Periodic actions on the Hilbert cube

by

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Abstract. We study the conjugation problem of two periodic homeomorphisms of the Hilbert cube. Typically we prove that if each has exactly one fixed point p and is trivial at p, then they are conjugate. Some applications to the quotient spaces are included. In particular we prove that the quotient spaces are Hilbert cube factors.

§ 1. Let $n>1$ be a fixed but arbitrary prime number. Let Q denote the Hilbert cube $\prod_{i=1}^{\infty} J_i$, where $J_i = J = [-1, 1]$. Writing Q as $\prod_{i=1}^{\infty} D_i$, a product of 2-disks D_i (in a natural way), let β_n denote the period n homeomorphism on Q by rotating each D_i through an angle of $2\pi/n$ degrees counterclockwise. (From here on β_n will always denote such a standard action on Q.) A homeomorphism $f\colon Q\to Q$ is said to be *trivial* at a point x of f provided that for any neighborhood U of x, there exists an open neighborhood V of x, V homotopically trivial, such that $f(V)=V$. One of our main results is the following

THEOREM 1. *Let $\alpha\colon Q\to Q$ be any periodic homeomorphism of period n having exactly one fixed point (say $O=(0, 0, \ldots)\in Q$). Then in order that α be conjugate to β_n it is necessary and sufficient that α be trivial at* 0.

α is *conjugate* to β_n means that there is a homeomorphism $h\colon Q\to Q$ for which $h\circ\alpha = \beta_n\circ h$. Some rather non-trivial examples of periodic actions on Q may be constructed using the following result of West ([12]):

(West) *The product of a countable infinite collection of (non-degenerate) compact, contractible polyhedra is homeomorphic with the Hilbert cube Q.*

A ready example is given by considering any homeomorphism $h\colon Y\times Q\to Q$, where Y denotes the triod. Let r be the period 3 homeomorphism of Y by rotating the arms of the triod. Let $\alpha_0 = r\times\beta_3\colon Y\times Q \to Y\times Q$ and $\alpha = h\circ\alpha_0\circ h^{-1}$. Clearly, α is trivial at the fixed point of α and is not trivially conjugate to the standard action β_3 on Q. We do not know whether every periodic action on Q having exactly one fixed point (say $O\in Q$) necessarily trivial at O. However, no example that I know of might indicate the contrary.

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A subset A of a space X is a *Z-set* provided Interior $(A) = \emptyset$ and for each homotopically trivial open set U in X, $U \setminus A$ remains homotopically trivial. A subset $X \subset Q$ is *symmetric* provided $(x_i) \in X$ implies $(-x_i) \in X$. Using Theorem 1 we can construct some interesting non-trivial symmetric subsets of Q. Namely, a Hilbert cube may be separated by a symmetric sub-Hilbert cube so that each of its components is itself a symmetric sub-Hilbert cube:

COROLLARY 1. *Let $A = \{(x_i) \in Q\colon\ x_1 \leqslant 0\}$ and $B = \{(x_i) \in Q\colon\ x_1 \geqslant 0\}$ be sub-Hilbert cubes in Q. Then there is a homeomorphism $h\colon Q \to Q$ such that $h(A)$, $h(B)$ and $h(A \cap B)$ are symmetric subsets of Q.*

Proof. Since $A \cap B$ is a Z-set in both A and B, using Anderson ([1]) we can easily construct an involution (that is, a period 2 homeomorphism) α on Q having exactly $O \in Q$ as fixed point and such that α is trivial at O, $\alpha(A) = A$ and $\alpha(B) = B$. By Theorem 1 there is a homeomorphism $h\colon Q \to Q$ for which $h \circ \alpha = \beta_2 \circ h$. h is what we wanted.

Throughout the following let α be any period n homeomorphism of Q having exactly O as fixed point and such that α is trivial at O. Let $Q_0 = Q \setminus \{O\}$, $\alpha_0 = \alpha|_{Q_0}$ and $\beta_0 = \beta_n|_{Q_0}$. By basic covering theory the orbit space Q_0/α_0 is an Eilenberg–Maclane space of type $(Z_n, 1)$; that is, the fundamental group $\pi_1(Q_0/\alpha_0)$ of Q_0/α_0 is isomorphic to Z_n, the integers modulo n, and are trivial in all other dimensions. Furthermore, the projection $P\colon Q_0 \to Q_0/\alpha_0$ is a n-fold covering map. In what follows let "$\cong$" denote "homeomorphic to".

