Closure-preserving covers
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Abstract. Let \( X \) be a paracompact space. If \( X \) is \( \sigma \)-locally compact (\( \sigma \)-discrete), then \( X \) has a closure-preserving cover consisting of compact (resp. finite) sets. If \( X \) has a closure-preserving cover by finite sets, then \( X \) is totally paracompact. These and some more general results are established by considering order locally finite and order star-finite covers.

The present paper is concerned with covering properties of topological spaces. We shall study the properties

1. the space \( X \) has a closure-preserving cover consisting of compact subsets of \( X \),
2. the space \( X \) has a closure-preserving cover consisting of finite subsets of \( X \), and some related ones.

H. Tamano [10] put forward the question whether (1) implies the paracompactness of \( X \). This question has been answered negatively by H. B. Potoczny [7]. In paper [8] he describes the remarkable structural characteristics of spaces possessing property (1). We dealt with property (1) and its related property in [12]. Paper [14] (announced in [13]) contains several results concerning (1) and (2) established by game-theoretical methods.

The topological terminology is that of [2]. Each space is assumed to be completely regular. Natural numbers are denoted by the letters \( m, n, k, \ldots \) and ordinal numbers are denoted by the letters \( \alpha, \beta, \gamma, \ldots \).

Recall that a collection \( \{ A_i : i \in I \} \) of subsets of a space \( X \) is said to be order locally finite if we can introduce a well-ordering \( \prec \) in the index set \( I \) so that for each \( i \in I \) the family \( \{ A_j : j \prec i \} \) is locally finite at each point of \( A_i \).

Since every well-ordered set is order-isomorphic to an initial segment of ordinal numbers, we shall use the notation \( \{ A_\xi : \xi \prec \alpha \} \) instead of \( \{ A_i : i \in I \} \). Order locally finite covers were introduced and studied by Y. Katuta [3].

We say that a collection \( \{ A_\xi : \xi < \alpha \} \) of subsets of a space \( X \) is order star-finite if for each \( \xi < \alpha \) the set \( A_\xi \) meets at most finitely many \( A_\eta \) with \( \eta < \xi \).
Clearly, every order star-finite collection of open sets in X is order locally finite.

Lemma 1. Let X be a paracompact space with dim X = 0. If X has two order locally finite covers (E}; η < β) and (U}; ζ < η), where E}; is closed in X and U}; is an open nbhd of E}; for each η < β, then X has two order star-finite covers (F}; η < β) and (V}; η < β), where (F}; η < β) refines (E}; η < β) and (V}; η < β) refines (U}; ζ < η), and F}; is closed in X and V}; is an open nbhd of E}; for each η < β.

Proof. We define, by induction with respect to ζ, two discrete families (E}; i ∈ I};) and (U}; i ∈ I}); of subsets of X so that E}; = \bigcup (E}; i ∈ I}); and U}; i ∈ I}); are subsets of U}; i ⊆ X for each i ∈ I};. We set (E}; i ∈ I}); = (E}; i ∈ I}); and (U}; i ∈ I}); = (U}; i ∈ I});. Let us assume that for some ζ < β and for each η < ζ the families (E}; i ∈ I}); and (U}; i ∈ I}); are defined. For each point x ∈ E}; there is an open nbhd U};, where U}; ⊆ X and U}; meets at most finitely many U};, with η < ζ. Thus the set X = \bigcup (E}; i ∈ I}); is a family of discrete (in X), where there exists an open nbhd U}; for a, where U}; ⊆ X and U}; meets at most finitely many U};, with η < ζ. Since (V}; i ∈ I}); is an open cover of E}; and X is a paracompact space with dim X = 0, there is a discrete family (U}; i ∈ I}); of closed-open subsets of X, where (U}; i ∈ I}); covers E}; and refines (V}; i ∈ I});. We put E}; = E}; i ∈ I} and (U}; i ∈ I}); have the desired properties. Thus (E}; i ∈ I}); and (U}; i ∈ I}); are defined. Let ζ < β be a well-ordering of I} where η < ζ. Then we define a well-ordering of \{(ζ, i) : η < i < ζ and η < ζ\} as follows: (ζ, i) < (η, j) if and only if η < i < ζ or i < j. Hence the covers (E}; i ∈ I}); and (U}; i ∈ I}); are order star-finite. Thus we may write E}; = E}; i ∈ I}); and U}; = U}; i ∈ I});, where i ∈ I} where η < ζ, η < β and \{(ζ, i) : i ∈ I} and η < ζ\} is order-isomorphic to (η, η < β).

