-manifold is a regular manifold for every $n \geqslant 6$.

(3.11) PROBLEM. *Suppose that M is a connected n-manifold and let X be a compactum and $M \neq X \subset M$. Is it true that* $\mathrm{Fd}(X) \leqslant n-1$?

§ 4. Fundamental dimension of the union of compacta. Let us prove the following:

(4.1) THEOREM. *Let X, Y be compacta. Then*

$$\mathrm{Fd}(X \cup Y) \leqslant \max\big(\mathrm{Fd}(X), \mathrm{Fd}(Y), \mathrm{Fd}(X \cap Y)+1\big).$$

The proof of Theorem (4.1) based on four lemmas:

(4.2) LEMMA. *Let X be a polyhedron and let X_0, X_1 and X_2 be subpolyhedra of X such that $X_0 = X_1 \cap X_2$ and $X = X_1 \cup X_2$. Then the set $X_1 \times \{-1\} \cup X_0 \times [-1,1] \cup X_2 \times \{1\}$ is a strong deformation retract of $X \times [-1,1]$.*

Proof. Let T be a fixed triangulation of X such that T induces triangulations of X_0, X_1 and X_2. It is evident that for every geometric simplex σ, the sets $\sigma \times \{-1\} \cup \dot{\sigma} \times [-1,1]$ and $\sigma \times \{1\} \cup \dot{\sigma} \times [-1,1]$ are strong deformation retracts of $\sigma \times [-1,1]$, where $\dot{\sigma}$ is the boundary of σ. Let $X^{(n)}$ be the n-skeleton of X and let

$$A_n = X_1 \times \{-1\} \cup X_2 \times \{1\} \cup (X^{(n)} \cup X_0) \times [-1,1] \quad \text{for} \quad n \geqslant 1.$$

We show that for each $n \geqslant 0$ the space A_{n-1} is a strong deformation retract of A_n. For each $i = -1, 1$ and for each geometric n-simplex $\sigma \subset \overline{X \setminus X_0}$ let $\varphi_\sigma^{(i)}\colon (\sigma \times [-1,1]) \times [0,1] \to \sigma \times [-1,1]$ be a homotopy satisfying the conditions:

$$\varphi_\sigma^{(i)}\big((x,s),t\big) = (x,s)$$

$$\text{for} \quad \big((x,s),t\big) \in (\sigma \times [-1,1]) \times \{0\} \cup (\dot{\sigma} \times [-1,1] \cup \sigma \times \{i\}) \times [0,1]$$

and

$$\varphi_\sigma^{(i)}\big((x,s),1\big) \in \sigma \times \{i\} \cup \dot{\sigma} \times [-1,1].$$

For $n \geqslant 0$ define a map

$$\varphi_n\colon A_n \times [0,1] \to A_n$$

by the conditions

$$\varphi_n|(\sigma \times [-1,1]) \times [0,1] = \varphi_\sigma^{(-1)} \quad \text{for an } n\text{-simplex } \sigma \subset \overline{X_1 \setminus X_0},$$

$$\varphi_n|(\sigma \times [-1,1]) \times [0,1] = \varphi_\sigma^{(1)} \quad \text{for an } n\text{-simplex } \sigma \subset \overline{X_2 \setminus X_0},$$

and

$$\varphi_n(x,t) = x \quad \text{for every } (x,t) \in A_{n-1} \times [0,1].$$

Then φ_n is well-defined and continuous and a map

$$r_n\colon A_n \to A_{n-1}$$

given by the formula

$$r_n(z) = \varphi_n(z,1) \quad \text{for every } z \in A_n$$

is a strong deformation retraction A_n to A_{n-1}. Let $n_0 = \dim X$. Then

$$A_{n_0} = X \quad \text{and} \quad A_{-1} = X_1 \times \{-1\} \cup X_0 \times [-1,1] \cup X_2 \times \{1\}.$$

Since A_i is a strong deformation retract of A_{i+1} for every $i = -1, 0, 1, \ldots, n_0-1$, A_{-1} is a strong deformation retract $X = A_{n_0}$ and this completes the proof.

