Representation of functions of two variables as sums of rectangular functions, I

by

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Abstract. It is shown that CH implies that every real $f(x, y)$ can be written $\sum g_n(x)h_n(y)$, with summation from 1 to $N(x, y) < \infty$, and that such a representation for $\exp(xy)$ implies CH and is impossible with a fixed number of non-zero terms.

In 1952, Michael P. Drazin asked me what was essentially the following question: does every real-valued function f of two real variables admit a representation in the form

$$f(x, y) = \sum_{n=1}^{\infty} g_n(x)\, h_n(y)\,? \tag{1}$$

The main aim of this paper is to present the proof (which I found in 1954) that if the continuum hypothesis is assumed then the answer is affirmative, and indeed there is a representation (1) with the property that for each point $(x, y) \in \boldsymbol{R}^2$ there are only finitely many non-zero terms in the series (Theorem 1). Moreover it will be shown that conversely this proposition implies the continuum hypothesis (Theorem 2); unfortunately, I see no way of deciding whether this converse remains valid when the finiteness assertion is omitted. Finally, it will be shown that "finitely many" in Theorem 1 cannot be replaced by any fixed integer N (Theorem 3).

It is easy to see that for non-negative f one cannot demand a representation (1) with non-negative g's and h's: for example, this is impossible in the case of the characteristic function of the diagonal, $1 - \operatorname{sgn}|x-y|$. It is hoped to discuss the possibility of representing a measurable f with measurable g's and h's in a subsequent paper.

THEOREM 1. *If X, Y are sets of cardinality $\aleph_1$, then given any function $f\colon X \times Y \to \boldsymbol{R}$ there exist two sequences of functions*

$$g_n\colon X \to \boldsymbol{R}\,, \quad h_n\colon Y \to \boldsymbol{R} \quad (n = 1, 2, \ldots)$$

such that (1) *holds for every $(x, y) \in X \times Y$, and moreover for each $(x, y) \in X \times Y$ there are only finitely many non-zero terms in the series.*

Proof. It will be convenient to regard a sequence of real-valued functions from a set E to $\boldsymbol{R}$ as a function from $N\times E$ to $\boldsymbol{R}$, and to denote by $\sum^*$ a sum in which there are only finitely many non-zero terms. Instead of (1) we shall thus write

$$(1') \qquad f(x, y) = \sum_{n=1}^{\infty}{}^{*} g(n, x)h(n, y).$$

We may and shall suppose that X and Y are disjoint. List the elements of $X \cup Y$ as a transfinite sequence (ζ_a) of type ω_1, and for each ordinal $a \leqslant \omega_1$ let

$$Z_a = \{\zeta_\beta\colon\ 0 \leqslant \beta < a\}, \quad X_a = Z_a \cap X, \quad Y_a = Z_a \cap Y.$$

Denote by $\mathfrak{I}$ the set of all ordered pairs of real-valued functions (g, h) satisfying the following conditions: for some ordinal $a = a(g, h)$, with $0 \leqslant a \leqslant \omega_1$, we have $\operatorname{dom} g = N\times X_a$, $\operatorname{dom} h = N\times Y_a$, and (1') holds for $(x, y) \in X_a \times Y_a$, and in addition

(a) given any two disjoint finite subsets K, K' of X_a and any finite subset L of Y_a, there exist infinitely many values of n for which simultaneously

$$g(n, \xi) = 0 \ (\xi \in K), \quad g(n, \xi') = 1 \ (\xi' \in K'), \quad h(n, \eta) = 0 \ (\eta \in L),$$

(b) given any two disjoint finite subsets L, L' of Y_a and any finite subset K of X_a, there exist infinitely many values of n for which simultaneously

$$h(n, \eta) = 0 \ (\eta \in L), \quad h(n, \eta') = 1 \ (\eta' \in L'), \quad g(n, \xi) = 0 \ (\xi \in K).$$

Partially order $\mathfrak{I}$ by the relation $\precsim$ defined as follows:

$$(g, h) \precsim (g', h') \quad \text{iff} \quad g \subseteq g' \ \& \ h \subseteq h'$$