The covers (F}; η < β) and (V}; η < β) satisfy our requirements.

Remark 1. Lemma 1 will remain true if we replace X is paracompact and E}; is a locally compact closed subset of X for each η < β.

Recall that a collection \{A}; i ∈ I}) of subsets of a space X is said to be closure-preserving if for each J ⊆ I} we have cl \bigcup (A}; i ∈ J}) = \bigcup (cl A}; i ∈ J}).

Lemma 2. If X has two order star-finite covers (E}; η < β) and (U}; ζ < η), where E}; is closed in X and U}; is an open nbhd of E}; for each η < β, then there is a family \{F}; η < β\} of finite subsets of \{ζ : η < β\} so that \bigcup (F}; η < β\} is a closure-preserving cover of X.

Proof. Let (E}; η < β) and (U}; ζ < η) be the order star-finite covers. We set T}; = \{ξ\} and T}; = \{η : η < ζ\} for each η < β. Assume that T}; is defined for some n > 0. Then we set T}; = \bigcup (T}; : η < T};). It is easy to prove (by induction with respect to ζ) that each T}; is a finite set. Now let us put \bigcup (T}; : η < T};. It is also easy to prove (by induction with respect to ζ) that each T}; is a finite set. Let us put \bigcup (T}; : η < T};, where ζ < η. We shall now prove that the family (F}; η < β) is closure-preserving. Let S ⊆ \{ζ : η < β\}, let

x ∈ cl \bigcup (F}; η ∈ S})

and let \beta = \inf(\{ζ : x ∈ E}; η\}). Since x ∈ E}; ⊆ X, it follows that U}; ∩ \bigcup (F}; η ∈ S} ≠ Ø. However, \bigcup (F}; η ∈ S}) = \bigcup (F}; η ∈ T});, n > 0 and \xi ∈ S\). Thus the set R = \{η : U}; ∩ E}; ∉ U};, n > 0 and \xi ∈ S\} is non-void.

Case 1. sup R < \beta. Then R is finite, because the family (F}; η < β) is order star-finite. Hence x ∈ \bigcup (F}; η ∈ E}; ∉ U});, \xi ∈ S\). Thus x ∈ \bigcup (F}; η ∈ S}). Therefore (F}; η < β) is closure-preserving.

Lemma 3 (K. Nagami [6]). For each paracompact space X there is a paracompact space X' with dim X' = 0, so that there is a perfect map f from X' onto X.

Theorem 1. If X has two order locally finite covers (E}; η < β) and (U}; ζ < η), where E}; is compact and U}; is an open nbhd of E}; for each η < β, then X has a closure-preserving cover by compact sets.

Proof. By a theorem of Y. Katuta [3] the space under the assumption of Theorem 1 is paracompact. Thus, by Lemma 3, there is a paracompact space X' with dim X' = 0, so that there is a perfect map f from X' onto X. It is easy to verify that \{f^{-1}(E}); η < β\} and \{f^{-1}(U}); η < β\} are order locally finite covers of X', where f^{-1}(E}); is compact and f^{-1}(U}); is an open nbhd of f^{-1}(E}); for each η < β. Applying Lemma 1, we get two order star-finite covers (F}; η < β) and (V}; η < β), where F}; is compact and V}; is an open nbhd of F}; for each η < β. Applying Lemma 2, we get the family (T}; η < β) of finite subsets of \{η : η < β\} so that the family (H}; η < β), where H}; = \bigcup (F}; η ∈ T});, is closure-preserving and covers X. Since f is continuous and compact, \{f( H}); η < β\} is a closure-preserving cover of X by compact sets (see [12], Theorem 2).

Example 1. A paracompact scattered space X' having a closure-preserving cover by finite sets and having no two order locally finite
covers \( \{ E_i : \xi < a \} \) and \( \{ U_i : \xi < a \} \), where \( E_i \) is compact and \( U_i \) is an open nbhd of \( E_i \) for each \( \xi < a \).

At first we shall construct a space \( X \) by a modification of the construction which has been used by H. B. Potočanov [7]. We set

\[
X = \{ (a, b) : a < b < a_0 \}, \quad D = \{ (a, a) : a < a_0 \},
\]

and

\[
\mathcal{G}_x = \{ (a, a) \} \cup \{ (a, \xi) : a < \xi < a_0 \} \cup \{ (a, \xi) : a < \xi < a \} \quad \text{for each } a < a_0.
\]

We define a topology of \( X \) as follows: if \( (a, \beta) \in X - D \), then the singleton \( \{ (a, \beta) \} \) is a basic open set; if \( (a, a) \in D \), then \( G_a - H_a \), where \( H_a \) is a countable subset of \( X - D \) is a basic open set. It is clear that the sets described above constitute a basis for a topology of \( X \).

