Concerning the shapes of $n$-dimensional spheres

by

Karol Borsuk (Warszawa)

Abstract. It is proved in this note that among compacta lying in the Euclidean $(n+1)$-space $E^{n+1}$ the compacta $X$ with the shape of the $n$-sphere are characterized by three following conditions:

1. $p_k(X) = 1$ and $X$ is acyclic in dimensions $k = 0, 1, \ldots, n-1$.
2. $X$ is approximatively 1-connected.
3. $X$ is an $FANR$-space.

One of the most important problems of the theory of shape is to find for a given compactum $X$ a system of shape invariants characterizing the shape of $X$. The aim of this note is to give a system of shape invariants characterizing the compacta with the shape of the $n$-dimensional sphere among all compacta lying in the Euclidean $(n+1)$-space $E^{n+1}$.

We assume as known the most elementary concepts and theorems of the theory of shape, in particular the notions of the shape $Sh(X)$ of a compactum $X$, of the $k$-dimensional fundamental group $\pi_k(X, x_0)$, where $x_0 \in X$, of the fundamental retraction, of the fundamental absolute neighborhood retract $FANR$, of the movability and of the approximative connectedness in dimensions $k$. The reader may find the definitions of these notions in [1] and in [3]. The homology notions for compacta are understood here in the sense of Vietoris (or, which is equivalent, in the sense of Čech). In particular, $p_k(X)$ denotes the $k$-dimensional Betti number of a compactum $X$.

§ 1. Infinite polyhedra adjacent to a compactum $X \subseteq E^{n+1}$. A set $Y \subseteq E^{n+1} \setminus X$ is said to be an infinite polyhedron adjacent to $X$ if $X \cup Y$ is a compactum and if there exists a locally finite, countable triangulation $T$ of $Y$ with diameters of its simplexes converging to zero. One sees easily that then every neighborhood of $X$ (in $E^{n+1}$) contains almost all simplexes of $T$. A triangulation $T$ of $Y$ satisfying these conditions is said to be appropriate. It is well known that if $A \subseteq E^{n+1}$ is a polyhedron containing in its interior $A$ a compactum $X \neq \emptyset$, then the set $Y = A \setminus X$ is an infinite polyhedron adjacent to $X$. 
Let us prove the following

(1.1) **Lemma.** Let $A \subset \mathbb{R}^{n+1}$ be a polyhedron containing in its interior a compactum $X \neq \emptyset$, but not containing any component of the set $\mathbb{R}^{n+1} \setminus X$. Let $T$ be an appropriate triangulation of an infinite polyhedron $Y \subset A$ adjacent to $X$. Then for every neighborhood $U_0$ of $X$ in $\mathbb{R}^{n+1}$ there exists an infinite polyhedron $Y_\delta$ adjacent to $X$ and satisfying the following conditions:

(a) $Y_\delta$ has an appropriate triangulation $T_\delta$ consisting of almost all simplices of $T$.

(b) $U_0$ contains all $(n+1)$-dimensional simplices of $T_\delta$.

(c) There is a retraction $r: X \times Y \to X \times Y_\delta$.

**Proof.** If $\delta$ is an $(n+1)$-dimensional simplex belonging to $T$, then one easily sees that there exists a finite system $A = A_1, A_2, \ldots, A_m$ of $(n+1)$-dimensional simplices of $T$ such that:

1. If $1 < i < j < m$ then $A_i \neq A_j$.
2. If $1 < i < m$ then $A_i \cap \mathbb{R}^{n+1} \setminus Y = \emptyset$ and $A_i \cap A_{i+1}$ is an $n$-dimensional simplex.
3. $A_m \cap \mathbb{R}^{n+1} \setminus Y$ contains an $n$-dimensional simplex.

One infers, by an inductive argument, that there exists an infinite polyhedron $Y' \subset A$ adjacent to $X$, having an appropriate triangulation consisting of almost all simplices of $T$ and such that $A$ is not contained in $Y'$. It easily follows that there exists a sequence $Y = Y_1 \supset Y_2 \supset \ldots$ of infinite polyhedra adjacent to $X$ such that $Y_i$ has (for every $i = 1, 2, \ldots$) an appropriate triangulation $T_i$ consisting of almost all simplices of $T$ and that $Y_{i+1}$ is a retract of $Y_i$ and that for every $(n+1)$-dimensional simplex $x$ of $T$ there is an index $i_x$ such that $x$ does not belong to $T_{i_x}$. Since almost all simplices of $T$ lie in $U_0$, we infer that one can select an index $i_x$ such that the infinite polyhedron $Y_x = Y_{i_x}$ satisfies the conditions (a), (b), and (c). Thus the proof of Lemma (1.1) is finished.