THEOREM 2. $Q_0/\alpha_0 \overset{h}{\cong} Q_0/\beta_0$. *Moreover, if $e \in \pi_1(Q_0/\alpha_0)$ and $e' \in \pi_1(Q_0/\beta_0)$ are generators (with respect to any base points), we may choose a homeomorphism h so that $h_{\#}(e) = e'$.*

Since the orbit space Q/α is an one-point compactification of Q_0/α_0 (similarly for Q/β_n), by Theorem 2 we have

THEOREM 3. $Q/\alpha \cong Q/\beta_n$.

By Lemma 9.9 of Borsuk [3, p. 118] it is not difficult to verify that Q/β_n is an absolute retract (AR). Hence Q/α is an AR. In fact, using a criterion of near homeomorphism given in Theorem 5 below, we may conclude that Q/α is a Hilbert cube factor:

THEOREM 4. $Q/\alpha \times Q \cong Q$.

Let X, Y be metric spaces. A surjection $f\colon X \to Y$ is a *near homeomorphism* provided that for each $\varepsilon > 0$, there exists a homeomorphism $g\colon X \to Y$ such that $d\big(g(x), f(x)\big) < \varepsilon$ for any $x \in X$. A special case of Theorem 4 is proven by Barit–Schori when $\alpha \doteq \beta_2$. Their proof uses a criterion of near homeomorphism given in Curtis ([7]). In the following we give a somewhat different criterion.

THEOREM 5. *Let K, L be locally finite simplicial complexes* (lfsc). *Let $f\colon |K| \to |L|$ be a surjection. Suppose that for each $\delta > 0$, there exists a triangulation L' of L of diameter less than δ such that for each $\sigma \in L'$, $f^{-1}(|\sigma|)$ is homeomorphic to a compact, contractible polyhedron and $f^{-1}\big(\mathrm{Bd}(|\sigma|)\big)$ is a Z-set in $f^{-1}(|\sigma|)$. Then $\mathrm{id} \times f\colon Q \times |K| \to Q \times |L|$ is a near homeomorphism.*

Remark. We shall see that it is unnecessary to assume K to be a lfsc. In some applications, $f^{-1}(|\sigma|)$ is homeomorphic to a n-cell or Q. Thus $f^{-1}(|\sigma|) \times Q \cong Q$. In such a case the proof goes through without using the result of West.

Proof. Let $\varepsilon > 0$ be given. Let i be a large number so that for $Q = J^i \times Q_1$, where $Q_1 = \prod_{k>i} J_k$, then $\mathrm{mesh}(Q_1) < \varepsilon/2$. Let $f_1 = \mathrm{id} \times f\colon Q_1 \times |K| \to Q_1 \times |L|$. By hypothesis there is a triangulation L' of L of mesh less than $\varepsilon/2$ such that for each $\sigma \in L'$, $f^{-1}(|\sigma|)$ is homeomorphic to a compact, contractible polyhedron and $f^{-1}\big(\mathrm{Bd}(|\sigma|)\big)$ is a Z-set in $f^{-1}(|\sigma|)$. By West's theorem $f_1^{-1}(Q_1 \times |\sigma|) \cong Q_1 \times f^{-1}(|\sigma|) \cong Q$ and by [14], $f_1^{-1}(Q_1 \times \mathrm{Bd}|\sigma|)$ is a Z-set in $f_1^{-1}(Q_1 \times |\sigma|)$. Let L_k denote the k-skeleton of L'. We now construct homeomorphisms

