that \( q_i \) isn’t accessible in \( D_1 \cap \bar{S} \subseteq g(f)(N) \), so that \( q_i \) isn’t an accessible point of \( S \). On the other hand, if \( x \in (-1, 1) - E \), then there is an infinite sequence \( \{q_i \mid w > 0 \} \) of non-zero integers such that \( x \) is in the closure of each \( D_{q_i} \), for \( f_k = \{q_k \mid k \leq n \} \). In this case, \( a = \bigcup \{f_d(q_d) \mid n \in e \} \) is an arc from \( \langle 0, 1 \rangle \) to \( \infty \) contained in \( S \).

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References


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Semigroups which admit few embeddings

by

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Abstract. \( S(X) \) is the semigroup, under composition, of all continuous selfmaps of the topological space \( X \). Two classes of spaces are given such that if \( X \) is from the first and \( Y \) is from the second and \( f \) is any isomorphism from \( S(X) \) into \( S(Y) \), then there is a unique idempotent \( e \) of \( S(Y) \) and a unique homeomorphism \( h \) from the range of \( e \) onto \( X \) such that \( f(f) = h \cdot f \cdot h^{-1} \circ e \) for each \( f \) in \( S(X) \). It follows from this that there is a fairly extensive class of spaces such that the semigroup of precisely three spaces from the class can be embedded in \( S(I) \) and the semigroups of precisely five can be embedded in \( S(B) \) where \( I \) and \( B \) denote respectively the closed unit interval and the space of real numbers.

1. Introduction. The symbol \( S(X) \) is used to denote the semigroup, under composition, of all continuous selfmaps of the topological space \( X \). It is well known that there exist semigroups \( S(X) \) into which many other such semigroups may be embedded. In fact, given any collection of semigroups, one need only choose a set \( X \) whose cardinality is not less than that of any of the semigroups and then each semigroup of the collection can be embedded in \( S(X) \) where \( X \) is given the discrete topology. In this case, \( S(X) \) is, of course, simply the full transformation semigroup on \( X \). The problem is made a bit more difficult by requiring that \( X \) satisfy various topological conditions and when we discuss some examples, we will see that for each collection of semigroups, one can produce an arcwise connected metric space \( X \) so that each semigroup of the collection can be embedded in \( S(X) \). However, such semigroups are really not our main concern here. We are much more interested in semigroups at the other end of the spectrum, that is, in semigroups of continuous functions into which very few other such semigroups can be embedded.

The main theorem of the paper is proven in section 4 and it gives two classes of spaces such that if \( X \) is from the first and \( Y \) is from the second, then for each homomorphism \( f \) from \( S(X) \) into \( S(Y) \), there exists a unique idempotent \( e \) of \( S(Y) \) and a unique homeomorphism \( h \) from \( X \) onto the range of \( e \) such that \( f(f) = h \cdot f \cdot h^{-1} \circ e \) for each \( f \) in \( S(X) \). We then look at some special cases in more detail and to give some idea of the type of result we get, we mention the essential ingredients of a result on \( S(I) \) and one on \( S(B) \) where \( I \) is the closed unit interval and
$E$ is the space of real numbers. Basically, we produce a fairly extensive class of spaces such that for any space $X$ in the class, $S(X)$ can be embedded in $S(I)$ if and only if $X$ is homeomorphic to either $I$, the two-point discrete space or the one-point discrete space. Moreover, $S(X)$ can be embedded in $S(R)$ if and only if $X$ is homeomorphic to either $E$, $I$, a half-open interval, the two-point discrete space or the one-point discrete space. In other words, there exists a fairly extensive class of semigroups of continuous selfmaps such that from this entire class, precisely three of the semigroups can be embedded in $S(I)$ and precisely five can be embedded in $S(R)$.

2. Some definitions and preliminary results.

**Definition (3.1).** A topological space $X$ is an $S^t$-space if it is $T_1$ and for each closed subset $H$ of $X$ and each point $p \in X - H$, there is a continuous selfmap $f$ of $X$ and a point $q \in X$ such that $f(x) = q$ for $x \in H$ and $f(p) \neq q$.

**Definition (3.2).** A space $X$ is a strong $S^t$-space if it is $T_1$ and for a pair of disjoint closed subsets $A$ and $B$ of $X$, there exists a continuous selfmap $f$ of $X$ and distinct points $p$ and $q$ of $X$ such that $f(x) = p$ for $x \in A$ and $f(x) = q$ for $x \in B$.

$S^t$-spaces were introduced in [4] and strong $S^t$-spaces in [3]. It is observed in these papers that any completely regular Hausdorff space which contains an arc is an $S^t$-space and a normal Hausdorff space which contains an arc is a strong $S^t$-space.