(that is: g', h' are extensions of g, h respectively). Now $(\emptyset, \emptyset) \in \mathfrak{I}$, and every totally ordered subset $\mathfrak{Q}$ of $\mathfrak{I}$ has an upper bound, namely

$$\big(\bigcup\{g\colon\ (g, h) \in \mathfrak{Q}\},\ \bigcup\{h\colon\ (g, h) \in \mathfrak{Q}\}\big),$$

and therefore by Zorn's Lemma $\mathfrak{I}$ contains a maximal element (g, h). It will now be sufficient to show that $a(g, h) = \omega_1$, since then $\operatorname{dom} g = N\times X$, $\operatorname{dom} h = N\times Y$, and (1') will hold for every $(x, y) \in X\times Y$.

Suppose if possible that $a(g, h) = a < \omega_1$: we shall show how to extend g or h. The element ζ_a belongs to X or Y; suppose the former. The sets X_a, Y_a are countable. Consequently we can list the elements of Y_a in a finite or infinite sequence

$$(2) \qquad y_1, y_2, \ldots,$$

and we can form an infinite sequence

$$(3) \qquad (K_1, K_1', L_1, L_1'),\ (K_2, K_2', L_2, L_2'),\ \ldots$$

consisting of all quadruples of which the first two members are disjoint finite subsets of X_a and the last two members are disjoint finite subsets of Y_a; each such quadruple being repeated infinitely often in the sequence (3).

Now define four infinite sequences of positive integers by induction as follows: $r_0 = 1$, and for $i = 1, 2, \ldots$

$p_i = r_{i-1}$ if y_i is undefined (i.e. if Y_a has $i-1$ or fewer elements), and otherwise p_i is the least integer $p > r_{i-1}$ for which

$$(4) \qquad h(p, y_1) = \ldots = h(p, y_{i-1}) = 0 \quad \text{and} \quad h(p, y_i) = 1;$$

q_i is the least integer $q > p_i$ for which

$$(5) \qquad g(q, \xi) = 0 \ (\xi \in K_i), \quad g(q, \xi') = 1 \ (\xi' \in K_i'), \quad \text{and} \quad h(q, \eta) = 0 \quad \text{for} \quad \eta \in \{y_1, \ldots, y_i\};$$

(6) q_i' is the least integer $q > q_i$ satisfying (5);

(7) r_i is the least integer $r > q_i'$ for which

$$h(r, \eta) = 0 \ (\eta \in L_i), \quad h(r, \eta') = 1 \ (\eta' \in L_i'), \quad g(r, \xi) = 0 \ (\xi \in K_i).$$

It is easy to verify that, because the pair (g, h) satisfies conditions (a) and (b), these integers all exist.

Define a function $g^*\colon N\times X_{a+1} \to \boldsymbol{R}$ by putting $g^*(n, x) = g(n, x)$ for $x \in X_a$, and defining $g^*(n, \zeta_a)$ by induction on n as follows:

$$(8) \qquad g^*(p_i, \zeta_a) = f(\zeta_a, y_i) - \sum_{n=1}^{p_i-1} g^*(n, \zeta_a)h(n, y_i) \quad \text{if} \quad p_i > r_{i-1},$$

$$(9) \qquad g^*(q_i, \zeta_a) = 1,$$

$$(10) \qquad g^*(n, \zeta_a) = 0 \quad \text{for all other values of } n.$$

Consider the pair (g^*, h); we shall show that it belongs to $\mathfrak{I}$, which contradicts the maximality of (g, h), since g^* is a proper extension of g. Observe that

$$\operatorname{dom} g^* = N\times (X_a \cup \{\zeta_a\}) = N\times X_{a+1}$$

and

$$\operatorname{dom} h = N\times Y_a = N\times Y_{a+1}.$$

Now we show that (1′) holds (with g replaced by g^*) for $(x, y) \in X_{\alpha+1} \times \times Y_{\alpha+1}$. Since $(g, h) \in \mathfrak{I}$, the only case to be considered is when $x = \zeta_\alpha$. Let y occur as y_k (say) in the sequence (2), and consider the sum

