Extension of closed mappings

by

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Abstract. The problem of the extension of closed mappings from closed sets is studied. The notion of the class $\mathfrak{ACR}(\mathcal{C})$ where $\mathcal{C}$ is the class of all metrizable or paracompact spaces is introduced as follows:

DEFINITION 1. $R \in \mathfrak{ACR}(\mathcal{C})$ if and only if for every locally compact space X from $\mathcal{C}$ and every A closed in X and homeomorphic to R there exists a closed retraction $r\colon X \to A$.

THEOREM 1. *The following conditions are equivalent*: (1) $R \in \mathfrak{ACR}(\mathcal{C})$, (ii) $R \in \mathfrak{AR}(\mathcal{C})$ *in the sense of Borsuk* [1], R *is locally compact and there exists a closed retraction* $r_0\colon R \times [0, 1) \to R \times \{0\}$.

If X and Y are locally compact spaces from $\mathcal{C}$ then the following condition is necessary for the existence of the closed extension over X of the mapping $f\colon A \to Y$ defined on the closed $A \subset X$ and closed:

(*) if $\gamma i(x) = \gamma i(y)$ for $x, y \in \gamma A \setminus A$ then $\gamma f(x) = \gamma f(y)$

where γ is the Freudenthal compactification and $i\colon A \to X$ is the inclusion.

The main theorem of the paper is

THEOREM 2. *If* X, Y *are locally compact,* $X \in \mathcal{C}$ *and* $Y \in \mathfrak{ACR}(\mathcal{C})$ *then the condition* (*) *is also sufficient for the existence of a closed extension* $F\colon X \to Y$ *of* $f\colon A \to Y$. *Some other facts about the class* $\mathfrak{ACR}(\mathcal{C})$ *are also proved.*

This paper is devoted to the study of the extensions of closed mappings from closed sets. We introduce the class $\mathfrak{ACR}$, which plays the same role with respect to closed mappings as the usual class $\mathfrak{AR}$ with respect to continuous mappings. Some characterizations of the class $\mathfrak{ACR}$ are given and the analogon of the Tietze-Urysohn theorem is proved.

We now recall some definitions:

DEFINITION 1. The mapping $f\colon X \to Y$ is *closed* if and only if for every subset A closed in X its image $f(A)$ is closed in Y.

DEFINITION 2. The mapping $f\colon X \to Y$ is *perfect* if and only if it is closed and for every $y \in Y$ the set $f^{-1}(y)$ is compact.

If $\mathcal{A} = \{A_s\}_{s \in S}$ is the family of mutually disjoint subsets of the space X, then we denote by $X/\mathcal{A}$ the space obtained by matching to a point every set A_s, which means the quotient space X/R, where xRy if and only if $x = y$ or $x, y \in A_s$ for some $s \in S$.

If $(rX, r\colon X \to rX)$ is a compactification of the space X then we denote shortly by $rX \setminus X$ the remainder $rX \setminus r(X)$.

Moreover, we denote by $\mathfrak{M}$ the class of all metrizable spaces and by $\mathfrak{PC}$ the class of all paracompact spaces.

All spaces are assumed to be Hausdorff (T_2) and all mappings are assumed to be continuous.

For all notions and notations not defined here see [2] and [3].

DEFINITION 3. We say that $R \in \mathfrak{ACR}(\mathfrak{C})$ if and only if for every locally compact space X from the class $\mathfrak{C}$ and every subset A of X closed in X and homeomorphic to R there exists a closed retraction $r\colon X \to A$. The class $\mathfrak{C}$ may be the class $\mathfrak{M}$ or $\mathfrak{PC}$.

