Representation of functions of two variables
as sums of rectangular functions, I

by

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Abstract. It is shown that CH implies that every real \( f(x, y) \) can be written
\[ \sum_{n=1}^{\infty} g_n(x) h_n(y), \]
with summation from 1 to \( N(x, y) < \infty \), and that such a representation
for \( \exp(xy) \) implies CH and is impossible with a fixed number of non-zero terms.

In 1932, Michael F. Drmkin asked me what was essentially the following
question: does every real-valued function \( f \) of two real variables
admit a representation in the form

\[ f(x, y) = \sum_{n=1}^{\infty} g_n(x) h_n(y). \]

The main aim of this paper is to present the proof (which I found in 1954)
that if the continuum hypothesis is assumed then the answer is affirmative,
and indeed there is a representation (1) with the property that for each
point \((x, y) \in \mathbb{R}^2\) there are only finitely many non-zero terms in the series
(Theorem 1). Moreover it will be shown that conversely this proposition
implies the continuum hypothesis (Theorem 2); unfortunately, I see no
way of deciding whether this converse remains valid when the finiteness
assertion is omitted. Finally, it will be shown that “finitely many” in
Theorem 1 cannot be replaced by any fixed integer \( N \) (Theorem 3).

It is easy to see that for non-negative \( f \) one cannot demand a represen-
tation (1) with non-negative \( g \)'s and \( h \)'s: for example, this is impos-
sible in the case of the characteristic function of the diagonal, \( 1- \text{sgn}[x-y] \).
It is hoped to discuss the possibility of representing a measurable \( f \) with
measurable \( g \)'s and \( h \)'s in a subsequent paper.

**Theorem 1.** If \( X, Y \) are sets of cardinality \( \aleph_1 \), then given any function
\( f: X \times Y \to \mathbb{R} \) there exist two sequences of functions
\[ g_n: X \to \mathbb{R}, \quad h_n: Y \to \mathbb{R} \quad (n = 1, 2, \ldots) \]
such that (1) holds for every \((x, y) \in X \times Y \), and moreover for each \((x, y) \in X \times Y \) there are only finitely many non-zero terms in the series.
Proof. It will be convenient to regard a sequence of real-valued functions from a set $E$ to $R$ as a function from $N \times E$ to $R$, and to denote by $\sum^\infty_{n=1} a$ a sum in which there are only finitely many non-zero terms. Instead of (1) we shall thus write

\[ f(x, y) = \sum_{n=1}^{\infty} g(n, x) h(n, y). \]

We may and shall suppose that $X$ and $Y$ are disjoint. List the elements of $X \cup Y$ as a transfinite sequence $(\xi_n)$ of type $\omega_1$, and for each ordinal $\alpha < \omega_1$ let

\[ Z_\alpha = \{ \xi_\beta; 0 < \beta < \eta \}, \quad X_\alpha = Z_\alpha \cap X, \quad Y_\alpha = Z_\alpha \cap Y. \]

Denote by $\mathcal{F}$ the set of all ordered pairs of real-valued functions $(g, h)$ satisfying the following conditions: for some ordinal $\alpha = (g, h)$, with $0 < \alpha < \omega_1$, we have $\text{dom } g = N \times X_\alpha$, $\text{dom } h = N \times Y_\alpha$, and (1') holds for $(x, y) \in X_\alpha \times Y_\alpha$, and in addition

(a) given any two disjoint finite subsets $K, K'$ of $X$, and any finite subset $L$ of $Y$, there exist infinitely many values of $n$ for which simultaneously

\[ g(n, \xi) = 0 \quad (\xi \in K), \quad g(n, \xi') = 1 \quad (\xi' \in K'), \quad h(n, \eta) = 0 \quad (\eta \in L), \]

(b) given any two disjoint finite subsets $L, L'$ of $Y$, and any finite subset $K$ of $X$, there exist infinitely many values of $n$ for which simultaneously

\[ h(n, \eta) = 0 \quad (\eta \in L), \quad h(n, \eta') = 1 \quad (\eta' \in L'), \quad g(n, \xi) = 0 \quad (\xi \in K). \]