$$g_k\colon f_1^{-1}(Q_1 \times |L_k|) \to Q \times |L_k|$$

inductively as follows. Let g_0 be any homeomorphism such that for any 0-simplex $\sigma \in L_0$, $g_0|_{f_1^{-1}(Q_1 \times |\sigma|)}$ is a homeomorphism between Hilbert cubes $f_1^{-1}(Q_1 \times |\sigma|)$ and $Q_1 \times |\sigma|$. Now let $\sigma_1 \in L_1$. Since $f_1^{-1}(Q_1 \times \mathrm{Bd}|\sigma_1|)$ is a Z-set in the Hilbert cube $f_1^{-1}(Q_1 \times |\sigma_1|)$, by Anderson's extension theorem ([1]) we can get a homeomorphism of $f_1^{-1}(Q_1 \times |\sigma_1|)$ onto $Q_1 \times |\sigma_1|$ which extends $g_0|_{f_1^{-1}(Q_1 \times \mathrm{Bd}|\sigma_1|)}$. Repeating the construction for every $\sigma_1 \in L_1$ we obtain a homeomorphism $g_1\colon f_1^{-1}(Q_1 \times |L_1|) \to Q_1 \times |L_1|$ which extends g_0. Define $g_2, g_3, \ldots$ similarly and let $\bar{g}\colon Q_1 \times |K| \to Q_1 \times |L|$ be the final homeomorphism. Let $g = \mathrm{id} \times \bar{g}\colon J^i \times (Q_1 \times |K|) \to J^i \times (Q_1 \times |L|)$. g is what we wanted.

Proof of Theorem 4. By Theorem 3 we may assume $\alpha = \beta_n$. Write Q as a product of 2-disks $D_1 \times D_2 \times \ldots$ Let c_i denote the radial contraction of D_i to $O \in D_i$. Denote $D^i = D_1 \times \ldots \times D_i$ and let $C_i = \mathrm{id} \times c_{i+1}\colon D^i \times D_{i+1} \to D^i \times \{O\}$. Thus (in a natural way) $Q \cong \varprojlim (D^i, C_i)$ and C_i induces mapping $C_i^*\colon D^{i+1}/\beta^{i+1} \to D^i/\beta^i$ such that $Q/\beta_n \cong \varprojlim (D^i/\beta^i, C_i^*)$, where β^i is the action of β_n on the component D^i. We verify routinely that for each $i \geqslant 1$, C_i^* satisfies the conditions of Theorem 5. Hence $C_i^* \times \mathrm{id}\colon D^{i+1}/\beta^{i+1} \times Q \to D^i/\beta^i \times Q$ is a near homeomorphism. By West's theorem each $D^i/\beta^i \times Q$ is homeomorphic to Q and by M. Brown ([4]), $\varprojlim (D^i/\beta^i \times Q, C_i^* \times \mathrm{id}) \cong Q$. Hence $Q/\beta_n \times Q \cong Q$ and the theorem is proved.

In an earlier paper ([13]) we obtain similar results in the setting of fixed point free actions in infinite dimensional normed linear spaces

$N \cong N^\infty$. Applications similar to that treated in [13] may be anticipated here.

§ 2. The Key Lemma. Throughout the following, let $\alpha: Q \to Q$ be a homeomorphism of period n having exactly O as fixed point and such that α is trivial at O.

A homotopy $f = \{f_t\}: X \times [0,1] \to Y$ is *limited* by a cover $\mathcal{U}$ of Y provided for each $x \in X$, there exists an $U \in \mathcal{U}$ for which $f(\{x\} \times [0,1]) \subset U$. In such case we also say f_0 is $\mathcal{U}$-homotopic to f_1. Suppose $A \subset Y$ is closed. A cover $\mathcal{U}$ of $Y \setminus A$ is *normal* (with respect to A) provided each map $g: Y \setminus A \to Y \setminus A$, g $\mathcal{U}$-close to id., extends to a map $g: Y \to Y$ which is the identity on A.

Let $P: Q \to Q/\alpha$ be the projection and let $p = P(0)$. We may think of Q/α as $Q_0/\alpha_0 \cup \{p\}$, the following is our key lemma.

LEMMA 1. $Q_0\alpha_0 \times [0,1) \cong Q_0/\alpha_0 \times [0,1]$. *This is readily seen to be true when* $\alpha = \beta_n$.

To give a prove we need several lemmas.