Every point of \( X - D \) is an isolated point and \( G_x \cap D = \{ (a, a) \} \) for each \( a < a_0 \). Therefore \( D \) is a closed discrete subset of \( X \). Thus the space \( X \) is scattered.

Every countable subset \( H \) of \( X \) is closed, because

\[
X - H = \bigcup (G_x - H : a < a_0).
\]

The basic open sets in \( X \) are simultaneously closed, because

\[
X - \{ (a, \beta) \} = \bigcup (G_x - \{ (a, \beta) \} : \gamma < a_0) \quad \text{whenever } a < \beta < a_0
\]

and

\[
X - (G_x - H_a) = H_a \cup \bigcup (G_x - \{ (a, \beta) : a < \beta < a_0 \}) \quad \text{for each } a < a_0.
\]

\( X \) is a Hausdorff space, because the basic open sets separate the points. Since each countable subset of \( X \) is closed, there is no infinite compact subset in \( X \); i.e. each compact subset of \( X \) is finite.

Since each set \( G_x \) has the Lindelöf property, \( X \) is a locally Lindelöf space.

\( \{ (a, a) : a < a_0 \} \) is a point-finite cover of \( X \), because \( G_x \cap G_y = \{ (a, \beta) \} \) where \( a < \beta < a_0 \).

For each countable subset \( H \) of \( X \) there is a closed-open Lindelöf sets \( G_x \).

\( \{ (a, a) : a < a_0 \} \) is the point-finite cover of \( X \) by the closed-open Lindelöf sets \( G_x \).

For each countable subset \( H \) of \( X \) there is a closed-open Lindelöf subset \( U \) of \( H \) such that \( H \subseteq U \) and \( U \cap D = H \cap D \). Namely, we set

\[
U = (H - D) \cup \{ (a, a) \in H \cap D \}.
\]

Since \( H - D \) is a countable set of isolated points, \( H - D \) is a closed-open set. Since \( \bigcup \{ G_x : (a, a) \in H \cap D \} \) is open, it remains to show that the set is closed. If \( (\beta, \beta) \in \partial D - H \), then

\[
G_x \cap \bigcup \{ (a, a) : (a, a) \in H \cap D \} \text{ is a countable subset of } X - D. \text{ Thus } G_x - \bigcup \{ (a, a) : (a, a) \in H \cap D \} \text{ is a basic nbhd of } (\beta, \beta).
\]

Clearly,

\[
X - \bigcup \{ (a, a) : (a, a) \in H \cap D \} = \bigcup \{ G_x - \bigcup \{ G_x : (a, a) \in H \cap D \} : (\beta, \beta) \in \partial D - H \}
\]

and hence the assertion follows.

We set \( F_{a,b} = \{ (a, a), (a, \beta), (\beta, \beta) \} \) for each \( (a, a), (a, \beta), (\beta, \beta) \in X - D \). The family \( \{ F_{a,b} : (a, a), (a, \beta), (\beta, \beta) \in X - D \} \) covers \( X \). We shall prove that the family is a closure-preserving one. Let \( S \subseteq X - D \) and let \( x \in \text{cl} \{ F_{a,b} : (a, a), (a, \beta), (\beta, \beta) \in S \} \).

If \( x \) is an isolated point of \( X \), then obviously \( x \in \bigcup \{ F_{a,b} : (a, a), (a, \beta), (\beta, \beta) \in S \} \).

Thus we may assume that \( x = (\gamma, \gamma) \in D \). Since \( G_x \) is a basic nbhd of \( (\gamma, \gamma) \), we have \( G_x \cap \bigcup \{ F_{a,b} : (a, a), (a, \beta), (\beta, \beta) \in S \} \neq 0 \). Hence \( G_x \cap F_{a,b} \neq 0 \) for some \( a < \beta < a_0 \). Since \( F_{a,b} \) is a three-point set, we shall consider three cases.

Case 1. \( (a, a) \in G_x \). Then \( a \in S \) and so \( (\gamma, \gamma) \in F_{a,b} \).