Remark. The condition of the local compactness of the space X is introduced since in the opposite case the one-point space would be the unique compact space belonging to $\mathfrak{ACR}(\mathfrak{M})$ and $\mathfrak{ACR}(\mathfrak{PC})$, as is shown in the following

EXAMPLE 1. Let $W = I \times N / \{\{0\} \times N, \{1\} \times N\}$, and for every subset M of N and every $k > 2$ we denote by $W_{M,k}$ the subset $[1/k, 1-1/k] \times M$ of W. It is clear that $W_{M,k} = \overline{W_{M,k}}$ for any M and k. Now, let R be the compact space containing two distinct points p and q. We denote by X the space $R \oplus W / \{\{[\{0\} \times N], p\}, \{[\{1\} \times N], q\}\}$. It is easy to check that the space X is paracompact and if R is metrizable, then so is X. We shall prove that there is no closed retraction from X onto R. In fact, assume that $r\colon X \to R$ is a closed retraction. We put $M_2 = N$ and write $A_2 = R \cup W_{M_2,3}$. The set A_2 is closed in X and locally compact, hence the mapping $r \in A_2$ is closed and we can apply ([6], Theorem 2), obtaining a compact set $Z \subset A_2$ such that $r(A_2 \setminus Z)$ is finite. It is clear that the set $A_2 \setminus Z$ contains infinitely many components of the set $W_{M_2,3}$ and hence there exists an infinite set $M_3 \subset M_2$ and a point $a \in R$ such that $r(W_{M_3,3}) = \{a\}$. Assume now that we have defined such an infinite set M_n that $r(W_{M_n,n}) = a$. We can apply the arguments given above to the set $A_{n+1} = W_{M_n,n+1} \cup R$ obtaining a new infinite set $M_{n+1} \subset M_n$ such that $r(W_{M_{n+1},n+1})$ is finite and thus equal to $\{a\}$. This completes the description of the inductive step. We put $D = \bigcup_{n=3}^{\infty} W_{M_n,n}$. It is clear that $r(D) = \{a\}$ and $p, q \in \overline{D}$. But p and q are distinct, hence either $p \notin \overline{r(D)}$ or $q \notin \overline{r(D)}$. This contradicts the continuity of r and therefore there is no closed retraction from X onto R.

We now prove the first main theorem of this paper.

THEOREM 1. *If $\mathfrak{C}$ is the class $\mathfrak{M}$ or $\mathfrak{PC}$, then the following conditions are equivalent:*

(i) $R \in \mathfrak{ACR}(\mathfrak{C})$,

(ii) $R \in \mathfrak{AR}(\mathfrak{C})$ *and there exists a closed retraction* $r_0\colon R \times [0,1) \to R \times \{0\}$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious.

We now prove the implication (ii) $\Rightarrow$ (i). Assume first that R is compact. Let $R \subset X \in \mathfrak{C}$ and let U_R be a neighbourhood of R with compact closure existing since X is locally compact. Denoting by Y the space obtained by matching to a point the set $X \setminus U_R$, we obtain a compact space from the class $\mathfrak{C}$ containing R as a closed subset. Let $r\colon Y \to R$ be some retraction and let $p\colon X \to Y$ be the quotient map. It follows from ([6], Proposition 1), that the composition $\bar{r} = r \circ p$ is the desired closed retraction. Observe, that, in fact, we prove here that every $\mathfrak{AR}(\mathfrak{C})$-space in the sense of ([1], V. 1) belongs to $\mathfrak{ACR}(\mathfrak{C})$.