In this paper, when we say a space is 0-dimensional, we mean simply that it is Hausdorff and has a basis of sets which are both closed and open. By a Lebesgue 0-dimensional space, we mean a Hausdorff space in which every finite open cover has a refinement by a partition of the space onto open sets. This agrees with the definition given in [1, p. 246] if one considers only normal spaces. It is well known that a Lebesgue 0-dimensional space is 0-dimensional but that a 0-dimensional space need not be a Lebesgue 0-dimensional. However, a 0-dimensional Lindelöf space is Lebesgue 0-dimensional [1, Theorem 16.17, p. 247].

**Proposition (2.3).** Every Lebesgue 0-dimensional space is a strong $S^t$-space.

**Proof.** Let $A$ and $B$ be disjoint closed subsets of the Lebesgue 0-dimensional space $X$. Then $\{X - A, (X - B)\}$ is an open cover of $X$ and hence there is a partition $\{V_i \}$ which refines it. Define $W = \bigcup \{V_i : V_i \subset X - A\}$.

The $W$ is both closed and open, $A \subset W$ and $B \subset W$. Select two distinct points $p$ and $q$ of $X$ and define a selfmap $f$ of $X$ by $f(x) = p$ for $x \in W$ and $f(x) = q$ for $x \in X - W$. The function $f$ is continuous and it follows that $X$ is a strong $S^t$-space.

**Definition (3.4).** A topological space $X$ is said to be strongly conformable if it is a first countable strong $S^t$-space and for each pair of compact, countable subspaces $A$ and $B$ of $X$ having exactly one limit point, there exists a continuous selfmap $f$ of $X$ mapping $A$ into $B$ such that $B - f(A)$ is finite.

If, in the latter definition, one replaces the requirement that $X$ be a strong $S^t$-space by the requirement that $X$ be merely an $S^t$-space, then one has the definition of a conformable space which was introduced in [6, Definition (3.3)]. The proof of the following result is essentially a combination of the techniques used in the proofs of Theorems (3.4) and (3.5) of [6] and will not be given.

**Proposition (2.5).** All locally Euclidean normal spaces and all Lebesgue 0-dimensional metric spaces are strongly conformable.

As in [6], we regard a space as being locally Euclidean if it is Hausdorff and each point has a neighborhood which is homeomorphic to some Euclidean $N$-space and there is no requirement that all of these neighborhoods have the same dimension.

Now let us recall that a space $X$ is said to be homogeneous if for each pair of points $p$ and $q$ in $X$, there exists a homeomorphism $h$ from $X$ onto $X$ such that $h(p) = q$. We weaken this requirement considerably in our next definition.

**Definition (3.6).** A topological space $X$ is said to be quasi-homogeneous if for each nonempty open set $G$ of $X$ and each point $p$ in $X$, there exist continuous selfmaps $f$ and $g$ of $X$ such that $g(p) \in G$ and $f \circ g = \text{id}$. $X$.

The next result gives a simple sufficient condition that a space be quasi-homogeneous. Before stating it, let us define a space $X$ to be an absolute retract of $G$ if $X$ is Hausdorff normal and for each closed subset $A$ of a normal space $Y$, each continuous map from $A$ into $X$ can be extended to a continuous map from $Y$ into $X$.

**Proposition (2.7).** Let $X$ be any absolute retract with the property that each nonempty open subset contains a copy of $X$ which is closed in $X$. Then $X$ is quasi-homogeneous.

**Proof.** Let $G$ be any nonempty open subset of $X$ and let $p$ be any point of $X$. By hypothesis, there exists a homeomorphism $g$ from $X$ onto a subspace $H$ of $G$ which is closed in $X$. Now any absolute retract is necessarily normal so the function $g^{-1}$ which maps $H$ continuously into $X$
has a continuous extension $f$ which maps $X$ into $X$. Evidently, both $f$ and $g$ belong to $S(X)$ and $f \circ g$ is the identity on $X$.

Since for each positive integer $N$, the closed unit ball $B^N$ in Euclidean $N$-space $E^N$ satisfies the hypothesis of the latter result, we immediately get

**Corollary (2.8).** Each closed unit ball $B^N$ is quasi-homogeneous.

Of course each $B^N$ is also quasi-homogeneous since it is homogeneous. The spaces $B^N$ are examples of quasi-homogeneous spaces which are not homogeneous.

**Proposition (2.9).** Any product of quasi-homogeneous spaces is quasi-homogeneous.

**Proof.** Let $X = \prod \{Y_a: a \in A\}$ where each $Y_a$ is quasi-homogeneous.

Let $P$ be any point of $X$ and $G$ any nonempty open subset of $X$. Then $G$ contains a basic open set of the form

$$P_{X1}^{-1}(H_1) \cap P_{X2}^{-1}(H_2) \cap \cdots \cap P_{XN}^{-1}(H_N),$$

where $P_i$ is the projection map from $X$ onto $Y_i$ and $H_i$ is a nonempty open subset of $Y_i$. Then there are continuous selfmaps $f_j$ and $g_j$ of $Y_j$ such that $g_j(p_j) = g_j(P_j(p_j)) \in H_i$ and $f_j \circ g_j = \iota_j$, the identity map on $Y_j$. Now define selfmaps $f$ and $g$ of $X$ by

$$
\begin{align*}
(f(t))_a &= f_j(t_j) & \text{for} & & j = 1, 2, \ldots, N, \\
(f(t))_a &= \iota & \text{for} & & a \neq 1, 2, \ldots, N, \\
g(t)_a &= g_j(t_j) & \text{for} & & j = 1, 2, \ldots, N, \\
g(t)_a &= \iota & \text{for} & & a \neq 1, 2, \ldots, N.
\end{align*}
$$

Then $f$ and $g$ are continuous selfmaps of $X$ such that $f \circ g$ is the identity on $X$ and $g(p) \in G$. Consequently, $X$ is quasi-homogeneous.