Case 2. \( (a, \beta) \in G_x \). Since \( a < \beta \), we have \( a = \gamma \). Thus \( (\gamma, \gamma) \in F_{a,b} \).

Case 3. \( (\beta, \beta) \in G_x \). Then \( \beta = \gamma \) and so \( (\gamma, \gamma) \in F_{a,b} \).

The space \( X \) is not normal, because the sets \( \{ (a, a) : a < a_0 \} \) and \( \{ (a, a) : a_0 < a < a_0 \} \) cannot be separated by open sets. In order to prove this, let \( U \) be an open nbhd of \( \{ (a, a) : a < a_0 \} \) in \( X \). For each \( a < a_0 \) there is a basic open set \( U_a \) with \( (a, a) \in U_a \subseteq U \). For each \( a < a_0 \) we have \( U_a = G_x - H_a \), where \( H_a \) is a countable subset of \( X - D \). Let us set

\[
S_1 = \{ (a, a) : (a, a) \in U_a \text{ for some } a < a_0 \text{ and some } a < a_0 \},
\]

\[
S_2 = \{ (a, b) : (b, b) \in U_b \text{ for some } b < a_0 \text{ and some } a < a_0 \}
\]

and

\[
S = \{ (a, \xi) : a < a_0 \} \cup S_1 \cup S_2.
\]

Then \( \text{cl} S \subseteq S_1 \). Hence there is an ordinal \( \eta < a_0 \) for which \( \text{cl} S \subset \eta \).

Clearly \( (\gamma, \gamma) \notin \{ (a, a) : a < \eta_0 \} \). Since \( \gamma \neq \eta \), we have \( (a, a) \notin H \) for each \( a < \eta \). Thus \( (a, a) \notin U_a \) for each \( a < \eta \). Finally, we shall show that \( (\gamma, \gamma) \in \text{cl} U \). Let \( U_0 = G_x - H_0 \) be any basic nbhd of \( (\gamma, \gamma) \).

Since \( H_0 \) is countable and \( \{ (a, a) : a < a_0 \} \) is an uncountable subset of \( G_x \), there exists an \( a < a_0 \) with \( (a, a) \in U_0 \). Thus \( U_1 \cup U_2 \neq 0 \). It follows that \( (\gamma, \gamma) \notin \text{cl} U \).

We define \( X^* \) as a one-point extension \( X \cup \{ p \} \) of \( X \) as follows:

\( U \) is a basic open nbhd of \( x \neq p \) in \( X^* \) if and only if \( U \) is a basic open nbhd of \( x \in X \); \( U \) is a basic open nbhd of \( p \) in \( X^* \) if and only if \( X^* - U \) is a closed-open Lindelöf set in \( X \).

It follows from the definition of \( X^* \) that the basic open sets are simultaneously closed and they separate points. Thus \( X^* \) is a regular space.
Let $U$ be a basic open set in $X^*$. Then $U$ has the Lindelöf property if and only if if $p \not\in U; X^* - U$ has the Lindelöf property if and only if if $p \in U$. Thus $X^*$ is a Lindelöf space. Since $X^*$ is a regular Lindelöf space, it is paracompact (see [2], p. 211).

Each countable subset of $X^*$ is closed. To prove this, let $H$ be a countable subset of $X^*$. Since $H - (p)$ is a countable subset of $X$, there exists a closed-open Lindelöf nbhd $U$ of $H - (p)$ in $X$. Since $H - (p)$ is closed in $U$ and $U$ is closed in $X^*$, the set $H - (p)$ is closed in $X^*$. Thus $H = (p) \cup (H - (p))$ is closed in $X^*$.

Since each countable subset of $X^*$ is closed, there is no infinite compact set in $X^*$. I.e. each compact subset of $X^*$ is a finite set.

Since $(F_{x}; (a, b) \times D) = d$ is a closure-preserving cover of $X$, the set $(p) \cup (F_{x}; (a, b) \times D)$ is a closure-preserving cover of $X^*$ for point sets.

Finally, let us suppose that $X^*$ has two order locally finite covers $(E_{x}; \xi < a)$ and $(U_{x}; \xi < a)$ where $E_{x}$ is compact and $U_{x}$ is an open nbhd of $E_{x}$ in $X^*$ for each $\xi < a$. Since $E_{x}$ is finite, $E_{x} - (p)$ is compact. Hence the space $X$ has two order locally finite covers $(E_{x}; \xi < a) - (p); \xi < a)$ and $(U_{x}; \xi < a) - (p); \xi < a)$ where $E_{x} - (p)$ is compact and $U_{x} - (p)$ is an open nbhd of $E_{x} - (p)$ in $X$ for each $\xi < a$. Thus $X$ is paracompact by a theorem of Y. Katuta (3). This is a contradiction, because $X$ is not normal (see [2], p. 207).

