Concerning the shapes of n-dimensional spheres

by

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Abstract. It is proved in this note that among compacta lying in the Euclidean $(n+1)$-space E^{n+1} the compacta X with the shape of the n-sphere are characterized by three following conditions:

1° $p_n(X) = 1$ and X is acyclic in dimensions $k = 0, 1, \ldots, n-1$,
2° X is approximatively 1-connected,
3° X is an FANR-space.

One of the most important problems of the theory of shape is to find for a given compactum X a system of shape invariants characterizing the shape of X. The aim of this note is to give a system of shape invariants characterizing the compacta with the shape of the n-dimensional sphere among all compacta lying in the Euclidean $(n+1)$-space E^{n+1}.

We assume as known the most elementary concepts and theorems of the theory of shape, in particular the notions of the *shape* $\mathrm{Sh}(X)$ of a compactum X, of the k-*dimensional fundamental group* $\underline{\pi}_k(X, x_0)$, where $x_0 \in X$, of the *fundamental retraction*, of the *fundamental absolute neighborhood retract* FANR, of the *movability* and of the *approximative connectedness in dimension* k. The reader may find the definitions of these notions in [1] and in [2]. The homology notions for compacta are understand here in the sense of Vietoris (or, which is equivalent, in the sense of Čech). In particular, $p_k(X)$ denotes the k-dimensional Betti number of a compactum X.

§ 1. Infinite polyhedra adjacent to a compactum $X \subset E^{n+1}$. A set $Y \subset E^{n+1} \setminus X$ is said to be an *infinite polyhedron adjacent to* X if $X \cup Y$ is a compactum and if there exists a locally finite, countable triangulation T of Y with diameters of its simplexes converging to zero. One sees easily that then every neighborhood of X (in E^{n+1}) contains almost all simplexes of T. A triangulation T of Y satisfying these conditions is said to be *appropriate*. It is well known that if $A \subset E^{n+1}$ is a polyhedron containing in its interior $\mathring{A}$ a compactum $X \neq \emptyset$, then the set $Y = A \setminus X$ is an infinite polyhedron adjacent to X.

Let us prove the following

(1.1) LEMMA. *Let $A \subset E^{n+1}$ be a polyhedron containing in its interior a compactum $X \neq \emptyset$, but not containing any component of the set $E^{n+1} \setminus X$. Let T be an appropriate triangulation of an infinite polyhedron $Y \subset A$ adjacent to X. Then for every neighborhood U_0 of X in E^{n+1} there exists an infinite polyhedron Y_0 adjacent to X and satisfying the following conditions:*

(a) *Y_0 has an appropriate triangulation T_0 consisting of almost all simplexes of T.*

(b) *U_0 contains all $(n+1)$-dimensional simplexes of T_0.*

(c) *There is a retraction s: $X \cup Y \to X \cup Y_0$.*

Proof. If Δ is an $(n+1)$-dimensional simplex belonging to T, then one easily sees that there exists a finite system $\Delta = \Delta_1, \Delta_2, \ldots, \Delta_m$ of $(n+1)$-dimensional simplexes of T such that:

1) If $1 \leqslant i < j \leqslant m$ then $\Delta_i \neq \Delta_j$.

(2) If $1 \leqslant i < m$ then $\Delta_i \cap \overline{E^{n+1} \setminus Y} = \emptyset$ and $\Delta_1 \cap \Delta_{i+1}$ is an n-dimensional simplex.

3) $\Delta_m \cap \overline{E^{n+1} \setminus Y}$ contains an n-dimensional simplex.

One infers, by an evident induction, that there exists an infinite polyhedron Y' adjacent to X, having an appropriate triangulation consisting of almost all simplexes of T and such that Δ is not contained in Y'. It easily follows that there exists a sequence $Y = Y_1 \supset Y_2 \supset \ldots$ of infinite polyhedrons adjacent to X such that Y_k has (for every $k = 1, 2, \ldots$) an appropriate triangulation T_k consisting of almost all simplexes of T and that Y_{k+1} is a retract of Y_k and that for every $(n+1)$-dimensional simplex Δ of T there is an index k_Δ such that Δ does not belong to T_{k_Δ}. Since almost all simplexes of T lie in U_0, we infer that one can select an index k_0 such that the infinite polyhedron $Y_0 = Y_{k_0}$ satisfies the conditions (a), (b) and (c). Thus the proof of Lemma (1.1) is finished.