Assume now that R is not compact. As usual, we denote by γX the Freudenthal compactification of X. Since R is connected, $\gamma(R \times [0,1)) = \omega(R \times [0,1))$ (see [6], Proposition 8) and hence $\gamma R = \omega R$. Assume now that $R = \bar{R} \subset X \in \mathfrak{C}$ and X is locally compact. It is well known ([2], Theorem I.9.10.5) that X can be expressed as the sum $\bigoplus_{s \in S} X_s$, where X_s is σ-compact for every $s \in S$, which means that $X_s = \bigcup_{n=1}^{\infty} A_n$ and every A_n is compact. Now, since R is connected, there exists an $s_0 \in S$ such that $R \subset X_{s_0}$. Hence, if $r'\colon X_{s_0} \to R$ is a closed retraction, then taking any $x_0 \in R$ and putting $r|X_{s_0} = r'$ and $r(\bigcup_{s \neq s_0} X_s) = \{x_0\}$ we obtain the closed retraction $r\colon X \to R$. So, without loss of generality, we can assume that X is σ-compact. It is easy to check that in this case the remainder $\{\omega\} = \omega X \setminus X$ is a G_δ-set in ωX and there exists a continuous function $g\colon \omega X \to I$ such that $\{\omega\} = g^{-1}(1)$. Now, let $J\colon \gamma X \to \omega X$ be the extension of identity over the Freudenthal compactification γX and let $G = g \circ J$. It is clear that $G^{-1}(1) = \gamma X \setminus X$ and the mapping $h = G|X\colon X \to [0,1)$ is perfect. Since $R \in \mathfrak{AR}(\mathfrak{C})$, there exists a retraction $r\colon X \to R$. The mapping r can be corrected to a closed retraction $\bar{r}\colon X \to R$ as follows:

Let $\mathcal{V} = \{V_n\}_{n=1}^{\infty}$ be a locally finite covering of R by open sets with compact closures. We put $F_n = \overline{V_n}$. It is clear that the family $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ is a locally finite covering of R. Moreover, we can refine the covering $\mathcal{V}$ to the covering $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ such that $A_n = \overline{A_n} \subset V_n$ for every n (see [3], Lemma 1 to Theorem 5.1.3). We now select for any n and for every $x \in A_n$ a set U_x^n open in X and such that $\overline{U_x^n}$ is a compact subset of $r^{-1}(F_n)$. We take as U_n the sum of some finite subfamily of $\{\overline{U_x^n}\}_{x \in A_n}$ covering A_n. It is clear that the family $\mathcal{U} = \{U_n\}_{n=1}^{\infty}$ satisfies the following conditions

(i) $\overline{U_n}$ is compact for every n,

(ii) $\overline{U_n} \subset r^{-1}(F_n)$ for every n,

(iii) $\bigcup_{n=1}^{\infty} U_n \supset R$.

We write $U = \bigcup_{n=1}^{\infty} U_n$ and we define f_0: $R \cup (X \setminus U) \to I$ as follows: $f_0(R) = 0$, $f_0 | X \setminus U = h | X \setminus U$, where h is defined above. Using the Tietze extension theorem, we can extend f_0 to the mapping f: $X \to I$ satisfying the condition $f(U) \subset (0, 1)$. We define the mapping F: $X \to R \times [0, 1)$ by the formula $F(x) = (r(x), f(x))$. The mapping $f | X \setminus U$ is perfect and, as is easy to verify, the mapping $F | X \setminus U$ is also perfect (see [3], Problem 3.X). Assume now that A is a closed subset of $\overline{U}$. Then $F(A) = \bigcup_{n=1}^{\infty} F(A \cap \overline{U_n})$. But $F(A \cap \overline{U_n})$ is compact and it is contained in $F_n \times \times [0, 1)$. Hence $F(A)$ is the sum of a locally finite family of compact sets and therefore $F(A)$ is closed. So $F | \overline{U}$ is closed, and, combining this with the above result about the mapping $F | X \setminus U$. we infer that the mapping F is closed. The mapping $\bar{r} = r_0 \circ F$ is the required closed retraction. In fact, $\bar{r}$ is closed as the superposition of two closed mappings r_0 and F and simultaneously $F(x) = (x, 0)$ for $x \in R$ and, since $r_0 | R \times \{0\} = \mathrm{id}_R$, the mapping $\bar{r}$ is a retraction.

This completes the construction of a closed retraction of X onto R and the proof of Theorem 1.

We now give some examples of non-compact spaces belonging to the class $\mathfrak{ACR}$.