(4.3) LEMMA. *Let X, Y be polyhedra and let A be a subpolyhedron of X. Let T_1, T_2 be fixed triangulations of X and Y, respectively, such that T_1 induces the triangulation $T_1|A$ of A. Suppose that $f\colon X \to Y$ is a map such that $f|A\colon A \to Y$ is the simplicial map and $\omega(f) \leqslant n$. Then there exist a map $\varphi\colon X \times [0,1] \to Y$ satisfying the conditions:*

(a) *$\varphi(x,0) = f(x)$ for $x \in X$,*

(b) *$\dim \varphi(X \times \{1\}) \leqslant n$,*

(c) *$\dim \varphi(A \times [0,1]) \leqslant \dim A + 1$.*

Proof. It is known (see [12], pp. 126 and 128) that there exist a positive integer N_1 and a simplicial map $g\colon X \to Y$ with respect to the triangulation $T_1^{(N_1)}$ of X homotopic to f relative to A (g is a simplicial approximation of f). The assumptions of (4.3) imply that there exists a map $f_1\colon X \to Y$ such that $f_1(X) \subset Y^{(n)}$ (by $Y^{(n)}$ we denote the n-skeleton of T_2). We can assume that f_1 is a simplicial map with respect to $T_1^{(N_2)}$ (where $N_2 \geqslant N_1$) and we can assume that g is a simplicial map with respect to $T_1^{(N_2)}$. Since $f_1 \simeq g$ and f_1, g are simplicial maps, one infers (see [12], pp. 131 and 132) that there exist $N_3 \geqslant N_2$ and a finite sequence $a_1, a_2, \ldots, a_k\colon X \to Y$ of simplical maps with respect to $T_1^{(N_3)}$ such that $a_1 = g$, $a_k = f_1$ and such that a_i and a_{i+1} are contiguous for every $i = 1, 2, \ldots, k-1$. It is clear that $\dim a_i(A) \leqslant \dim A$ for every $i = 1, 2, \ldots,$

$\ldots, k-1$ and that for every $i = 0, 1, \ldots, k-1$ there exists a homotopy $\varphi_i\colon X\times\left[\frac{i}{k}, \frac{i+1}{k}\right]\to Y$ such that

$$\varphi_0(x, 0) = f(x) \quad \text{and} \quad \varphi_0\left(x, \frac{1}{k}\right) = g(x) \quad \text{for} \quad x \in X$$

and

$$\varphi_i\left(x, \frac{i}{k}\right) = a_i(x) \quad \text{and} \quad \varphi_i\left(x, \frac{i+1}{k}\right) = a_{i+1}(x) \quad \text{for every } x \in X \text{ and } i = 1, 2, \ldots, k-1$$

and

$$\dim\varphi_i\left(A\times\left[\frac{i}{k}, \frac{i+1}{k}\right]\right) \leqslant \dim A + 1 \quad \text{for every } i = 1, 2, \ldots, k-1 .$$

Setting $\varphi(x, t) = \varphi_i(x, t)$ for every $(x, t)\in X\times\left[\frac{i}{k}, \frac{i+1}{k}\right]$, we get a homotopy satisfying conditions (a), (b), (c) of (4.3). Thus the proof of the lemma is completed.

(4.4) LEMMA. *Let $X = \varprojlim\{Z_k, f_k^{k+1}\}$, and let B_k be a strong deformation retract of Z_k for every $k = 1, 2, \ldots$ Suppose that $Z_k, B_k \in \mathrm{ANR}$, and $f_k^{k+1}(B_{k+1})\subset B_k$ for every $k = 1, 2, \ldots$ Then*

$$\mathrm{Sh}(X) = \mathrm{Sh}(\varprojlim\{B_k, f_k^{k+1}|B_{k+1}\}) .$$

The simple proof of this lemma may be left to the reader.