§ 2. A lemma on extending maps. Now let us prove the following

(2.1) **Lemma.** Let $X$ be a movable compactum lying in a space $M \in AB(3)$. If $X$ is approximately 1-connected and acyclic in dimensions $k = 0, 1, \ldots, n-1$, then for every neighborhood $U$ of $X$ in $M$ there exists a neighborhood $U_0 \subset U$ of $X$ (in $M$) such that if $U_0$ is a closed subset of a space $U$ such that $\partial X_0$ is a polyhedron of dimension $< n$, then every map $g: G_0 \to M$ with all values in $U_0$ can be extended to a map $\hat{g} : G \to M$ with all values in $U$.

**Proof.** If $n = 0$ then $\partial X_0$ is a finite set. Setting $U_0 = U$, we get the required extension $\hat{g}$ of $g$ if we assign to every point $x \in \partial X_0$ an arbitrary point $\hat{g}(x) \in U$.

Now let us assume that $n = m+1$ and that for $n = m$ the lemma holds true. Let $T$ be a triangulation of the polyhedron $\partial X_0$ and let $X'$ denote the union of $G_0$ and of all simplices of $T$ with dimensions $< m = n-1$. By our hypothesis, for every neighborhood $U_0$ of $X'$ (in $M$) there exists a neighborhood $U_0$ of $X$ (in $M$) such that every map $g: G_0 \to M$ satisfying the condition $g(\partial G_0) \subset U_0$, can be extended to a map $\hat{g} : G \to M$ with all values in $U'$.

Since $X$ is movable and approximately 1-connected an acyclic in dimensions $k = 0, 1, \ldots, n-1 = m$, we infer from the "modified theorem of Hurewicz" proved by Mrs. K. Kuperberg [3], p. 26, that for every point $x \in X$ the fundamental group $\pi_1(X, x_0)$ is trivial and consequently (see [3], p. 191) the compactum $X$ is approximately $m$-connected. This means that if $U$ is an arbitrarily given neighborhood of $X$ (in $M$) then the neighborhood $U_0$ of $X$ can be selected so that every map of the boundary of any simplex $G$ of dimension $m+1 = n$ into $U'$ has a continuous extension onto $U$ with all values in $U$. It follows that every map $g: G_0 \to M$, satisfying the condition $g(\partial G) \subset U_0$, can be extended to a map $g' : G \to M$ with values in $U'$, and the map $g'$ can be extended onto each $n$-dimensional simplex $A \subset T$ to a map with all values in $U$. Thus we get a map $\hat{g} : G \to M$ being an extension of the map $g$ and satisfying the condition $\hat{g}(\partial G) \subset U$. Hence the proof of Lemma (2.1) is finished.

§ 3. **Main theorem.** Now let us pass to the main goal of this note:

(3.1) **Theorem.** The shape of a compactum $X \subset \mathbb{R}^{n+1}$ is the same as the shape of the $n$-dimensional sphere $S^n$ if and only if three following conditions are satisfied:

1. $\pi_n(X) = 1$ and $X$ is acyclic in dimensions $k = 0, 1, \ldots, n-1$.
2. $X$ is approximately 1-connected.
3. $X \times S^1$ is FRANR.

**Proof.** It is clear that $X = S^n$ satisfies the conditions 1, 2, and 3. Since these conditions are shape invariants, we infer that each compactum $X$ with $\text{Sh}(X) = \text{Sh}(S^n)$ satisfies them.

Now let us assume that $X$ is a compactum lying in $\mathbb{R}^{n+1}$ and satisfying the conditions 1, 2, and 3. By 1 the set $\mathbb{R}^{n+1} \setminus X$ has two components: one bounded component $G$ and the other unbounded $G'$. Consider two $(n+1)$-dimensional simplices $A, A'$ lying in $\mathbb{R}^{n+1}$ and such that $A \subset G$ and that $X$ lies in the interior of $A$. Let $\partial A$ denote the interior of $A$. Then the set

$$A = A \setminus \partial A$$

is a polyhedron containing $X$ in its interior and the boundary $\partial A = \partial A \setminus A$ of $A$ is a deformation retract of $A$. It follows that

$$\text{Sh}(A) = \text{Sh}(\partial A) = \text{Sh}(S^n).$$
The set \( Y = A \setminus X \) is an infinite polyhedron adjacent to \( X \). Let \( I \) be an appropriate triangulation of \( Y \). By \( 3^8 \) there exists a compact neighborhood \( U \subset A \) of \( X \) in \( E^{n+1} \) and a fundamental retraction
\[
t = (\tau, U, \mathcal{U})_{E^{n+1}, E^{n+1}}.
\]