Recall that $X$ is said to be $\sigma$-locally compact if $X$ has a countable cover $(X_{n}; n \geq 0)$ where each $X_{n}$ is a locally compact closed subset of $X$. These spaces were studied by K. Morita [5] and A. H. Stone [9].

**Lemma 4.** (Y. Katuta [3]). If $X$ is a paracompact and $\sigma$-locally compact, then $X$ has two order locally finite covers $(E_{x}; \xi < a)$ and $(U_{x}; \xi < a)$ where $E_{x}$ is compact and $U_{x}$ is an open nbhd of $E_{x}$ in $X$ for each $\xi < a$.

**Theorem 2.** If $X$ is paracompact and $\sigma$-locally compact, then $X$ has a closure-preserving cover by compact sets.

**Proof.** By Lemma 4 the space $X$ has two order locally finite covers $(E_{x}; \xi < a)$ and $(U_{x}; \xi < a)$ where $E_{x}$ is compact and $U_{x}$ is an open nbhd of $E_{x}$ for each $\xi < a$. Thus, by Theorem 1, the space $X$ has a closure-preserving cover by compact sets.

Recall that $X$ is said to be $\sigma$-discrete if $X$ has a countable cover $(X_{n}; n \geq 0)$ where each $X_{n}$ is a discrete closed subset of $X$.

**Example 2.** A $\sigma$-discrete, locally compact, scattered, non-normal space $X$ with $\text{ind} X = 0$ that has no closure-preserving cover by compact sets. As an example of the space $X$ it suffices to take the space defined in (5), p. 167. This space $X$ is zero-dimensional, locally compact, $\sigma$-discrete, separable and contains a closed discrete uncountable subset $P; X - P$ is countable, isolated and dense in $X$.

**Lemma 5.** Each closure-preserving family of pairwise disjoint closed sets is discrete.

The proof is immediate.

Recall that $X$ is said to be $G$-scattered if for each non-void closed subset $B$ of $X$ there is a point $x \in B$ and an open nbhd $U$ of $x$ for which $B \cap \text{cl} U$ is compact. $G$-scattered spaces were studied in [11].

**Theorem 3.** Let $X$ be a paracompact space. If $X$ has two order locally finite covers $(E_{x}; \xi < a)$ and $(U_{x}; \xi < a)$ where $E_{x}$ is a $G$-scattered closed subset of $X$ and $U_{x}$ is an open nbhd of $E_{x}$ for each $\xi < a$, then $X$ has a countable cover $(X_{n}; n \geq 0)$ where $X_{n}$ is a $G$-scattered closed subset of $X$ for each $n \geq 0$.

**Proof.** By Lemma 3 there is a paracompact space $X'$ with dim $X' = 0$ and a perfect map $f$ from $X'$ onto $X$. The set $E_{x}' = f^{-1}(E_{x})$ is $G$-scattered (see [11], Theorem 1.3) and closed in $X'$ and $U_{x}' = f^{-1}(U_{x})$ is an open nbhd of $E_{x}'$ in $X'$ for each $\xi < a$. It is easy to check that $(E_{x}; \xi < a)$ and $(U_{x}; \xi < a)$ are order locally finite covers of $X'$. By Lemma 1 there exist two order star-finite covers $(F_{x}; \eta < \beta)$ and $(Y_{x}; \eta < \beta)$ where $(F_{x}; \eta < \beta)$ refines $(E_{x}; \xi < a)$, $(F_{x}; \eta < \beta)$ refines $(U_{x}; \xi < a)$, and $F_{x}$ is closed in $X'$ and $Y_{x}$ is an open nbhd of $F_{x}$ for each $\eta < \beta$. Since $F_{x}$ is a closed subset of some $E_{x}'$, it is $G$-scattered. By Lemma 2 there is a collection $(Z_{x}; \eta < \beta)$ of finite subsets of $(\eta; \eta < \beta)$ such that $(H_{x}; \eta < \beta)$, where $H_{x} = \bigcup (Z_{x}; \eta < \beta)$, is a closure-preserving cover of $X'$. Since $H_{x}$ is the union of a finite family of $G$-scattered sets, it is also $G$-scattered (see [11], Theorem 1.1). Let us remark that card $T_{x} \geq 1$, because $\eta \in T_{x}$. For each $n \geq 0$ we set $A_{n} = \{\eta < \beta: \text{card} T_{x} < n\}$, $X_{n}' = \bigcup (H_{x}; \eta \in A_{n})$ and $X'_{n} = f(X_{n}')$.