(4.5) LEMMA. *Let X, Y be compacta and $Z = X\cup Y$. Then there exist inverse sequences $\{Z_k^1, p_k^{k+1}\}$, $\{Z_k^2, q_k^{k+1}\}$, $\{Z_k^0, s_k^{k+1}\}$ and $\{Z_k, r_k^{k+1}\}$ such that for every $k = 1, 2, \ldots$ the following conditions are satisfied:*

(i) *$Z_k = Z_k^1\cup Z_k^2$, Z_k^1, Z_k^2, $Z_k^0 = Z_k^1\cap Z_k^2$ are polyhedra, (Z_k^1, Z_k^0), (Z_k^2, Z_k^0), (Z_k, Z_k^1), (Z_k, Z_k^2) are polyhedral pairs and there exists a triangulation T_k of Z_k which induces triangulations of Z_k^1, Z_k^2, Z_k^0 and such that s_k^{k+1} is a simplicial map with respect to T_k and T_{k+1}.*

(ii) *$s_k^{k+1}(z) = p_k^{k+1}(z) = q_k^{k+1}(z) = r_k^{k+1}(z)$ for every $z\in Z_{k+1}^0$,*

(iii) *$r_k^{k+1}(z) = p_k^{k+1}(z)$ for every $z\in Z_k^1$ and $r_k^{k+1}(z) = q_k^{k+1}(z)$ for every $z\in Z_k^2$,*

(iv) *$\dim s_k^{k+1}(Z_{k+1}^0)\leqslant \mathrm{Fd}(X\cap Y)$,*

(v) *$\omega(p_k^{k+1})\leqslant\mathrm{Fd}(X)$ and $\omega(q_k^{k+1})\leqslant \mathrm{Fd}(Y)$,*

(vi) *$\mathrm{Sh}(X) = \mathrm{Sh}(\varprojlim\{Z_k^1, p_k^{k+1}\})$,*
$\mathrm{Sh}(Y) = \mathrm{Sh}(\varprojlim\{Z_k^2, q_k^{k+1}\})$,
$\mathrm{Sh}(Z) = \mathrm{Sh}(\varprojlim\{Z_k, r_k^{k+1}\})$ and
$\mathrm{Sh}(X\cap Y) = \mathrm{Sh}(\varprojlim\{Z_k^0, s_k^{k+1}\})$.

Proof. We can assume that $X\cup Y\subset Q$. Then there exist a sequence $l_1\leqslant l_2\leqslant\ldots$ of positive integers and sequences $\{Z_k^1\}_{k=1}^\infty$, $\{Z_k^2\}_{k=1}^\infty$ and $\{Z_k^0\}_{k=1}^\infty$ such that Z_k^1, Z_k^2, $Z_k^0 = Z_k^1\cap Z_k^2$ are subpolyhedra of $[0,1]^{l_k}$ and

$$\text{(4.6)}\qquad \begin{aligned} &\tau_{l_k}^{-1}(Z_k^1)\supset\tau_{l_{k+1}}^{-1}(Z_{k+1}^1)\supset X ,\\ &\tau_{l_k}^{-1}(Z_k^2)\supset\tau_{l_{k+1}}^{-1}(Z_{k+1}^2)\supset Y ,\end{aligned}$$

$$X = \bigcap_{k=1}^\infty\tau_{l_k}^{-1}(Z_k^1) \quad\text{and}\quad Y = \bigcap_{k=1}^\infty\tau_{l_k}^{-1}(Z_k^2) .$$

This implies that

$$\text{(4.7)}\qquad \tau_{l_k}^{-1}(Z_k^0)\supset\tau_{l_{k+1}}^{-1}(Z_{k+1}^0) \quad\text{and}\quad X\cap Y = \bigcap_{k=1}^\infty\tau_{l_k}^{-1}(Z_k^0) .$$

Let $Z_k = Z_k^1\cup Z_k^2$. From (4.6) we get

$$\text{(4.8)}\qquad \tau_{l_k}^{-1}(Z_k)\supset\tau_{l_{k+1}}^{-1}(Z_{k+1})\supset X\cup Y \quad\text{and}\quad X\cup Y = \bigcap_{k=1}^\infty\tau_{l_k}^{-1}(Z_k) .$$