Example 2. Let R be a closed Euclidean half-space, which means that $R = \{(x_1, \dots, x_n) \in E^n\colon x_1 \geqslant 0\}$. The space R belongs to $\mathfrak{ACR}(\mathfrak{M})$ and $\mathfrak{ACR}(\mathfrak{PR})$. In fact, since R is a retract of E^n, we have $R \in \mathfrak{AR}(\mathfrak{M})$ and $R \in \mathfrak{AR}(\mathfrak{PC})$ (see [1], Theorem IV.2.1). We define the retraction r_0: $R \times [0, 1) \to R \times \{0\}$ by the formula

$$r_0(x_1, \dots, x_n, t) = (x_1 + t/(1-t), x_2, \dots, x_n, 0)\,.$$

It can easily be checked that r_0 is closed and hence R satisfies condition (ii) of Theorem 1.

Example 3. Let $R_1 \in \mathfrak{ACR}(\mathfrak{C})$ and let $R \subset R_1$ be a c-retract of R_1, that is, let R be the image of R_1 under a closed retraction r_1. Then $R \in \mathfrak{ACR}(\mathfrak{C})$. In fact, let $X = R[0, 1) \cup R_1 \times \{0\}$. It is clear that if R_1 is locally compact, then so is X and $X \in \mathfrak{C}$. Let r_1: $X \to R_1$ and r_2: $R_1 \to R$ be closed retractions. The mapping $r_0 = r_2 \circ r_1$ is the required closed retraction from $R \times [0, 1)$ onto R. On the other hand, $R \in \mathfrak{AR}(\mathfrak{C})$, as is shown in [1], IV.2.2.

We now give some corollaries to Theorem 1.

Corollary 1. *If R is a non-compact space from the class $\mathfrak{ACR}(\mathfrak{M})$ or $\mathfrak{ACR}(\mathfrak{PC})$, then the retraction r_0: $R \times [0, 1) \to R$ from the condition* (ii) *of Theorem* 1 *is perfect and $\gamma R = \omega R$.*

The proof follows immediately from the first part of the proof of Theorem 1 in the case of a non-compact R and the well known properties of perfect mappings, (see [3], Problem 3.X).

Corollary 2. *If R_n is an $\mathfrak{AR}(\mathfrak{M})$-space for $n = 1, 2, \dots$, then $\boldsymbol{P}_{n=1}^{\infty} R_n \in \mathfrak{ACR}(\mathfrak{M})$.*

Corollary 3. *If R_s is an $\mathfrak{AR}(\mathfrak{PC})$-space for every $s \in S$, then $\boldsymbol{P}_{s \in S} R_s \in \mathfrak{ACR}(\mathfrak{PC})$.*

Corollary 4. *If $R_n \in \mathfrak{AR}(\mathfrak{M})$ for $n = 1, 2, \dots$, $\boldsymbol{P}_{n=1}^{\infty} R_n$ is locally compact and $R_k \in \mathfrak{ACR}(\mathfrak{M})$ for some non-compact R_k, then $\boldsymbol{P}_{n=1}^{\infty} R_n \in \mathfrak{ACR}(\mathfrak{M})$.*

Proof. It is clear that $\boldsymbol{P}_{n=1}^{\infty} R_n \in \mathfrak{AR}(\mathfrak{M})$ (see [1], Theorem IV.7.1). Let r_k: $R_k \times [0, 1) \to R_k \times \{0\}$ be the closed retraction from condition (ii) of Theorem 1. It follows from Corollary 1 that r_k is perfect and hence, by ([2], Corollary I.10.2.3) the product

$$r_k \times \boldsymbol{P}_{n \neq k} \mathrm{id}_{R_n}\colon \boldsymbol{P}_{n=1}^{\infty} R_n \times [0, 1) \to \boldsymbol{P}_{n=1}^{\infty} R_n \times \{0\}$$

is the perfect retraction required in condition (ii) of Theorem 1.