§ 2. A lemma on extending of maps. Now let us prove the following

(2.1) LEMMA. *Let X be a movable compactum lying in a space $M \in \mathrm{AR}(\mathfrak{M})$. If X is approximatively 1-connected and acyclic in dimensions $k = 0, 1, \ldots, n-1$, then for every neighborhood U of X (in M) there exists a neighborhood $U_0 \subset U$ of X (in M) such that if C_0 is a closed subset of a space C such that $\overline{C \setminus C_0}$ is a finite polyhedron of dimension $\leqslant n$, then every map g: $C_0 \to M$ with all values in U_0 can be extended to a map $\hat{g}$: $C \to M$ with all values in U.*

Proof. If $n = 0$ then $C \setminus C_0$ is a finite set. Setting $U_0 = U$, we get the required extension $\hat{g}$ of g if we assign to every point $x \in C \setminus C_0$ an arbitrary point $\hat{g}(x) \in U$.

Now let us assume that $n = m+1$ and that for $n = m$ the lemma holds true. Let T be a triangulation of the polyhedron $\overline{C \setminus C_0}$ and let C' denote the union of C_0 and of all simplexes of T with dimensions $\leqslant m = n-1$. By our hypothesis, for every neighborhood U' of X (in M) there exists a neighborhood U_0 of X (in M) such that every map g: $C_0 \to M$ satisfying the condition $g(C_0) \subset U_0$, can be extended to a map g': $C' \to M$ with all values in U'.

Since X is movable and approximatively 1-connected an acyclic in dimensions $k = 0, 1, \ldots, n-1 = m$, we infer by the "modified theorem of Hurewicz", proved by Mrs. K. Kuperberg [5], p. 26, that for every point $x_0 \in X$ the fundamental group $\underline{\pi}_m(X, x_0)$ is trivial and consequently (see [3], p. 191) the compactum X is approximatively m-connected. This means that if U is an arbitrarily given neighborhood of X (in M) then the neighborhood U' of X can be selected so that every map of the boundary of any simplex Δ of dimension $m+1 = n$ into U' has a continuous extension onto Δ with all values in U. It follows that every map g: $C_0 \to M$, satisfying the condition $g(C_0) \subset U_0$, can be extended to a map g': $C' \to M$ with values in U', and the map g' can be extended onto each n-dimensional simplex $\Delta \in T$ to a map with all values in U. Thus we get a map $\hat{g}$: $C \to M$ being an extension of the map g and satisfying the condition $\hat{g}(C) \subset U$. Hence the proof of Lemma (2.1) is finished.

§ 3. Main theorem. Now let us pass to the main goal of this note:

(3.1) THEOREM. *The shape of a compactum $X \subset E^{n+1}$ is the same as the shape of the n-dimensional sphere S^n if and only if three following conditions are satisfied:*

1° *$p_n(X) = 1$ and X is acyclic in dimensions $k = 0, 1, \ldots, n-1$.*

2° *X is approximatively 1-connected.*

3° *$X \in$ FANR.*

Proof. It is clear that $X = S^n$ satisfies the conditions 1°, 2° and 3°. Since these conditions are shape invariants, we infer that each compactum X with $\mathrm{Sh}(X) = \mathrm{Sh}(S^n)$ satisfies them.

Now let us assume that X is a compactum lying in E^{n+1} and satisfying the conditions 1°, 2° and 3°. By 1° the set $E^{n+1} \setminus X$ has two components: one bounded component G and the other unbounded G'. Consider two $(n+1)$-dimensional simplexes Δ, Δ' lying in E^{n+1} and such that $\Delta \subset G$ and that X lies in the interior of Δ'. Let $\dot{\Delta}$ denote the interior of Δ. Then the set

$$A = \Delta' \setminus \dot{\Delta}$$

is a polyhedron containing X in its interior and the boundary $\dot{\Delta} = \Delta \setminus \dot{\Delta}$ of Δ is a deformation retract of A. It follows that

$$\mathrm{Sh}(A) = \mathrm{Sh}(\dot{\Delta}) = \mathrm{Sh}(S^n).$$

The set $Y = A \setminus X$ is an infinite polyhedron adjacent to X. Let T be an appropriate triangulation of Y. By 3° there exists a compact neighborhood $U \subset A$ of X in E^{n+1} and a fundamental retraction

$$\underline{r} = \{r_k, U, X\}_{E^{n+1}, E^{n+1}}.$$

By Lemma (1.1) there exists an infinite polyhedron Y_0 adjacent to X and satisfying the conditions (a), (b) and (c). Let U_0 be a neighborhood of X (in $M = E^{n+1}$) satisfying the conditions of Lemma (2.1). Let C_0 denote the union of the set X and of all simplexes of the triangulation T_0 of Y_0 (given by the condition (a)), lying in U_0. Setting

$$C = X \cup Y_0,$$

we infer by the condition (b) and by Lemma (2.1) that the inclusion map $g\colon C_0 \to E^{n+1}$ can be extended to a map $\hat{g}\colon X \cup Y_0 \to E^{n+1}$ such that