Since $\bigcup \{A_{n}; n \geq 0\} = \{\eta; \eta < \beta\}$, it follows that $\bigcup (X_{n}' ; n \geq 0) = X'$. Thus $\bigcup (X_{n}' ; n \geq 0) = X$. Since the family $(H_{x}; \eta < \beta)$ is closure-preserving and each $H_{x}$ is closed, the set $X_{n}'$ is closed for each $n \geq 0$. Thus each $X_{n}'$ is closed, because $f$ is a closed map. By Theorem 1.3 of [11] $X_{n}'$ is $G$-scattered if and only if $X'_{n}$ is $G$-scattered. Now we prove (by induction with respect to $n$) that $X'_{n}$ is $G$-scattered.

$X'_{1} = \bigcup (H_{x}; \eta \in A_{1}) = \bigcup (F_{x}; \eta \in A_{1})$.

The family $(F_{x}; \eta \in A_{1})$ is closure-preserving and its members are pairwise disjoint. Therefore, by Lemma 5, it is a discrete family. Hence $X_{1}'$ is locally $G$-scattered and therefore $G$-scattered. Assume that for some $n \geq 0$ the set $X_{n}'$ is $G$-scattered. Let us set $R_{n+1} = \{\eta < \beta: \text{card} T_{x} = n + 1\}$. Then $A_{n+1} = A_{n} \cup R_{n+1}$, and let us set $Y_{n+1} = \bigcup (F_{x}; \eta \in A_{n+1})$ and $X_{n+1}' = \bigcup (F_{x}; \eta \in T_{x} - \{\xi\} \text{ and } \xi \in R_{n+1})$.

Then $X_{n+1}' = X_{n}' \cup Y_{n+1}'$. We claim that $Z_{x}' \subseteq X_{n}'$. If $\eta \in T_{x} - \{\xi\}$ and $\xi \in R_{n+1}$, then $\eta < \xi$. Hence $T_{x} \subseteq T_{x}'$ and $\text{card} T_{x}' < \text{card} T_{x} = n + 1$. Thus
η ∈ A_n and F_n ⊆ H_n ⊆ X_n. We claim that Y_n is C-scattered. Since F_n with η ∈ H_n is C-scattered, it suffices to prove that the family \((V_{ij}; \ i ∈ I_{ij})\) consists of pairwise disjoint sets. Let us suppose that we have two \(i\) and \(j\) in \(I_{ij}\) with \(i < j\) and \(V_i ∩ V_j \neq ∅\). Then \(T_i ⊆ T_j\) and so card \(T_j > n + 2\). This is a contradiction. Hence \(Y_n\) is C-scattered. Since \(X_n = \cup Y_n\) is the union of C-scattered sets, \(X_n\) is also C-scattered (see [11], Theorem 1.1).

Remark 2. If \(\{E_n; n ≥ 0\}\) is a countable cover of \(X\) with \(E_n\) closed, then \(X\) has two order locally finite covers \(\{E_n; n ≥ 0\}\) and \(\{U_n; n > 0\}\) where \(U_n = X\setminus E_n\) for each \(n > 0\).

Remark 3. Theorem 3 is surprising if we look at Theorem 2.5 of [11]; roughly speaking, the countable covers are sufficient to define the class of spaces.

As a corollary of Theorem 3 we have

Theorem 4. If \(X\) has two order locally finite covers \(\{B_i; i < α\}\) and \(\{V_i; i < α\}\) where \(B_i\) is compact and \(U_i = X\setminus B_i\) is an open nbhd of \(B_i\) for each \(i < α\), then \(X\) has a countable cover \(\{X_n; n ≥ 0\}\) where \(X_n\) is a C-scattered closed subset of \(X\) for each \(n ≥ 0\).

Theorem 5. Each paracompact \(α\)-discrete space has a closure-preserving cover by finite sets.