From (4.6), (4.7) and (4.8) we infer that for every $k = 1, 2, \ldots$ there are maps $\hat p_k^{k+1}\colon Z_{k+1}^1\to Z_k^1$, $\hat q_k^{k+1}\colon Z_{k+1}^2\to Z_k^2$, $\hat s_k^{k+1}\colon Z_{k+1}^0\to Z_k^0$ and $\hat r_k^{k+1}\colon Z_{k+1}\to Z_k$ such that

$$\text{(4.9)}\qquad \hat s_k^{k+1}(z) = \hat p_k^{k+1}(z) = \hat q_k^{k+1}(z) = \hat r_k^{k+1}(z) \quad\text{for every } z\in Z_{k+1}^0 ,$$

$$\text{(4.10)}\qquad \hat r_k^{k+1}(z) = \hat p_k^{k+1}(z) \text{ for } z\in Z_{k+1}^1 \quad\text{and}\quad \hat r_k^{k+1}(z) = \hat q_k^{k+1}(z) \text{ for } z\in Z_{k+1}^2$$

and

$$\text{(4.11)}\qquad \begin{aligned} \mathrm{Sh}(X) &= \mathrm{Sh}(\varprojlim\{Z_k^1, \hat p_k^{k+1}\}) ,\\ \mathrm{Sh}(Y) &= \mathrm{Sh}(\varprojlim\{Z_k^2, \hat q_k^{k+1}\}) ,\\ \mathrm{Sh}(X\cup Y) = \mathrm{Sh}(Z) &= \mathrm{Sh}(\varprojlim\{Z_k, \hat r_k^{k+1}\}) ,\\ \mathrm{Sh}(X\cap Y) &= \mathrm{Sh}(\varprojlim\{Z_k^0, \hat s_k^{k+1}\}) .\end{aligned}$$

By Theorem (1.5) we can assume that $\omega(\hat p_k^{k+1})\leqslant\mathrm{Fd}(X)$, $\omega(\hat q_k^{k+1})\leqslant\mathrm{Fd}(Y)$, $\omega(\hat s_k^{k+1})\leqslant\mathrm{Fd}(X\cap Y)$, $\omega(\hat r_k^{k+1})\leqslant\mathrm{Fd}(Z)$.

It follows by the simplicial approximation theorem that there exist a triangulation T_k of Z_k which induces triangulations of Z_k^i for $i = 0, 1, 2$ and simplicial maps $s_k^{k+1}\colon Z_{k+1}^0\to Z_k^0$ (with respect to T_k and T_{k+1}) homotopic to $\hat s_k^{k+1}$ and such that

$$\dim\hat s_k^{k+1}(Z_{k+1}^0)\leqslant\mathrm{Fd}(X\cap Y) .$$

Using Borsuk's homotopy extension theorem to the homotopy joining $\hat s_k^{k+1}$ with s_k^{k+1} in $Z_k^0\subset Z_k^i$ for $i = 1, 2$, we infer that there exist maps $p_k^{k+1}\colon Z_{k+1}^1\to Z_k^1$, $q_k^{k+1}\colon Z_{k+1}^2\to Z_k^2$ and $r_k^{k+1}\colon Z_{k+1}\to Z_k$ such that conditions (i)–(vi) of (4.5) are satisfied.

Proof of Theorem (4.1). Suppose that $\mathrm{Fd}(X)$, $\mathrm{Fd}(Y)$, $\mathrm{Fd}(X\cap Y) < +\infty$. Let $\{Z_k^1, p_k^{k+1}\}$, $\{Z_k^2, q_k^{k+1}\}$, $\{Z_k^0, s_k^{k+1}\}$ and $\{Z_k, r_k^{k+1}\}$ be sequences from Lemma (4.5). Consider the inverse sequence $\{V_k, \bar r_k^{k+1}\}$ where $V_k = Z_k\times[-1,1]$ and $\bar r_k^{k+1}\colon V_{k+1}\to V_k$ is a map given by the formula

$$\bar r_k^{k+1}(x,t) = \big(r_k^{k+1}(x), t\big) \quad \text{for every } (x,t)\in V_{k+1} = Z_{k+1}\times[-1,1].$$