$$\sum_{n=1}^{\infty} g^*(n, \zeta_\alpha) h(n, y_k) . \tag{11}$$

In view of (10), we have $g^*(n, \zeta_\alpha) = 0$ unless n is of the form p_i (with $p_i > r_{i-1}$) or q_i, while by the definitions of p_i and q_i (see (4) and (5)) we also have $h(p_i, y_k) = 0$ for $i > k$, $h(p_k, y_k) = 1$, and $h(q_i, y_k) = 0$ for $i \geqslant k$. Consequently the sum (11) has only finitely many non-zero terms, and reduces to

$$\sum_{n=1}^{p_k - 1} g^*(n, \zeta_\alpha) h(n, y_k) + g^*(p_k, \zeta_\alpha) ,$$

which by (8) is equal to $f(\zeta_\alpha, y_k)$. We have thus established that (1′) holds.

Finally, we must verify conditions (a) and (b), with g replaced by g^* and α by $\alpha+1$.

Condition (a). Let K, K' be disjoint finite subsets of $X_{\alpha+1}$, and let L be a finite subset of $Y_{\alpha+1} = Y_\alpha$.

Case 1. $\zeta_\alpha \notin K \cup K'$. Then $K \cup K' \subseteq X_\alpha$, and the required infinitely many values of n exist because the pair (g, h) satisfies (a).

Case 2. $\zeta_\alpha \in K$. Then for every occurrence of $(K \setminus \{\zeta_\alpha\}, K', L, \emptyset)$ as a term (K_i, K_i', L_i, L_i') in the sequence (3), it follows from the definitions of g^* and q_i' (see (6), (10)) that

$$g^*(q_i', \xi) = 0 \ \ (\xi \in K) , \quad g^*(q_i', \xi) = g(q_i', \xi') = 1 \ \ (\xi' \in K') ,$$
$$h(q_i', \eta) = 0 \ \ (\eta \in L) ,$$

provided that i is so large that $L \subseteq \{y_1, \ldots, y_i\}$, and there are infinitely many such occurrences.

Case 3. $\zeta_\alpha \in K'$. Then for every occurrence of $(K, K' \setminus \{\zeta_\alpha\}, L, \emptyset)$ as a term (K_i, K_i', L_i, L_i') in the sequence (3), it follows from the definitions of g^* and q_i (see (5), (9)) that

$$g^*(q_i, \xi) = g(q_i, \xi) = 0 \ \ (\xi \in K) , \quad g^*(q_i, \xi') = 1 \ \ (\xi' \in K') ,$$
$$h(q_i, \eta) = 0 \ \ (\eta \in L) ,$$

provided that i is so large that $L \subseteq \{y_1, \ldots, y_i\}$, and there are infinitely many such occurrences.

Condition (b). Let K be a finite subset of $X_{\alpha+1}$ and let L, L' be disjoint finite subsets of $Y_{\alpha+1} = Y_\alpha$.

Case 1. $\zeta_\alpha \notin K$. Then $K \subseteq X_\alpha$, and the required infinitely many values of n exist because the pair (g, h) satisfies (b).

Case 2. $\zeta_\alpha \in K$. Then for every occurrence of $(K \setminus \{\zeta_\alpha\}, \emptyset, L, L')$ as a term (K_i, K_i', L_i, L_i') in the sequence (3), it follows from the definitions of g^* and r_i (see (7), (10)) that

$$h(r_i, \eta) = 0 \ \ (\eta \in L) , \quad h(r_i, \eta') = 1 \ \ (\eta' \in L') , \quad g^*(r_i, \xi) = 0 \ \ (\xi \in K) ,$$

and there are infinitely many such occurrences.

The proof that $(g^*, h) \in \mathfrak{I}$ has now been completed, and with it the proof of Theorem 1.

THEOREM 2. *The existence of a representation*

$$e^{xy} = \sum_{n=1}^{N(x,y)} g_n(x) h_n(y) , \tag{12}$$

where $N(x, y)$ is a positive integer for each point $(x, y) \in \mathbf{R}^2$, implies the continuum hypothesis.