By Lemma (1.1) there exists an infinite polyhedron \( Y_0 \) adjacent to \( X \) and satisfying the conditions (a), (b) and (c). Let \( U_0 \) be a neighborhood of \( X \) (in \( M = E^{n+1} \)) satisfying the conditions of Lemma (2.1). Let \( C_0 \) denote the union of the set \( X \) and of all simplexes of the triangulation \( \mathcal{U} \) of \( Y_0 \) (given by the condition (a)), lying in \( U_0 \). Setting
\[
\mathcal{C} = X \cup Y_0,
\]
we infer by the condition (b) and by Lemma (2.1) that the inclusion map \( g: C_0 \rightarrow E^{n+1} \) can be extended to a map \( \tilde{g}: X \cup Y_0 \rightarrow E^{n+1} \) such that
\[
\tilde{g}(X \cup Y_0) \subset U.
\]
Setting
\[
f = \tilde{g} g,
\]
we obtain a map \( f \) of the set \( A = X \cup Y \) into \( E^{n+1} \) and all values of this map belong to \( U \). Since \( A \) is a closed subset of \( E^{n+1} \), we can extend \( f \) to a map
\[
\tilde{f}: E^{n+1} \rightarrow E^{n+1}.
\]
Then \( \tilde{f}(A) = f(A) \subset U \) and \( \tilde{f}(x) = x \) for every point \( x \in X \). Setting
\[
\tilde{r}_x = \tilde{r}_x \tilde{f}
\]
for every \( x = 1, 2, \ldots \),
we get a sequence of maps \( \tilde{r}_x: E^{n+1} \rightarrow E^{n+1} \) such that
\[
\tilde{r}_x(x) = \tilde{r}_x(x) = x \quad \text{for every point } x \in X.
\]

If we recall that \( r \) is a fundamental retraction, we infer that for every neighborhood \( V \) of \( X \) (in \( E^{n+1} \)) there exists a neighborhood \( W \) of \( U \) in \( E^{n+1} \) such that
\[
\tilde{r}_x(W) \subset W \quad \text{in } V \text{ for almost all } x.
\]

But since the values of \( f \) belong to \( U \) and since \( \tilde{f} \) is an extension of \( f \), we infer that there exists a neighborhood \( W \) of \( A \) (in \( E^{n+1} \)) such that \( \tilde{f}(W) \subset W \). Consequently
\[
\tilde{r}_x(W) = \tilde{r}_x \tilde{f}(W) = \tilde{r}_x \tilde{f}(W) = \tilde{r}_x \tilde{f}(W) = \tilde{r}_x \tilde{f}(W) = \tilde{r}_x \tilde{f}(W) = \tilde{r}_x \tilde{f}(W) \quad \text{in } V \text{ for almost all } x.
\]

Hence \( \tilde{r} = (\tilde{r}_x, A, X)_{E^{n+1}, E^{n+1}} \) is a fundamental retraction of \( A \). It follows that

\[
\text{Sh}(X) = \text{Sh}(A) = \text{Sh}(S^n).
\]

Moreover \( \text{Sh}(X) \) is not trivial, because \( p_0(X) = 1 \). However it is known (11, p. 359) that there exist only two shapes \( \text{Sh}(S^n) \), actually the trivial shape and \( \text{Sh}(S^n) \) itself. Hence \( \text{Sh}(X) = \text{Sh}(S^n) \) and the proof of Theorem (3.1) is finished.

\section*{§ 4. Remarks and problems}

The condition that \( X \) lies in the space \( E^{n+1} \) (appearing in Theorem (3.1) is not a shape invariant. However it is easy to modify the formulation of Theorem (3.1) in order to give to it a purely shape-theoretical form:

Let us assign to every compactum \( X \) a number \( e(X) \) defined as follows:

If there exist natural numbers \( k \) such that the space \( E^k \) contains a subset \( Y \subset \text{Sh}(X) \), then \( e(X) \) is the minimum of all such numbers \( k \).

If none of spaces \( E^k \) contains a subset \( Y \subset \text{Sh}(X) \) then \( e(X) = \infty \).

It is clear that \( e(X) \) is a shape invariant and that \( e(X) \leq n \) implies that \( X \) is acyclic in all dimensions \( \geq n \). Using this number \( e(X) \), we can re-formulate Theorem (3.1) as follows:

\begin{enumerate}
    \item[4.1] \textbf{Theorem.} In order \( X \subset \text{Sh}(S^n) \) it is necessary and sufficient that \( X \) is a compactum satisfying the following conditions:
    \begin{enumerate}
        \item \( p_0(X) = 1 \) and \( X \) is acyclic in dimensions \( k = 0, 1, \ldots, n-1 \).
        \item \( X \) is approximatively 1-connected.
        \item \( X \subset \text{FANR} \).
        \item \( e(X) = n+1 \).
    \end{enumerate}
\end{enumerate}

In this formulation only shape-invariants are involved.

The following problems remain open:

\begin{enumerate}
    \item[4.2] Does Theorem (3.1) remain true if one replaces in it the condition \( 3^* \) by the weaker one, that \( X \) is movable?
    \item[4.3] Does Theorem (4.1) remain true if one replaces in it the condition \( 4^* \) by the hypothesis that \( X \) is acyclic in all dimensions \( \geq n \)?
\end{enumerate}

This last problem may be considered as a question corresponding in the theory of shape to the famous conjecture of Poincaré.

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K. Borsuk


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