It is evident that $\mathrm{Sh}(X\cup Y) = \mathrm{Sh}(\varprojlim\{V_k, \bar r_k^{k+1}\})$. From Lemma (4.2) we infer that the set $V_k' = Z_k^1\times\{-1\}\cup Z_k^2\times\{1\}\cup Z_k^0\times[-1,1]$ is a strong deformation retract of V_k. By Lemma (4.4) we infer that

$$\mathrm{Sh}(X\cup Y) = \mathrm{Sh}(\varprojlim\{V_k', \bar r_k^{k+1}|V_{k+1}'\}).$$

Conditions (i) and (iv) of (4.5) imply that

$$s_k^{k+1}(Z_{k+1}^0)\subset A_k, \tag{4.12}$$

where by A_k we denote the $\mathrm{Fd}(X\cap Y)$-dimensional skeleton of Z_k^0 (with respect to T_k). Consider now the sequence $\{U_k, u_k^{k+1}\}$, where $U_k = Z_k^1\times\{-1\}\cup Z_k^2\times\{1\}\cup A_k\times[-1,1]$ and $u_k^{k+1}\colon U_{k+1}\to U_k$ is given by the formula

$$u_k^{k+1}(z) = \bar r_k^{k+1}(z) \quad \text{for} \quad z\in U_{k+1}.$$

(4.12) follows (compare the proof of Lemma (1.3)) that

$$\varprojlim\{V_k', \bar r_k^{k+1}|V_{k+1}'\} = \varprojlim\{U_k, u_k^{k+1}\}.$$

Condition (i)–(vi) of (4.5) and Lemma (4.3) imply that for every $k = 1, 2, \ldots$, there exist homotopies

$$\varphi_1^k\colon (Z_{k+1}^1\times\{-1\})\times[0,1]\to Z_k^1\times\{-1\}$$

and

$$\varphi_2^k\colon (Z_{k+1}^2\times\{1\})\times[0,1]\to Z_k^2\times\{1\}$$

such that

$$\varphi_1^k\big((z,-1),0\big) = u_k^{k+1}(z,-1) = \big(p_k^{k+1}(z),-1\big) \quad \text{for every } z\in Z_k^1, \tag{4.13}$$

$$\varphi_2^k\big((z,1),0\big) = u_k^{k+1}(z,1) = \big(q_k^{k+1}(z),1\big) \quad \text{for every } z\in Z_k^2, \tag{4.14}$$

$$\dim\varphi_1^k\big((Z_{k+1}^1\times\{-1\})\times\{1\}\big),\ \dim\varphi_2^k\big((Z_{k+1}^2\times\{1\})\times\{1\}\big) \leqslant \max\big(\mathrm{Fd}(X), \mathrm{Fd}(Y)\big), \tag{4.15}$$

$$\dim\varphi_1^k\big((A_{k+1}\times\{-1\})\times[0,1]\big) \leqslant \dim A_{k+1}+1 \leqslant \mathrm{Fd}(X\cap Y)+1, \tag{4.16}$$

$$\dim\varphi_2^k\big((A_{k+1}\times\{1\})\times[0,1]\big) \leqslant \mathrm{Fd}(X\cap Y)+1. \tag{4.17}$$

Let $\alpha\colon [-1,1]\times[0,1]\to[-1,1]\times\{0\}\cup\{-1,1\}\times[0,1] = L$ be a map such that

$$\alpha(0,t) = (0,0),$$
$$\alpha(-1,t) = (-1,t),$$
$$\alpha(1,t) = (1,t),$$
$$\alpha(b,0) = (b,0) \quad \text{for every } b\in[-1,1],$$

$\alpha|[-1,1]\times\{t\}\to L$ is a homeomorphic embedding for every $t\in[0,1]$.