Proof. If \(X\) is paracompact and \(α\)-discrete, then \(\dim X = 0\) (see [2], p. 214). \(X\) is \(α\)-discrete, i.e., \(X = \cup \{X_n; n ≥ 0\}\) where each \(X_n\) is a discrete closed subset of \(X\). Hence we may assume, without loss of generality, that \(X_n \cap X_{m} = ∅\) for \(m ≠ n\). There is a well-ordering \(\{x_1 < x_2 < \ldots\}\) of \(X\) where \(x < x'\) holds whenever \(x < x'\) holds for \(i < j\) in \(X\). Let us put \(E_i = \{x_i\}\) for each \(x_i < x\). The family \(\{E_i; x_i \in X\}\) is discrete for each \(n ≥ 0\). Since \(X\) is \(α\)-discrete, it is collectionwise normal (see [2], p. 214). Thus there is a \(\alpha\)-discrete family \(\{U_i; x_i \in X\}\) of open sets in \(X\) with \(U_j \subseteq X_j\). It is easy to check that the covers \(\{U_i; x_i < x\}\) and \(\{U_i; x < x\}\) of \(X\) are order locally finite. By Lemma 1 we get two order star-finite covers \(\{F_1, \ldots, F_n\; \eta < β\}\) and \(\{V_1, \ldots, V_n\; \eta < β\}\) of \(X\) where \(F_1, \ldots, F_n\) contains at most one point and \(V_1, \ldots, V_n\) is an open nbhd of \(F_i\) for each \(i < β\). By Lemma 2 there is a family \(\{T_1, \ldots, T_n\; \eta < β\}\) of finite subsets of \(\{η; \eta < β\}\) for which \((\bigcup \{F_i; x_i < x\}\; \eta < β\) is a closure-preserving cover of \(X\). Thus \(X\) has a closure-preserving cover by finite sets.

Lemma 6. For any family \(\{E_i; i ∈ I\}\) of non-void sets there exists a \(J ⊆ I\) for which the family \(\{E_i; j ∈ J\}\) is pairwise disjoint and where for each \(i ∈ I\) there exists a \(j ∈ J\) with \(E_i \cap E_j = ∅\).

The proof is a standard application of the Kuratowski–Zorn Lemma.

Theorem 6. If \(X\) has a closure-preserving cover by finite sets, then \(X\) has a countable cover \(\{X_n; n ≥ 0\}\) where \(X_n\) is a scattered closed subset of \(X\) and it is the union of \(n\) discrete (not necessarily closed) subsets for each \(n ≥ 0\).

Proof. Let \(\{E_i; i ∈ I\}\) be a closure-preserving cover of \(X\), where \(E_i\) is finite for each \(i ∈ I\). Let us set \(I_0 = \{i ∈ I; card E_i < n\}\) and \(X_n = \bigcup \{E_i; i ∈ I_0\}\) for each \(n ≥ 0\). Then \(X_n\) is a closed subset of \(X\), because \(\{E_i; i ∈ I\}\) is a closure-preserving family of closed subsets of \(X\). Now it suffices to prove the following auxiliary statement \(S(n)\): If \(X\) has a closure-preserving cover \(\{E_i; i ∈ I\}\) with \(card E_i < n\) for each \(i ∈ I\), then \(X\) is scattered and it is the union of \(n\) discrete subsets. Clearly, \(S(0)\) holds. By Lemma 5, \(S(1)\) holds. Assume that \(S(n)\) holds for some \(n ≥ 0\). We shall prove that \(S(n+1)\) also holds. So let \(X\) be a space with a closure-preserving cover \(\{E_i; i ∈ I\}\) where \(card E_i < n+1\) for each \(i ∈ I\). We may assume, without loss of generality, that \(E_i \neq \emptyset\) for each \(i ∈ I\). By Lemma 6 there exists a \(J ⊆ I\) for which the family \(\{E_j; j ∈ J\}\) is constituted by pairwise disjoint sets and where for each \(i ∈ I\) there exists a \(j ∈ J\) with \(E_i \cap E_j = ∅\). Let us put \(Y = \bigcup \{E_j; j ∈ J\}\). Since \(\{E_i; i ∈ I\}\) is a closure-preserving family of closed sets, \(Y\) is a closed subset of \(X\). By Lemma 5 the family \(\{E_j; j ∈ J\}\) is discrete. Thus \(Y\) is a discrete closed subset of \(X\). Let us fix \(i ∈ I\). Since \(E_i \cap Y = ∅\), \(card (E_i \setminus Y) < n\). Thus \(X \setminus Y\) has the closure-preserving cover \(\{E_i \setminus Y; i ∈ I\}\) with \(card (E_i \setminus Y) < n\) for each \(i ∈ I\). Hence, by \(S(n)\), \(X \setminus Y\) is a scattered space and it is the union of \(n\) discrete subsets. Thus \(X = X \setminus Y \cup Y\) is also scattered, because the union of two scattered spaces is a scattered space (see [4], p. 79). Clearly, \(X\) is the union of \(n+1\) discrete subsets.