LEMMA 2. (A) *Let M be a Q-manifold and $A \subset M$ be a closed Z-set. Then for any open cover $\mathcal{U}$ of M there is a closed map $f: M \to M$ $\mathcal{U}$-homotopic to* id *such that $f|_A =$ id and $f(M)$ is a Z-set of M;*

(B) *Let X be a compact metric space and $A \subset X$ be closed. Suppose f is a map of X into a Q-manifold M such that $f(X)$ is a Z-set. Then there is a map $g: X \to M$ such that $g(X)$ is a Z-set, $g|_A = f|_A$ and $g(X \setminus A) \cap g(A) = \emptyset$.*

Proof. (A) follows from the usual techniques using [5, Theorem 7.2] (see for example [15, Lemma 3.3]).

(B) We may write M as $M \times Q \times Q$ and assume that $f(X) \subset M \times \{O\} \times \{O\}$. Let $h: X \to Q$ be any imbedding such that all coordinates of each $h(x)$ are positive. Let $\lambda: X \to [0,1]$ and $\lambda_1: M \to [0,1]$ be maps satisfying $\lambda^{-1}(0) = A$ and $\lambda_1^{-1}(0) = f(A)$. Define $g: X \to M \times Q \times Q$ by $g(x) = \big(g(x), \lambda(x)h(x), \lambda_1(g(x))h(x)\big)$. g is what we wanted.

LEMMA 3. *Let X be a metric absolute neighborhood retract* (ANR). *Then for any open cover $\mathcal{U}$ of X, there is a simplicial complex K and maps $f: X \to |K|$, $g: |K| \to X$ such that $g \circ f$ is $\mathcal{U}$-homotopic to* id. *Moreover, if X is a Q-manifold, we may choose g so that* $\mathrm{cl}\big(g(|K|)\big)$ *is a Z-set in X.*

Proof. The first part follows from Palais [11, Lemma 6.4] and S. T. Hu [8, p. 111]. As for a proof of the second part, we may use the first part to obtain f, g_1 and $|K|$ with respect to an open cover $\mathcal{U}_1$ of X such that $\mathcal{U}_1$ is a star refinement of $\mathcal{U}$. By Lemma 2(A) there is a map $g_2: X \to X$, $\mathcal{U}_1$-homotopic to id such that $g_2(X)$ is a closed Z-set of X. Then $g = g_2 \circ g_1$ satisfies the condition of the lemma.

In the following let $P: Q \to Q/\alpha$ be the projection and let $p = P(0)$.

LEMMA 4. *Let K be a simplicial complex. Suppose $f: |K| \times \{0,1\} \to Q/\alpha$ is a map such that $f^{-1}(p) = |K| \times \{1\}$ and $\big(f(|K| \times \{o\})\big)$ is contained in a closed Z-set of Q_0/α_0. Then f has an extension $F: |K| \times [0,1] \to Q/\alpha$ such that $F^{-1}(p) = |K| \times \{1\}$ and for each neighborhood V of p there is a neighborhood V_1 of p such that $F(\{y\} \times [0,1]) \subset V$ whenever $F(y,0) \in V_1$.*

Proof. Let $\mathcal{U}$ be an open cover of $Q_0/\alpha_0 = (Q/\alpha) \setminus \{p\}$ which is normal with respect to p. In particular, the diameter of $W \in \mathcal{U}$ becomes uniformly small as W approaches p. Given K a finer triangulation if necessary we may suppose that for each $\sigma \in K$, $f(|\sigma|, 0)$ is contained in some member of $\mathcal{U}$. By hypothesis of p there is a neighborhood system $\{U_i\}_{i \geqslant 0}$ at O consisting of homotopically trivial open sets U_i such that $U_0 = Q$ and for all i, $U_{i+1} \subset U_i$ and $\alpha(U_i) = U_i$. Thus $V_i = P(U_i)$ is a neighborhood system at $p = P(0)$. Let K_i denote the ith-skeleton of K. We now construct maps

$$f_i: |K_i| \times [0,1] \to Q/\alpha$$

inductively as follows. For any 0-simplex, $\sigma \in K_0$, let $i(\sigma)$ be the largest integer for which $f(|\sigma|) \in V_{i(\sigma)}$. Let $g_1: |\sigma| \times \{0,1\} \to Q$ be any map such that $P \circ g_1 = f|_{|\sigma| \times \{0,1\}}$. Clearly $g_1\big((|\sigma|, 0)\big) \in U_{i(\sigma)}$. Since $U_{i(\sigma)}$ is homotopically trivial, we can extend g_1 to a map, again called g_1, of $|\sigma| \times [0,1] \to Q$ such that $g_1(|\sigma| \times [0,1]) \subset U_{i(\sigma)}$. By Lemma 2 we may assume $g_1(|\sigma| \times [0,1])$ is a Z-set and $g_1^{-1}(0)$ is the point $(|\sigma|, 1)$. Repeating this process for every $\sigma \in K_0$ we obtain a map $G_1: |K_0| \times [0,1] \to Q$ satisfying:

(A_1) For any $\sigma \in K_0$, $G_1(|\sigma| \times [0,1])$ is a Z-set contained in $U_{i(\sigma)}$ where $i(\sigma)$ is the largest index containing the point $G_1(|\sigma|, 0)$;

(B_1) $P \circ G(x) = f(x)$ for $x \in |K_0| \times \{0,1\}$; and

(C_1) $G_1^{-1}(0) = |K_0| \times \{1\}$.

Let $f_1 = P \circ G_1$.

Suppose $\sigma_1 \in K_1$. Define a map $\lambda: \mathrm{Bd}(|\sigma_1| \times [0,1]) \to Q/\alpha$ by $\lambda(x) = f(x)$ for $x \in |\sigma_1| \times \{0,1\}$ and $\lambda(x) = f_1(x)$ for $x \in \mathrm{Bd}(|\sigma_1|) \times [0,1]$. Clearly, λ lifts to a map g_2 (that is, $P \circ g_2 = \lambda$) such that $g_2\big(\mathrm{Bd}(|\sigma_1| \times [0,1])\big)$ is a Z-set in Q. Let $i(\sigma_1)$ be the largest integer for which $g_2\big(\mathrm{Bd}(|\sigma_1| \times [0,1])\big) \subset U_{i(\sigma_1)}$. For reasons similar to that above, we can extend g_2 to a map, again called g_2, of $|\sigma_1| \times [0,1] \to Q$ such that $g_2(|\sigma_1| \times [0,1])$ is a Z-set contained in $U_{i(\sigma_1)}$ and $g_2^{-1}(0) = |\sigma_1| \times \{1\}$. Repeating this process we obtain a map $G_2: |K_1| \times [0,1] \to Q$ satisfying:

(A_2) For any $\sigma_1 \in K_1$, $G_2(|\sigma_1| \times [0,1])$ is a Z-set contained in $U_{i(\sigma_1)}$ where $i(\sigma_1)$ is the largest index containing $G_2\big(\mathrm{Bd}(|\sigma_1| \times [0,1])\big)$;

(B_2) $P \circ G_2(x) = f(x)$ for $x \in |K_1| \times \{0,1\}$;

(C_2) $G_2^{-1}(0) = |K_1| \times \{1\}$; and

(D_2) $P \circ G_2|_{|K_0| \times [0,1]} = f_1$.

Let $f_2 = P \circ G_2$.

Define $f_3, f_3, \ldots$ analogously and let $F\colon |K|\times[0,1]\to Q/\alpha$ be the final mapping. It is routine to verify that F is a continuous extension of f fulfilling the requirements of the lemma.

Proof of Lemma 1. Let V be a neighborhood of $p = P(0)$, where $P\colon Q\to Q/\alpha$ is the projection map. By virtue of Lemma 3 there is a simplicial complex K, maps $f\colon Q_0/\alpha_0\to|K|$, $g\colon |K|\to Q_0/\alpha_0$ and a homotopy $F=\{f_t\}$: $Q/\alpha\times[0,1]\to Q/\alpha$ such that $f_0=\mathrm{id}$, $f_1|_{Q_0/\alpha_0}=g\circ f$, $f_t^{-1}(p)=p$ for all t and $\mathrm{cl}(g(|K|))$ is a Z-set in Q_0/α_0.