Corollary 5. *If $R_s \in \mathfrak{AR}(\mathfrak{PC})$ for every $s \in S$, $\boldsymbol{P}_{s \in S} R_s$ is locally compact and paracompact and $R_{s_0} \in \mathfrak{ACR}(\mathfrak{PC})$ is non-compact for some $s_0 \in S$, then $\boldsymbol{P}_{s \in S} R_s \in \mathfrak{ACR}(\mathfrak{PC})$.*

The proof is quite analogous to the proof of Corollary 4.

We can now formulate and prove some facts about the extension of closed mappings from closed subsets of locally compact spaces. As usual, we denote by $\mathfrak{C}$ the class of all metrizable spaces $\mathfrak{M}$ or the class of all paracompact spaces $\mathfrak{PC}$.

We define first the notion of an admissible pair of closed mappings.

Definition 4. Let both X and Y be locally compact spaces. Let i: $A \to X$ be a closed embedding (we can regard A as the closed subset of X) and let f: $A \to Y$ be a closed mapping. The pair (i, f) is *admissible* if and only if the following condition is satisfied:

$$(*) \quad \text{if} \quad \gamma i(x) = \gamma i(y) \text{ for } x, y \in \gamma A \setminus A\,, \quad \text{then} \quad \gamma f(x) = \gamma f(y)\,.$$

Observe that condition $(*)$ is equivalent to the existence of a mapping g: $\gamma X \setminus X \to \gamma Y$ such that $\gamma f | \gamma A \setminus A = g \circ (\gamma i | \gamma A \setminus A)$. This follows from the compactness of the set $\gamma A \setminus A$.

To check that condition $(*)$ is not always satisfied, it is sufficient to observe that the pair $(i\colon E^1 \to E^2, f = \mathrm{id}\colon E^1 \to E^1)$, where $i(x) = (x, 0)$, is not admissible. It is easy to verify that both i and f are closed and that $\gamma i(\gamma E^1\setminus E^1) = \{\omega\} = \gamma E^2\setminus E^2 = \omega E^2\setminus E^2$ (see [6], corollary to Proposition 8) and simultaneously $\gamma f(\gamma E^1\setminus E^1)$ is a two-point set.

It is clear that if $\gamma Y = \omega Y$ and $f\colon A\to Y$ is perfect, then for every closed embedding $i\colon A\to X$ the pair (i, f) is admissible. In fact, if A is compact, then there is nothing to prove, and if A is non-compact, then $\gamma f(\gamma A\setminus A) = \{\omega\} = \gamma Y\setminus Y$, hence condition $(*)$ is satisfied for every closed embedding $i\colon A\to X$.

We first prove the following

PROPOSITION 1. *Let $i\colon A\to X$ be a closed embedding into a locally compact space X from $\mathfrak{C}$ and let $f\colon A\to Y$ be a perfect mapping into the space $Y\in\mathfrak{A}\mathfrak{C}\mathfrak{R}(\mathfrak{C})$. Then f can be extended to a closed mapping $F\colon X\to Y$.*