Partially order $\mathcal{F}$ by the relation $\subseteq$ defined as follows:

\[ (g, h) \subseteq (g', h') \iff g \subseteq g' \text{ and } h \subseteq h', \]

(that is: $g'$, $h'$ are extensions of $g$, $h$ respectively). Now $(\emptyset, \emptyset) \in \mathcal{F}$, and every totally ordered subset $\mathcal{S}$ of $\mathcal{F}$ has an upper bound, namely

\[ \bigcup \{ (g, h) \in \mathcal{S}; \bigcup (h, (g, h) \in \mathcal{S}) \}, \]

and therefore by Zorn's Lemma $\mathcal{F}$ contains a maximal element $(g, h)$. It will now be sufficient to show that $a(g, h) = \alpha_1$, since then dom $g = N \times X$, $\text{dom } h = N \times Y$, and (1') will hold for every $(x, y) \in X \times Y$.

Suppose if possible that $a(g, h) = a < \alpha_1$; we shall show how to extend $g$ or $h$. The element $\xi_n$ belongs to $X$ or $Y$; suppose the former. The sets $X_\alpha, Y_\alpha$ are countable. Consequently we can list the elements of $Y_\alpha$ in a finite or infinite sequence

\[ Y_\alpha, Y_\beta, \ldots, \]

and we can form an infinite sequence

\[ (K_1, K'_1, L_1, L'_1), (K_2, K'_2, L_2, L'_2), \ldots \]

consisting of all quadruples of which the first two members are disjoint finite subsets of $X$, and the last two members are disjoint finite subsets of $Y$; each such quadruple being repeated infinitely often in the sequence (3).

Now define four infinite sequences of positive integers by induction as follows: $r_1 = 1$, and for $i = 1, 2, \ldots$ \[ p_i = r_{i-1} \text{ if } y_i \text{ is undefined (i.e. if } Y \text{ has } i-1 \text{ or fewer elements), and otherwise } p_i \text{ is the least integer } p > r_{i-1} \text{ for which} \]

\[ g_i(p, y_i) = \ldots = g_i(p, y_{i-1}) = 0 \quad \text{ and } \quad g_i(p, y_i) = 1; \]

$g_i$ is the least integer $g > p_i$ for which

\[ g_i(q, \xi) = 0 \quad (\xi \in K_i), \quad g_i(q, \xi) = 1 \quad (\xi' \in K'_i), \quad \text{ and} \quad h_i(q, \eta) = 0 \quad \text{ for } \eta \in Y_i, \ldots, Y_1; \]

\[ q_i \text{ is the least integer } q > q_i \text{ satisfying (5);} \]

\[ r_i = \text{ the least integer } r > q_i \text{ for which} \]

\[ h_i(q, \eta) = 0 \quad (\eta \in L_i), \quad h_i(q, \eta') = 1 \quad (\eta' \in L'_i), \quad g_i(q, \xi) = 0 \quad (\xi \in K_i). \]

It is easy to verify that, because the pair $(g, h)$ satisfies conditions (a) and (b), these integers all exist.

Define a function $g^* : N \times X_{\alpha+1} \rightarrow R$ by putting $g^*(n, x) = g(n, x)$ for $x \in X$, and defining $g^*(n, \xi_n)$ by induction on $n$ as follows:

\[ g^*(n, \xi_n) = f(n, y_n) - \sum_{n=1}^{\infty} g(n, \xi_n) h(n, y_i) \quad \text{ if } \quad p_i > r_{i-1}, \]

\[ g^*(n, \xi_n) = 1, \]

\[ g^*(n, \xi_n) = 0 \quad \text{ for all other values of } n. \]

Consider the pair $(g^*, h)$; we shall show that it belongs to $\mathcal{F}$, which contradicts the maximality of $(g, h)$, since $g^*$ is a proper extension of $g$. Observe that

\[ \text{dom } g^* = N \times (X_\alpha \cup \xi_n) = N \times X_{\alpha+1} \]

and

\[ \text{dom } h^* = N \times Y_\alpha = N \times Y_{\alpha+1}. \]
Now we show that (1') holds (with $g$ replaced by $g^*$) for $(a_1, y) \in X_{a+1} \times Y_{a+1}$. Since $(g, h) \in \mathcal{S}$, the only case to be considered is when $x = z_\omega$.