By Lemma 4 there is a homotopy $G=\{g_t\}$: $|K|\times[0,1]\to Q/\alpha$ such that $g_0=g$, $G^{-1}(p)=|K|\times\{1\}$ and for each neighborhood U of p there is a neighborhood U_1 of p such that $G(\{y\}\times[0,1])\subset U$ whenever $G(y,0)\in U_1$. This last condition implies that the homotopy $\{g_t\circ f\}_{t\in[0,1]}$ extends to a homotopy $\{g'_t\}$ of Q/α into itself satisfying $g'_0=f_1$, $g'_1(Q/\alpha)=p$ and for all $t<1$, $(g'_t)^{-1}(p)=p$. $\{f_t\}$ follows by $\{g'_t\}$ implies that there is a homotopy $H=\{h_t\}$: $Q/\alpha\times[0,1]\to Q/\alpha$ such that $h_t^{-1}(p)=p$ for all t, $h_0=\mathrm{id}$ and $h_1(Q/\alpha)\subset V$. Thus the restriction $H'\colon Q_0/\alpha_0\times[0,1]\to Q_0/\alpha_0$ is a proper map (*proper* means the inverse of any compact set is compact).

Now we consider Q/α as the subspace $Q/\alpha\times\{1\}$ in the product $Q/\alpha\times[0,1]$. Let $W=Q/\alpha\times(r,1]$ be an open neighborhood of $Q/\alpha\times\{1\}$ in $Q/\alpha\times[0,1]$ and let W_0 be any neighborhood of $(p,1)\in Q/\alpha\times\{1\}$. By what we have just shown and by the Mapping Replacement Theorem of [2, Theorem 3.1] there is a proper imbedding $\lambda=\{\lambda_t\}$ of $(Q_0/\alpha_0\times\{1\})\times[0,1]$ into $Q_0/\alpha_0\times[0,1]$ such that $\lambda_0=\mathrm{id}$, the image $\lambda\big((Q_0/\alpha_0\times\{1\})\times[0,1]\big)$ a Z-set of $Q_0/\alpha_0\times[0,1]$ contained in W and $\lambda_1(Q_0/\alpha_0\times\{1\})\subset W_0$. By Theorem 6.1 of [2] and the manner in which it was proved, λ_1 extends to a homeomorphism h of $Q/\alpha\times[0,1]$ onto itself satisfying $h(x)=x$ for $x\in(\{p\}\times[0,1])\cup(Q/\alpha\times[0,r])$ and $h(Q/\alpha\times\{1\})\subset W_0$.

Now using the usual techniques (see for example, Theorem 1 of [4']) we can shrink $Q/\alpha\times\{1\}$ to the point $(p,1)$ by means of a surjection $\varphi\colon Q/\alpha\times[0,1]\to Q/\alpha\times[0,1]$ such that $\varphi(x)=x$ for $x\in\{p\}\times[0,1]$, $\varphi^{-1}(x)=\{\text{point}\}$ for $x\neq(p,1)$ and $\varphi^{-1}(a,1)=Q/\alpha\times\{1\}$. Then $\varphi_0=\varphi|_{Q_0/\alpha_0\times[0,1)}$ is a homeomorphism of $Q_0/\alpha_0\times[0,1)$ onto $Q_0/\alpha_0\times[0,1]$.

§ 3. Proofs.

Proof of Theorem 2. Q_0/α_0 and Q_0/β_0 are Q-manifolds which are dominated by CW complexes. Hence they have the homotopy types of CW complexes [9, 127]. Consider homeomorphisms

$$Q_0/\alpha_0 \xrightarrow{h_1} Q_0/\alpha_0\times[0,1] \xrightarrow{h_2} Q_0/\alpha_0\times[0,1)\,,$$
$$Q_0/\beta_0 \xrightarrow{h'_1} Q_0/\beta_0\times[0,1] \xrightarrow{h'_2} Q_0/\beta_0\times[0,1)\,,$$