Proof. We denote by $X\cup_f Y$ the space $X\oplus Y/\{f^{-1}(y)\cup\{y\}\}_{y\in Y}$ and by $\varphi\colon X\oplus Y\to X\cup_f Y$ the standard quotient mapping. We prove that the mapping φ is perfect. First, let $x\in X\cup_f Y$. Clearly, if $x\in X\setminus A$ then $\varphi^{-1}(x) = \{x\}$ and if $x\in Y$ then $\varphi^{-1}(x) = \{x\}\cup f^{-1}(x)$; hence in both cases $\varphi^{-1}(x)$ is compact. Now, let M be a closed subset of $X\oplus Y$. It is clear that $\varphi(M) = \varphi(M\cap X)\cup\varphi(M\cap Y)$ and $\varphi(M\cap Y) = M\cap Y$ is closed in $X\cup_f Y$. So we can regard only the case $M\subset X$. To verify that $\varphi(M)$ is closed take a point $x\in X\cup_f Y\setminus\varphi(M\cap X)$. If $x\in\varphi\big(Y\setminus f(A)\big)$, then $x\in U = (X\cup_f Y)\setminus\varphi(X)\subset X\cup_f Y\setminus\varphi(M)$ and U is an open neighbourhood of x. If $x\in\varphi(X\setminus A)$, then $x\in U = \varphi\big(X\setminus(A\cup M)\big)$, and if $x\in\varphi(A)$, then $x\in U = \varphi\big(Y\setminus f(M)\big)$. It is easy to check that in all the cases described above U is an open neighbourhood of x not intersecting $\varphi(M)$ and that they are all the cases possible for $x\notin\varphi(M)$. Hence we have proved that the mapping φ is perfect. Therefore we infer, using the results of [4] and [5], that $X\cup_f Y\in\mathfrak{C}$ and, as can easily be verified, $X\cup_f Y$ is locally compact. It now remains to observe that Y is closed in $X\cup_f Y$ and hence there exists a closed retraction $r\colon X\cup_f Y\to Y$. The superposition $F = r\circ\varphi|X$ is the required closed extension of f. In fact, if $x\in A$ then $\varphi(x) = f(x)\in X\cup_f Y$, and hence $r\circ\varphi(x) = f(x)$. On the other hand, F is closed as a superposition of two closed mappings.

We can now prove the main theorem of this paper.

THEOREM 2. *Let $(i\colon A\to X,\ f\colon A\to Y)$ be an admissible pair. If $X\in\mathfrak{C}$ and $Y\in\mathfrak{A}\mathfrak{C}\mathfrak{R}(\mathfrak{C})$ then f can be extended to a closed mapping $F\colon X\to Y$.*

Proof. Notice first that since both A and X are locally compact and paracompact and the mapping i is a closed embedding, it is easy to check that $\gamma i(\gamma A\setminus A)$ is a closed subset of the compact, zero-dimensional space $\gamma X\setminus X$. ($\operatorname{ind}(\gamma X\setminus X) = 0 = \operatorname{Ind}(\gamma X\setminus X)$ by [3], Theorem 7.1.10).

Notice now that since Y is locally compact, paracompact and connected (see [1], Corollary IV.2.3), it is, by [2], Theorem 1.9.10.5 σ-compact, i.e. $Y = \bigcup_{n=1}^{\infty} Z_n$ and every Z_n is compact. Moreover, since every compact subset of a locally compact space has an open neighbourhood with compact closure, we can obtain a family $\{R_n\}_{n=1}^{\infty}$ of compact sets such that

(i) $R_n\subset\operatorname{Int}R_{n+1}$ for every n,

(ii) $Y = \bigcup_{n=1}^{\infty} R_n$,

putting $R_0 = \emptyset$ and taking as R_n the compact closure of the neighbourhood $U_{R_{n-1}\cup Z_n}$.

Write $F_n = f^{-1}(R_n)$ and let $f_n\colon F_n\to R_n$ be the restriction of f. Since f_n is closed and R_n is compact, we obtain by [6], Theorem 2 compact sets $\widetilde{G}_n\subset F_n$ such that $f_n(F_n\setminus\widetilde{G}_n)$ is finite for every n. Denote $G_n = \bigcup_{k=1}^{n}\widetilde{G}_k$ and let $f_n(F_n\setminus G_n) = \{y_1, \ldots, y_m\}$. Clearly, we can assume that $f_n^{-1}(y_i)$ is not compact for every n and i. Now, since $f_n(F_n\setminus G_n)\subset f_{n+1}(F_{n+1}\setminus G_{n+1})$, we can order the set $\{y_1, y_2, \ldots\} = \bigcup_{n=1}^{\infty} f_n(F_n\setminus G_n) = f(A\setminus\bigcup_{n=1}^{\infty} G_n)$ in such a manner that the function $n(k) = \min\{n\colon y_k\subset\operatorname{Int}R_n\}$ is not decreasing. Write $B_k = \gamma f^{-1}(y_k)\cap(\gamma A\setminus A)$ and $\widetilde{C}_k = \gamma i(B_k)$. It follows from the definition of the admissible pair that the sets $\widetilde{C}_k$ are closed and mutually disjoint.