Let $y$ occur as $y_k$ (say) in the sequence (2), and consider the sum

$$\sum_{n=1}^{\infty} g^*(a_n, z_\omega) h(n, y_k).$$

In view of (10), we have $g^*(a_n, z_\omega) = 0$ unless $n$ is of the form $p_i$ (with $p_i > r_{\omega-1}$) or $s_i$, while by the definitions of $p_i$ and $s_i$ (see (6) and (9)) we also have $h(p_i, y_k) = 0$ for $i > k$, $h(p, y_k) = 1$ for some $i$, and $h(q_i, y_k) = 0$ for $i > k$. Consequently the sum (11) has only finitely many non-zero terms, and reduces to

$$\sum_{n=1}^{N} g^*(a_n, z_\omega) h(n, y_k) + g^*(a, z_\omega),$$

which by (8) is equal to $f(z_\omega, y_n)$. We have thus established that (1') holds.

Finally, we must verify conditions (a) and (b), with $g$ replaced by $g^*$ and $a$ by $a+1$.

Condition (a). Let $K, K'$ be disjoint finite subsets of $X_{a+1}$, and let $L$ be a finite subset of $Y_{a+1} = Y_a$.

Case 1. $z_\omega \in K \cup K'$. Then $K \cup K' \subseteq X_a$, and the required infinitely many values of $n$ exist because the pair $(g, h)$ satisfies (a).

Case 2. $z_\omega \in K$. Then for every occurrence of $(K \setminus z_\omega, K', L, \emptyset)$ as a term $(K_i, K'_i, L_i, L'_i)$ in the sequence (3), it follows from the definitions of $g^*$ and $q_i$ (see (6), (9)) that

$$g^*(q_i, \xi) = 0 \ (\xi \in K), \quad g^*(q_i, \zeta) = g(q_i, z_\omega) = 1 \ (\zeta \in K')$$

provided that $i$ is large enough that $L \subseteq (y_1, ..., y_i)$, and there are infinitely many such occurrences.

Case 3. $z_\omega \in K'$. Then for every occurrence of $(K, K' \setminus z_\omega, L, \emptyset)$ as a term $(K_i, K'_i, L_i, L'_i)$ in the sequence (3), it follows from the definitions of $g^*$ and $q_i$ (see (6), (9)) that

$$g^*(q_i, \xi) = g(q_i, z_\omega) = 0 \ (\xi \in K), \quad g^*(q_i, \zeta) = g(q_i, z_\omega) = 1 \ (\zeta \in K')$$

provided that $i$ is large enough that $L \subseteq (y_1, ..., y_i)$, and there are infinitely many such occurrences.

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Condition (b). Let $K$ be a finite subset of $X_{a+1}$ and let $L, L'$ be disjoint finite subsets of $Y_{a+1} = Y_a$.

Case 1. $z_\omega \in K$. Then $K \subseteq X_a$, and the required infinitely many values of $n$ exist because the pair $(g, h)$ satisfies (b).

Case 2. $z_\omega \in K$. Then for every occurrence of $(K \setminus z_\omega, O, L, L')$ as a term $(K_i, K'_i, L_i, L'_i)$ in the sequence (3), it follows from the definitions of $g^*$ and $q_i$ (see (7), (10)) that

$$h(q_i, \eta) = 0 \ (\eta \in L), \quad h(q_i, \zeta) = 1 \ (\zeta \in L'), \quad g^*(q_i, \xi) = 0 \ (\xi \in K),$$

and there are infinitely many such occurrences.

The proof that $(g^*, h) \in \mathcal{S}$ has now been completed, and with it the proof of Theorem 1.