Let

$$\alpha(b,t) = \big(\alpha_1(b,t), \alpha_2(b,t)\big)\in L \quad \text{for every } (b,t)\in[-1,1]\times[0,1].$$

Let $\psi^k\colon U_{k+1}\times[0,1]\to U_k$ be a map given by the formula

$$\psi^k\big((z,b),t\big) = \begin{cases} \varphi_1^k\big((z,b),t\big) & \text{for } \big((z,b),t\big)\in(Z_{k+1}^1\times\{-1\})\times[0,1],\\ \varphi_2^k\big((z,b),t\big) & \text{for } \big((z,b),t\big)\in(Z_{k+1}^2\times\{1\})\times[0,1],\\ \varphi_1^k\big((z,-1),\alpha_2(b,t)\big) & \text{for } \big((z,b),t\big)\in(A_{k+1}\times[-1,1])\times[0,1) \\ & \quad\text{such that } \alpha_1(b,t) = -1,\\ \varphi_2^k\big((z,1),\alpha_2(b,t)\big) & \text{for } \big((z,b),t\big)\in(A_{k+1}\times[-1,1])\times[0,1] \\ & \quad\text{such that } \alpha_1(b,t) = 1,\\ \big(s_k^{k+1}(z),\alpha_1(b,t)\big) & \text{for } \big((z,b),t\big)\in(A_{k+1}\times[-1,1])\times[0,1] \\ & \quad\text{such that } |\alpha_1(b,t)|<1. \end{cases}$$

From (4.13), (4.14) and (4.15) we have

$$\psi^k(U_{k+1}\times\{1\})\subset A_k\times[-1,1]\cup\varphi_1^k\big((Z_{k+1}^1\times\{-1\})\times\{1\}\big)\cup \cup\varphi_2^k\big((Z_{k+1}^2\times\{1\})\times\{1\}\big)\cup\varphi_1^k\big((A_{k+1}\times\{-1\})\times[0,1]\big)\cup \cup\varphi_2^k\big((A_{k+1}\times\{1\})\times[0,1]\big).$$

Conditions (4.15), (4.16), (4.17) imply that

$$\dim\psi^k(U_{k+1}\times\{1\}) \leqslant \max\big(\mathrm{Fd}(X), \mathrm{Fd}(Y), \mathrm{Fd}(X\cap Y)+1\big).$$

From this we infer that

$$\omega(u_k^{k+1}) \leqslant \max\big(\mathrm{Fd}(X), \mathrm{Fd}(Y), \mathrm{Fd}(X\cap Y)+1\big) \quad \text{for every } k = 1, 2, \ldots,$$

and from Theorem (1.5) we have

$$\mathrm{Fd}(X\cup Y) \leqslant \max\big(\mathrm{Fd}(X), \mathrm{Fd}(Y), \mathrm{Fd}(X\cap Y)+1\big).$$

This completes the proof of Theorem (4.1).

Remark. Easy examples show that the fundamental dimension of the union of two compacta X, Y may be equal to $\max\big(\mathrm{Fd}(X), \mathrm{Fd}(Y)\big)$ and also to $\mathrm{Fd}(X\cap Y)+1$, and it can be less than either of these numbers, as well.

Using Theorem (4.1) one can obtain easily some theorems:

(4.18) THEOREM. *Let X, A be compacta and $A \subset X$. Then* $\mathrm{Fd}(X/A) \leqslant \max(\mathrm{Fd}(X), \mathrm{Fd}(A)+1)$.

Proof. T. A. Chapman (see [7], p. 653) has proved that if A_1, A_2, B and Y are compacta such that $A_1, A_2, \subset B \subset Y$, $\mathrm{Sh}(A_1) = \mathrm{Sh}(A_2)$ and $\mathrm{Sh}(B) = \mathrm{Sh}(\{\text{point}\})$, then $\mathrm{Sh}(Y/A_1) = \mathrm{Sh}(Y/A_2)$. Using this theorem in the case where $Y = (X\times\{0\} \cup A\times[0,1])/A\times\{1\}$ and $A_1 = B = a(A\times\times[0,1])$, $A_2 = \{a_0\}$, where $a_0 \in A_1$ and $a\colon X\times\{0\}\cup A\times[0,1]\to Y$ is the natural projection, we get

$$\mathrm{Sh}(X/A) = \mathrm{Sh}\big(Y/a(A\times[0,1])\big) = \mathrm{Sh}(Y/A_1) = \mathrm{Sh}(Y)$$