We now prove that there exists a family $\{C_k\}_{k=1}^{\infty}$ of mutually disjoint open-and-closed sets in $\gamma X\setminus X$ such that $\widetilde{C}_k\subset C_k$ for every k. Take

$$V_k = (\gamma X\setminus X)\setminus\gamma i\Big(\gamma A\setminus(\gamma f)^{-1}\big(\operatorname{Int}R_{n(k)}\cup\{y_l\colon\ l\neq k,\ n(l)\leqslant n(k)\}\big)\Big).$$

Clearly $\widetilde{C}_k\subset V_k$ and, since for every k there exists only a finite number of such l that $n(l)\leqslant n(k)$, the set V_k is open and $V_k\cap\widetilde{C}_l = \emptyset$ for $l\neq k$. We define the sets C_k inductively. Let C_1 be an open-and-closed subset of $\gamma X\setminus X$ such that $\widetilde{C}_1\subset C_1\subset V_1$. Such a set exists since $\operatorname{Ind}(\gamma X\setminus X) = 0$. Assume now that we have defined the sets $C_1, \ldots, C_{k-1}$ and take as C_k an open-and-closed subset of $\gamma X\setminus X$ satisfying the following condition: $\widetilde{C}_k\subset C_k\subset V_k\setminus\bigcup_{i=1}^{k-1} C_i$. The family $\{C_k\}_{k=1}^{\infty}$ satisfies the above-mentioned conditions.

We can now construct a family $\{U_n\}_{n=1}^{\infty}$ of open subsets of γX satisfying the following conditions:

(i) $\overline{U_i}\cap\overline{U_j} = \emptyset$ if $i\neq j$,

(ii) $C_i\subset U_i$ } for every i.

(iii) $\operatorname{Fr}(U_i)\subset X$ } for every i.

(iv) $f(U_i\cap A) = \{y_i\}$ } for every i.

The inductive construction of the sets U_i is rather complicated and proceeds as follows:

Put

$$H_1 = \big((\gamma X \setminus X)\setminus C_1\big) \cup$$

$$\cup\, \gamma i(\gamma f)^{-1}\big(\gamma Y \setminus \big(\mathrm{Int}(R_{n(1)}) \cup \{y_i\colon\ i \neq 1,\ n(i) = n(1)\}\big)\big) \cup G_{n(1)}\,.$$

It follows from the definition of C_1 that the set H_1 is closed and, clearly $H_1 \cap C_1 = \emptyset$. Moreover, $C_1 \cup H_1 \supset \gamma X \setminus X$. Since γX is normal, there exists a set U_1, open in γX and such that $C_1 \subset U_1 \subset \overline{U}_1 \subset \gamma H \setminus H_1$. It is easy to check that the set U_1 satisfies conditions (ii)-(iv) given above and, moreover,

$$\overline{U}_1 \cap \bigcup_{i=2}^{\infty}\big(C_i \cup \gamma i\big((\gamma f)^{-1}(y_i)\big)\big) = \emptyset\,.$$

Assume now that we have defined the sets $U_1, \ldots, U_{k-1}$ satisfying conditions (i)-(iv) and, moreover,

$$\bigcup_{i=1}^{k-1} \overline{U}_i \cap \bigcup_{i=k}^{\infty}\big(C_i \cup \gamma i\big(\gamma f^{-1}(y_i)\big)\big) = \emptyset\,.$$