where h_1, h'_1 are given by Anderson–Schori ([2']) and h_2, h'_2 are given by Lemma 1. Let $e\in\pi_1(Q_0/\alpha_0)$ and $e'\in\pi_1(Q_0/\beta_0)$ be given. Let $p\colon Q_0/\alpha_0\times[0,1)\to Q_0/\alpha_0$ and $p'\colon Q_0/\beta_0\times[0,1)\to Q_0/\beta_0$ denote the projections and let $e_1=(h_2\circ h_1)_{\#}(e)$ and $e'_1=(h'_2\circ h'_1)_{\#}(e')$. By [10, Theorem 4] there is a homotopy equivalence $f\colon Q_0/\alpha_0\to Q_0/\beta_0$ such that $f_{\#}(p_{\#}(e_1))=p'_{\#}(e'_1)$. By a result of Chapman [6, Theorem 5] there is a homeomorphism $g\colon Q_0/\alpha_0\times[0,1)\to Q_0/\beta_0\times[0,1)$ such that g is homotopic to $f\times\mathrm{id}\colon Q_0/\alpha_0\times[0,1)\to Q_0/\beta_0\times[0,1)$. Thus $g_{\#}=(f\times\mathrm{id})_{\#}$ and we have the following commutative diagram:

$$\begin{array}{ccc}
\pi_1(Q_0/\alpha_0\times[0,1)) & \xrightarrow{p_{\#}} & \pi_1(Q_0/\alpha_0)\\
{\scriptstyle g_{\#}=(f\times\mathrm{id})_{\#}}\downarrow & & \downarrow{\scriptstyle f_{\#}}\\
\pi_1(Q_0/\beta_0\times[0,1)) & \xrightarrow[p'_{\#}]{} & \pi_1(Q_0/\beta_0)
\end{array}$$

Hence $g_{\#}(e_1)=p'^{-1}_{\#}\circ f_{\#}\circ p_{\#}(e_1)=e'_1$. Let $h\colon Q_0/\alpha_0\to Q_0/\beta_0$ be the homeomorphism defined by $h=h'^{-1}_1\circ h'^{-1}_2\circ g\circ h_2\circ h_1$. Then

$$h_{\#}(e)=(h'^{-1}_1\circ h'^{-1}_2)_{\#}\circ g_{\#}\circ(h_2\circ h_1)_{\#}(e)$$
$$=(h'_2\circ h'_1)^{-1}_{\#}\circ g_{\#}(e_1)=(h'_2\circ h'_1)^{-1}_{\#}(e'_1)=e'.$$

h is what we wanted.

Proof of Theorem 1. Since Q is the one-point compactification of $Q_0=Q\setminus\{o\}$, it suffices to show that there is a homeomorphism $h\colon Q_0\to Q_0$ satisfying $h\circ\alpha=\beta_n\circ h$. Let $b\in Q_0$ and suppose λ, λ_1: $([0,1],0)\to(Q_0,b)$ are maps (preserving base points) such that $\lambda(1)=\alpha(b)$ and $\lambda_1(1)=\beta_n(b)$. Let $P\colon Q_0\to Q_0/\alpha_0$ and $P_1\colon Q_0\to Q_0/\beta_0$ denote the projections. Then $e=[P\circ\lambda]\in\pi_1(Q_0/\alpha_0)$ and $e'=[P_1\circ\lambda_1]\in\pi_1(Q_0/\beta_0)$ are generators. It follows from Theorem 2 that there is a homeomorphism $h'\colon \big(Q_0/\alpha_0, P(b)\big)\to\big(Q_0/\beta_0, P_1(b)\big)$ such that $h'_{\#}(e)=e'$. The function h' then induces a fibre homeomorphism $h\colon (Q_0,b)\to(Q_0,b)$ satisfying $P_1\circ h=h'\circ P$ and $h\circ\alpha(b)=\beta_n\circ h(b)$. For each $x\in Q_0$, since the set $\{h(x), h\circ\alpha(x)\}\in P^{-1}\big(h'\circ P(x)\big)$, there is an $1\leqslant i\leqslant n$ for which $h\circ\alpha(x)=\beta_n^i\circ h(x)$. Let $A_i=\{x\in Q_0\colon h\circ\alpha(x)=\beta_n^i\circ h(x)\}$. We verify easily that each A_i is closed and $\{A_i\}$ are pairwise disjoint. Since Q_0 is connected and $A_1\neq\emptyset$, $A_1=Q_0$.

References

[1] R. D. Anderson, *On topological infinite deficiency*, Mich. Math. J. 14 (1967), pp. 365–383.

[2] — and T. A. Chapman, *Extending homeomorphism to Hilbert cube manifolds*, Pacific J. Math. 38 (1971), pp. 281–293.