(because $Y/B \simeq X/A$). It is evident that

$$\mathrm{Fd}\big(a(A\times[0,1])\big) = \mathrm{Fd}(A_1) = 0 \quad \text{and} \quad \mathrm{Fd}\big(a(X)\cap A_1\big) = \mathrm{Fd}(A)\,.$$

From Theorem (4.1) we infer

$$\mathrm{Fd}(X/A) = \mathrm{Fd}(Y) \leqslant \max\big(\mathrm{Fd}(a(X)), \mathrm{Fd}(a(X)\cap A_1)+1\big) = \max\big(\mathrm{Fd}(X), \mathrm{Fd}(A)+1\big)\,.$$

This proves (4.18).

Remark. Easy examples show that the fundamental dimension of the space X/A, where X and A are compacta and $X\supset A$, may be equal to $\mathrm{Fd}(X)$ and also to $\mathrm{Fd}(A)+1$, and it can be less than either of these numbers, as well.

Suppose that (X, x_0) and (Y, y_0) are two pointed compacta. It is clear that there exists a pointed compactum (Z, z_0) such that $Z = Z'\cup Z''$, where $Z'\cup Z'' = \{z_0\}$ and there exist two homeomorphisms:

$$h'\colon (X, x_0)\to(Z', z_0) \quad \text{and} \quad h''\colon (Y, y_0)\to(Z'', z_0)\,.$$

It is evident that the topological properties of (Z, z_0) depend only on the topological properties of (X, x_0) and of (Y, y_0). Thus we can say (see [4], p. 234) that the topological type of (Z, z_0) is the sum of the topological types of (X, x_0) and of (Y, y_0) and we write shortly: $(Z, z_0) = (X, x_0) \underset{\text{top}}{+} (Y, y_0)$.

(4.19) THEOREM. *Suppose that X, Y are compacta and $X\cap Y$ is of trivial shape. Then* $\mathrm{Fd}(X\cup Y) = \max(\mathrm{Fd}(X), \mathrm{Fd}(Y))$.

The above-mentioned result of Chapman implies that if A is a compactum of trivial shape and $A\supset A$, then $\mathrm{Sh}(X) = \mathrm{Sh}(X/A)$. Using this theorem in the case where X, Y are compacta and $X\cap Y$ is a compactum of trivial shape, we conclude that

$$\mathrm{Sh}(X\cup Y) = \mathrm{Sh}(X\cup Y/X\cap Y)\,, \quad \mathrm{Sh}(X) = \mathrm{Sh}(X/X\cap Y)\,,$$
$$\mathrm{Sh}(Y) = \mathrm{Sh}(Y/X\cap Y)\,.$$

Hence, Theorem (5.1) is a consequence of the following

(4.20) THEOREM. *Suppose that X, Y are non-empty compacta and $(Z, z_0) = (X, x_0) \underset{\text{top}}{+} (Y, y_0)$. Then* $\mathrm{Fd}(Z) = \max(\mathrm{Fd}(X), \mathrm{Fd}(Y))$.

This theorem gives a positive answer to the question raised by Professor K. Borsuk (see [4], p. 240).

Proof of Theorem (4.20). Theorem (4.1) implies that $\mathrm{Fd}(Z) \leqslant \max(\mathrm{Fd}(X), \mathrm{Fd}(Y), 1)$. Since the sets X, Y are retracts of Z, we infer that $\mathrm{Fd}(Z) \geqslant \max(\mathrm{Fd}(X), \mathrm{Fd}(Y))$. Suppose that $\mathrm{Fd}(X) = \mathrm{Fd}(Y) = 0$. Theorem (2.1) implies that we may assume that X, Y are continua and it is evident that the one-point union of compacta of trivial shape is a compactum of trivial shape.

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