Proof. (I owe the idea of this to P. Erdös.) Suppose if possible that there exists a representation (12) but $2^{\aleph_0} > \aleph_1$: naturally, we are assuming the axiom of choice. Let Q be a subset of $\mathbf{R}$ of cardinality $\aleph_1$. For each $x \in \mathbf{R}$ and each positive integer N, let $Q(N, x) = \{y \in Q\colon N(x, y) = N\}$; then $Q = \bigcup_N Q(N, x)$, and therefore $Q(N, x)$ is infinite for some integer $N = N(x)$; select an $(N+1)$-element subset $S(x)$ of $Q(N(x), x)$. For each non-empty finite subset S of Q, let $P(S) = \{x \in \mathbf{R}\colon S(x) = S\}$; then $\mathbf{R} = \bigcup_S P(S)$, and therefore $P(S)$ is infinite for some $S = S_0$, say. Let S_0 have cardinality $N+1$, and select an $(N+1)$-element subset R_0 of $P(S_0)$. Then for every point $(x, y) \in R_0 \times S_0$ we have

$$e^{xy} = \sum_{n=1}^{N} g_n(x) h_n(y) ,$$

and it follows ([1], [2]) that

$$\det[e^{x_i y_j}] = 0 , \quad \text{where} \quad R_0 = \{x_1, \ldots, x_{N+1}\}, \ S_0 = \{y_1, \ldots, y_{N+1}\} .$$

But ([3], p. 9) such a determinant never vanishes, and we have a contradiction.

THEOREM 3. *There exists no positive integer N such that e^{xy} admits a representation*

$$e^{xy} = \sum_{n=1}^{\infty} g_n(x) h_n(y) \tag{13}$$

with the property that for each point $(x, y) \in \mathbf{R}^2$ there are no more than N non-zero terms in the series.

Proof. Suppose if possible that there exists a representation (13) of this kind. Let N_0 be the least integer such that there exist sets $A, B \subseteq \mathbf{R}$ of cardinality $2^{\aleph_0}$, with the property that for each point $(x, y) \in A \times B$ there are no more than N_0 non-zero terms in the series in (13). Thus $1 \leqslant N_0 \leqslant N$. Take any such sets A, B, and for each N_0-element subset E of the set of positive integers let

$$P(E) = \{(x, y) \in A \times B:\ g_n(x) h_n(y) \neq 0 \text{ for } n \in E\}.$$

Given any E, let $Q(E)$ be a maximal collection of points of $P(E)$ such that no two lie on the same horizontal or vertical line. Now $|Q(E)| \leqslant N_0$, because otherwise if we select an (N_0+1)-element subset

$$\{(x_i, y_i):\ i = 1, \ldots, N_0+1\}$$

of $Q(E)$ then for each i, j we have $g_n(x_i) h_n(y_j) \neq 0$ for $n \in E$ and (since there are no more that N_0 non-zero terms in the series) $e^{x_i y_j} = \sum_{n \in E} g_n(x_i) h_n(y_j)$, whence $\det[e^{x_i y_j}] = 0$, which is impossible. Let $A_0(E)$, $B_0(E)$ be the projections of $Q(E)$ on the axes. Then $P(E) \subseteq [A_0(E) \times B] \cup [A \times B_0(E)]$, and

$$|A_0(E)| \leqslant N_0, \quad |B_0(E)| \leqslant N_0.$$

It follows that

$$\Big|\bigcup_E A_0(E)\Big| \leqslant \aleph_0, \quad \Big|\bigcup_E B_0(E)\Big| \leqslant \aleph_0,$$

and therefore $|A'| = |B'| = 2^{\aleph_0}$, where

$$A' = A \setminus \bigcup_E A_0(E), \quad B' = B \setminus \bigcup_E B_0(E).$$

But for each point $(x, y) \in A' \times B'$ there are no more than N_0-1 non-zero terms in the series (13), and this contradicts the definition of N_0.

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