Problem 1. Let \(X\) be a paracompact space with a closure-preserving cover by compact sets. Does \(X\) have a countable cover by \(C\)-scattered closed subsets?

Remark 4. We may consider Theorem 4 (according to Theorem 1) and Theorem 6 as partial solutions of Problem 1.

Example 3. A compact scattered space \(X\) that has no closure-preserving cover by finite sets. We take \(X = [\tau; \tau < α]\) endowed with the order (interval) topology. \(X\) is a compact scattered space. Assume that \(\{E_i; i ∈ I\}\) is a cover of \(X\) by finite sets. The set \(\{i ∈ I; card E_i = 1\}\) is finite, because \(X\) is compact. Let us set \(I' = \{i ∈ I; card E_i > 1\}\). We define \(f: I' → X\) as follows: \(f(i) = max(α < α' < α; E_i ∩ E_{α'} = ∅)\). Clearly, there is a sequence \(\{i_n; n ≥ 0\}\) ⊆ \(I'\) for which \(f(i_n) = f(i_{n+1}) < f(i_{n+2}) < \ldots\). Let \(\alpha = sup(\{f(i_n); n ≥ 0\}\). Then \(\alpha ε \{E_i; \alpha ≥ 0\}\) and \(\alpha ∈ \bigcup \{E_i; \alpha ≥ 0\}\). Hence \(\{E_i; i ∈ I\}\) is not closure-preserving.

Remark 5. Let \(X\) be the Aleksandrov compactification of the space defined in Example 3. Then \(X\) is another example of a compact scattered space that has no closure-preserving cover by finite sets.
Recall that $X$ is said to be totall para-compact if each open basis of $X$ contains a locally finite cover of $X$.

**Theorem 7.** If $X$ is para-compact and if $X$ has a closure-preserving cover by finite sets, then $X$ is totally para-compact.

**Proof.** By Theorem 6 the space $X$ has a countable cover by its scattered closed subsets. By Theorem 3.1 of [11] each para-compact scattered space is absolutely para-compact. By Theorem 1.7 of D. Curtis [1] a para-compact space having a countable cover by its closed absolutely para-compact subsets is totally para-compact.

**Theorem 8.** If $X$ has two order locally finite covers $\{E_i; \xi < \alpha\}$ and $\{U_i; \xi < \alpha\}$ where $E_i$ is compact and $U_i$ is an open nbhd of $E_i$ for each $\xi < \alpha$, then $X$ is totally para-compact.

**Proof.** By a theorem of Y. Katuta [3] the space $X$ is para-compact. By Theorem 4 the space $X$ has a countable cover by its $\xi$-covered closed subsets. By Theorem 3.1 of [11] and Theorem 1.7 of [1] the space $X$ is totally para-compact if it is para-compact and if it has a countable cover by its $\xi$-covered closed subsets.

**Problem 2.** Let $X$ be a para-compact space with a closure-preserving cover by compact sets. Is $X$ totally para-compact?

**Remark 6.** Theorem 7 and Theorem 8 (according to Theorem 1) can be considered as partial solutions of Problem 2.

Recall that a family $\{E_i; i \in I\}$ is said to be $\sigma$-closure-preserving if $I$ is the union of such a countable family $\{I_n; n \geq 0\}$ that $\{E_i; i \in I_n\}$ is closure-preserving for each $n \geq 0$.

**Example 4.** A para-compact scattered space $X$ for which the following holds: (a) $X$ has no closure-preserving cover by compact sets, but (b) $X$ has a $\sigma$-closure-preserving cover by compact sets. $X$ is the space $S_3$ defined in [11], p. 71. This space $X$ is para-compact (even Lindelöf) and scattered. It is uncountable and has a countable dense open subset $N$. Each compact subset of $X$ is finite. Hence (a) follows. The family $\{(p_n, \mathcal{P}_n); n < \omega\}$ is closure-preserving and covers $X - N$. Obviously, $Y$ has a cover by singletons. Hence (b) follows.

**Remark 7.** Let $X$ be the Aleksandrov compactification of the space defined in Example 2. Then $X$ is an example of a compact space satisfying (a) and (b).

**References**