[2'] — and R. Schori, *Factors of infinite-dimensional manifolds*, Trans. Amer. Math. Soc. 142 (1969), pp. 315–330.

[3] K. Borsuk, *Theory of Retracts*, Warszawa 1967.

[4] M. Brown, *Some applications of an approximation theorem for inverse limits* Proc. Amer. Math. Soc. 11 (1960), pp. 478–483.

[4'] — *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. 66 (1960), pp. 74–76.

[5] T. A. Chapman, *Dense sigma-compact subsets of infinite-dimensional manifolds*, Trans. Amer. Math. Soc. 154 (1971), pp. 399–426.

[6] — *On the structure of Hilbert cube manifold*, Compositio Math. 24 (1972), pp. 329–353.

[7] D. Curtis, *Near homeomorphisms and fine homotopy equivalence*, Manuscript.

[8] S. T. Hu, *Theory of Retracts*, 1965.

[9] A. T. Lundell and S. Weingram, *The Topology of CW Complexes*, 1969.

[10] R. Mosher and M. Tangora, *Cohomology Operations and Applications in Homotopy Theory*, 1969.

[11] R. Palais, *Homotopy theory of infinite dimensional manifolds*, Topology 5 (1966), pp. 1–16.

[12] J. E. West, *Infinite products which are Hilbert cubes*, Trans. Amer. Math. Soc. 150 (1970), pp. 1–25.

[13] R. Y. T. Wong, *Periodic actions on* (I-D) *normed linear spaces*, Fund. Math. 80 (1973), pp. 133–139.

[14] — *Homotopy negligible subsets of bundles*, Compositio Mathematica 24 (1972), pp. 119–128.

[15] — *On homeomorphisms of infinite dimensional bundles, I*, submitted to Trans. Amer. Math. Soc.

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Some properties of fundamental dimension

by

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Abstract. By a fundamental dimension of compactum X we understand the number $\mathrm{Fd}(X) = \min(\dim Y\colon \mathrm{Sh}(X) \leqslant \mathrm{Sh}\, Y)$.

The main purpose of this paper is to give characterizations of compacta with fundamental dimension $\leqslant n$, where n is a positive integer or 0, and to apply these characterizations to the study of some problems.

By a fundamental dimension of compactum X we understand the number $\mathrm{Fd}(X) = \min(\dim Y\colon \mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y))$.

This notion has been introduced by K. Borsuk in [3] (see also [5], p. 31). In the theory of shape it has a similar role to that of dimension in topology. Therefore if is one of the most important invariants of the theory of shape.

The main purpose of this paper is to give characterizations of compacta with fundamental dimension $\leqslant n$ (where n is a positive integer or 0) and to apply these characterizations to the study of the following problems:

(1) Suppose that all components of the compactum X have fundamental dimension $\leqslant n$. Is it true that $\mathrm{Fd}(X) \leqslant n$?

(2) Suppose that $X = \varprojlim\{X_k, p_k^{k+1}\}$ and $\mathrm{Fd}(X_k) \leqslant n$ for every $k = 1, 2, \ldots$ Is it true that $\mathrm{Fd}(X) \leqslant n$?

(3) Suppose that X, Y are compacta. Is it true that $\mathrm{Fd}(X \cup Y) \leqslant \max(\mathrm{Fd}(X), \mathrm{Fd}(Y), 1+\mathrm{Fd}(X \cap Y))$?

(4) Suppose that X, A are compacta and $X \supset A$. Is it true that $\mathrm{Fd}(X/A) \leqslant \max(\mathrm{Fd}(X), \mathrm{Fd}(A)+1)$?

(5) Suppose that M is a connected n-manifold and let $X \subsetneqq M$ be a compactum. Is it true that $\mathrm{Fd}(X) \leqslant n-1$?

Theorems (2.1), (3.2), (4.1) and (4.18) give positive answers to problems (1), (2), (3), and (4). Theorem (3.6) gives a partial answer to problem (5). The above-mentioned characterizations are given in Theorems (1.5), (1.7) and (1.9). The first section contains also Theorem (1.6), which implies that $\mathrm{Fd}(X) = \min(\dim Y\colon \mathrm{Sh}(X) = \mathrm{Sh}(Y))$. This result was obtained by W. Holsztyński as early as 1968.