$$\hat{g}(X \cup Y_0) \subset U.$$

Setting

$$f = \hat{g}s,$$

we obtain a map f of the set $A = X \cup Y$ into E^{n+1} and all values of this map belong to U. Since A is a closed subset of E^{n+1}, we can extend f to a map

$$\hat{f}\colon E^{n+1} \to E^{n+1}.$$

Then $\hat{f}(A) = f(A) \subset U$ and $\hat{f}(x) = x$ for every point $x \in X$. Setting

$$\hat{r}_k = r_k \hat{f} \quad \text{for every } k = 1, 2, \ldots,$$

we get a sequence of maps $\hat{r}_k\colon E^{n+1} \to E^{n+1}$ such that

$$\hat{r}_k(x) = r_k(x) = x \quad \text{for every point } x \in X.$$

If we recall that $\underline{r}$ is a fundamental retraction, we infer that for every neighborhood V of X (in E^{n+1}) there exists a neighborhood $\hat{W}$ of U in E^{n+1} such that

$$r_k/\hat{W} \simeq r_{k+1}/\hat{W} \quad \text{in } V \text{ for almost all } k.$$

But since the values of f belong to U and since $\hat{f}$ is an extension of f, we infer that there exists a neighborhood W of A (in E^{n+1}) such that $\hat{f}(W) \subset \hat{W}$. Consequently

$$\hat{r}_k/W = r_k\hat{f}/W \simeq r_{k+1}\hat{f}/W = \hat{r}_{k+1}/W \quad \text{in } V \text{ for almost all } k.$$

Hence $\hat{\underline{r}} = \{\hat{r}_k, A, X\}_{E^{n+1}, E^{n+1}}$ is a fundamental retraction of A. It follows that

$$\mathrm{Sh}(X) \leqslant \mathrm{Sh}(A) = \mathrm{Sh}(S^n).$$

Moreover $\mathrm{Sh}(X)$ is not trivial, because $p_n(X) = 1$. However it is known ([4], p. 389) that there exist only two shapes $\leqslant \mathrm{Sh}(S^n)$, actually the trivial shape and $\mathrm{Sh}(S^n)$ itself. Hence $\mathrm{Sh}(X) = \mathrm{Sh}(S^n)$ and the proof of Theorem (3.1) is finished.

§ 4. Remarks and problems. The condition that X lies in the space E^{n+1} (appearing in Theorem (3.1) is not a shape invariant. However it is easy to modify the formulation of Theorem (3.1) in order to give to it a purely shape-theoretical form:

Let us assign to every compactum X a number $e(X)$ defined as follows:

If there exist natural numbers k such that the space E^k contains a subset $Y \in \mathrm{Sh}(X)$, then $e(X)$ is the minimum of all such numbers k.

If none of spaces E^k contains a subset $Y \in \mathrm{Sh}(X)$ then $e(X) = \infty$. It is clear that $e(X)$ is a shape invariant and that $e(X) \leqslant n$ implies that X is acyclic in all dimensions $\geqslant n$. Using this number $e(X)$, we can reformulate Theorem (3.1) as follows:

(4.1) THEOREM. *In order $X \in \mathrm{Sh}(S^n)$ it is necessary and sufficient that X is a compactum satisfying the following conditions*:

1° $p_n(X) = 1$ *and X is acyclic in dimensions* $k = 0, 1, \ldots, n-1$.

2° *X is approximatively* 1-*connected.*

3° $X \in$ FANR.

4° $e(X) = n+1$.

In this formulation only shape-invariants are involved.

The following problems remain open:

(4.2) *Does Theorem* (3.1) *remain true if one replaces in it the condition* 3° *be the weaker one, that X is movable?*

(4.3) *Does Theorem* (4.1) *remain true if one replaces in it the condition* 4° *by the hypothesis that X is acyclic in all dimensions $> n$?*

This last problem may be considered as a question corresponding in the theory of shape to the fameous conjecture of Poincaré.

References

[1] K. Borsuk, *Concerning the notion of the shape of compacta*, Proc. of the Intern. Symposium on Topology and its Applications, Herceg–Novi 1968, Beograd 1969, pp. 98–104.

[2] K. Borsuk, *A note on the theory of shape for compacta*, Fund. Math. 67 (1970), pp. 265–278.

[3] — *Some remarks on shape properties of compacta*, Fund. Math. 85 (1974), pp. 185–195.

[4] — and W. Holsztyński, *Hopf classification theorem in the shape theory*, Bull. Acad. Polon. Sci. 19 (1971), pp. 387–390.

[5] K. Kuperberg, *An isomorphism theorem of the Hurewicz-type in Borsuk's theory of shape*, Fund. Math. 77 (1972), pp. 21–32.

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