Similarly to the above, we take the set

$$H_k = \big((\gamma X \setminus X)\setminus C_k\big) \cup$$

$$\cup\, \gamma i(\gamma f)^{-1}\big(\gamma Y \setminus \big(\mathrm{Int}(R_{n(k)}) \cup \{y_i\colon\ i \neq k,\ n(i) \leqslant n(k)\}\big)\big) \cup G_{n(k)} \cup \bigcup_{i=1}^{k-1} \overline{U}_i\,.$$

It is easy to check, using the above arguments, that the set H_k is closed, $C_k = (\gamma X \setminus X)\setminus H_k$ and that

$$\bigcup_{\substack{i=1\\ i\neq k}}^{\infty} \gamma i\big((\gamma f)^{-1}(y_i)\big) \cap H_k = \emptyset\,.$$

We can hence select a set U_k open in γX and such that $C_k \subset U_k \subset \overline{U}_k \subset \gamma X \setminus H_k$. It is easy to check that the family $\{U_1, \ldots, U_k\}$ satisfies all the conditions (i)-(iv) and

$$\bigcup_{i=1}^{k} \overline{U}_i \cap \bigcup_{i=k+1}^{\infty} C_i \cup \gamma i(\gamma f)^{-1}(y_i) = \emptyset\,,$$

which completes the description of the inductive step.

It is clear that the sets $U_1, U_2, \ldots$ satisfy the condition (i)-(iv).

Now select for every i a point $x_i \in A \cap \overline{U}_i$ and define the space $\widetilde{X}$ as the quotient space $X/\{\overline{U}_i\}_{i=1}^{\infty}$. We prove that $\widetilde{X}$ is locally compact and $\widetilde{X} \in \mathfrak{C}$. Notice first that the standard quotient mapping $\varphi\colon X \to \widetilde{X}$ is closed as we can check considering the extension of φ over γX and using [6], Theorem 5. On the other hand, the space $X' = (X \setminus \bigcup_{i=1}^{\infty} U_i) \cup \{x_1, x_2, \ldots\}$ is a closed subset of X: hence X' is locally compact and belongs to the class $\mathfrak{C}$. It is clear that $\widetilde{X} = X'/\{\mathrm{Fr}(U_i) \cup \{x_i\}\}_{i=1}^{\infty}$ and, the fibres of the quotient mapping $\varphi'\colon X' \to \widetilde{X}$ are either singular points or compact sets $\mathrm{Fr}(U_i) \cup \{x_i\}$, hence the mapping φ' is perfect. It follows then from the results of [4] and [5] that $\widetilde{X} \in \mathfrak{C}$ and, as can easily be verified, $\widetilde{X}$ is locally compact. Notice now that it follows from condition (iv) given above that there exists a mapping $\tilde{f}\colon \widetilde{A} = \varphi i(A) \to Y$ such that $f = \tilde{f} \circ \varphi \circ i$. It is sufficient to put $f([x]) = f(x)$ if $x \in \varphi i(A) \setminus \{[x_1], [x_2], \ldots\}$ and $\tilde{f}([x_n]) = f(x_n)$ for every n. It is clear that f is closed and, moreover, since

$$\tilde{f}^{-1}(R_n) = i(F_n) \setminus \bigcup_{i\colon n(i)\leqslant n} (U_i \setminus \{x_i\}) / \{\mathrm{Fr}(U_i) \cap i(A) \cup \{x_i\}\}_{i\colon n(i)\leqslant n}$$

and the closure of U_i in γX contain all the image $\gamma i(\gamma A \setminus A)$, $\tilde{f}^{-1}(R_n)$ is compact for every n. It follows that the mapping f is perfect and we can apply Proposition 1 to obtain a closed extention $\widetilde{F}\colon X \to Y$ of the mapping $\tilde{f}$.

It now remains to put $F = \widetilde{F} \circ \varphi$ and to observe that the mapping F is the required closed extension of f.

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