Theorem 2. The existence of a representation

$$e^u = \sum_{n=1}^{N} g_n(x) h_n(y),$$

where $N(x, y)$ is a positive integer for each point $(x, y) \in \mathcal{R}$, implies the continuum hypothesis.

Proof. (I owe the idea of this to P. Erdös.) Suppose if possible that there exists a representation (12) but $2^\omega > \omega$: naturally, we are assuming the axiom of choice. Let $Q$ be a subset of $\mathcal{R}$ of cardinality $\omega$. For each $x \in \mathcal{R}$ and each positive integer $N$, let $Q(N, x) = \{y \in \mathcal{R}: Q(x, y) = N\}$; then $Q = \bigcup Q(N, x)$, and therefore $Q(x, y)$ is finite for some integer $N = N(x, y)$; select an $(N+1)$-element subset $S(x)$ of $Q(N, x)$. For each non-empty finite subset $S$ of $Q$, let $P(S) = \{x \in \mathcal{R}: S(x, y) = S\}$; then $R = \bigcup P(S)$, and therefore $P(S)$ is infinite for some $S = S_n$, say. Let $S_n$ have cardinality $N+1$, and select an $(N+1)$-element subset $R_n$ of $P(S_n)$. Then for every point $(x, y) \in R_n \times S_n$ we have

$$e^{x,y} = \sum_{n=1}^{N} g_n(x) h_n(y),$$

and it follows [11, 2] that

$$\det(e^{x,y}) = 0, \quad \text{where} \quad R_n = \{x_1, ..., x_{N+1}\}, \quad S_n = \{y_1, ..., y_{N+1}\}.$$

But (3), p. 9 such a determinant never vanishes, and we have a contradiction.
THEOREM 3. There exists no positive integer \( N \) such that \( \sigma^N \) admits a representation

\[
\sigma^N = \sum_{n=1}^\infty g_n(x) h_n(y)
\]

with the property that for each point \((x, y) \in \mathbb{R}^2\) there are no more than \( N \) non-zero terms in the series.

Proof. Suppose if possible that there exists a representation (13) of this kind. Let \( N_0 \) be the least integer such that there exist sets \( A, B \subseteq \mathbb{R} \) of cardinality \( 2^{N_0} \), with the property that for each point \((x, y) \in A \times B\) there are no more than \( N_0 \) non-zero terms in the series in (13). Thus \( 1 \leq N_0 \leq N \). Take any such sets \( A, B \), and for each \( N_0 \)-element subset \( E \) of the set of positive integers let

\[
P(E) = \{(x, y) \in A \times B: g_n(x)h_n(y) \neq 0 \text{ for } n \in E\}
\]

Given any \( E \), let \( Q(E) \) be a maximal collection of points of \( P(E) \) such that no two lie on the same horizontal or vertical line. Now \( |Q(E)| \leq N_0 \), because otherwise if we select an \((N_0+1)\)-element subset

\[
\{(x_i, y_j): i = 1, ..., N_0+1\}
\]

of \( Q(E) \) then for each \( i, j \) we have \( g_n(x_i)h_n(y_j) \neq 0 \) for \( n \in E \) and (since there are no more than \( N_0 \) non-zero terms in the series) \( \sum g_n(x_i)h_n(y_j) = 0 \), whence \( \det([\sigma^{x_iy_j}]) = 0 \), which is impossible. Let \( A_d(E), B_d(E) \) be the projections of \( Q(E) \) on the axes. Then \( P(E) \subseteq \{A_d(E) \times B_d(E)\} \) and

\[
|A_d(E)| \leq N_0, \quad |B_d(E)| = N_0.
\]

It follows that

\[
|\bigcup E A_d(E)| \leq N_0, \quad |\bigcup E B_d(E)| \leq N_0,
\]

and therefore \( |A'| = |B'| = 2^{N_0} \), where

\[
A' = A \setminus \bigcup E A_d(E), \quad B' = B \setminus \bigcup E B_d(E).
\]

But for each point \((x, y) \in A' \times B'\) there are no more than \( N_0 - 1 \) non-zero terms in the series (13), and this contradicts the definition of \( N_0 \).