By a retract of a space $X$, we simply mean any subspace $Y$ of $X$ which is the range of an idempotent continuous selfmap of $X$.

**Definition (2.10).** A topological space $X$ is a spray if it is Hausdorff, connected, first countable and, in addition satisfies the following three conditions:

(2.10.1) A discrete subspace can be at most countable.

(2.10.2) Each nondegenerate connected subset has nonempty interior.

(2.10.3) Let $\mathcal{A}_d: \delta \in d$ be an uncountable collection of retracts of $X$ such that each has more than one point. Then there is at least one whose boundary intersects the interior (with respect to $X$) of another.

Conditions (2.10.2) and (2.10.3) are really quite stringent. For example, if $X$ and $Y$ are two connected $T_i$ spaces each with more than one point, then $X \times Y$ will not satisfy (2.10.2). Merely choose any $a \in X$ and let

$$A_a = \{(a, y): y \in Y\}.$$

Then $A_a$ is a connected subset of $X \times Y$ but it has empty interior. As for (2.10.3) suppose that $X$ is Hausdorff and has uncountably many points and that $Y$ is also Hausdorff and has at least two points. Now $A_a$ as defined above is a retract of $X \times Y$ and since $X \times Y$ is Hausdorff, it is closed. Thus, it contains its boundary. Since $\mathcal{A}_d: a \in X$ is mutually disjoint and uncountable, it follows that (2.10.3) is not satisfied.

Thus, it follows for one reason or another that very few products of spaces are sprays. In the next section, however, we will see a number of examples of sprays. In fact, we will look at a method for constructing various examples which will be useful to us.

3. A method for constructing examples. Let $\{(Y_a, p_a): a \in A\}$ be a collection of ordered pairs where, for each $a \in A$, $Y_a$ is a metric space with metric $d_a$ and $p_a$ is a point of $Y_a$. For purposes of discussion, it will be convenient to assume that all of these spaces are mutually disjoint although this is not really necessary and one will easily see the appropriate modifications to make when they are not mutually disjoint. Choose some point $q$ which is not in any of the $Y_a$ and let

$$X = \left(\bigcup \{Y_a - \{p_a\}: a \in A\}\right) \cup \{q\}.$$

Define a metric $d$ on $X$ as follows:

$$
\begin{align*}
d(q, y) &= d_q(p_a, y) & \text{when} & & y \in Y_a, \\
d(x, y) &= d_x(p_a, y) & \text{when} & & x, y \in Y_a, \\
d(x, y) &= d_x(p_a) + d_y(p_a, y) & \text{when} & & x \in Y_a \text{ and } y \in Y_b.
\end{align*}
$$

One shows in a routine manner that $d$ actually is a metric on $X$.

**Definition (3.1).** The space $X$ with the metric $d$ is referred to as the **bounded union** of the pairs $\{(Y_a, p_a): a \in A\}$. For each $a$, the point $p_a$ is referred to as the **bonding point** of $Y_a$ and the point $q$ in $X$ is referred to as the **exceptional point** of $X$.

If the family happens to be finite in number, the bonded union is nothing more than the quotient space obtained by first taking the free union of the spaces in the family and then identifying the bonding points. This is not necessarily true, however, when the family is infinite. For example, let

$$L_n = \{(x, y) \in R \times R: y = x/n \text{ and } 0 \leq x \leq 1\}.$$
where \( n \) is a positive integer and let
\[
I_n = \bigcup \{ I_m : m \leq n \}
\]
where the topology on \( I_n \) is that which it inherits from the Euclidean plane. One shows easily that \( I_n \) is homeomorphic to the bounded union of a countably infinite number of copies of the closed unit interval \( I \) where for each copy, the bonding point is one of the endpoints. This differs from the space \( I_n \) which is the free union of the spaces \( I_m \) with the origins identified, for the set
\[
\bigcup \{ (x, y) : y = x/n \text{ and } 0 \leq x < 1/n \}
\]
is open in \( I_n \) but not in \( I_n \).

The construction of \( I_n \) can be generalized by letting \( a \) be any cardinal number and taking the bonded union of \( a \) copies of \( I \) where for each copy, the bonding point is one of the endpoints. We do the same sort of thing with the space \( J \) of non-negative real numbers and the unit circle \( C \) of the Euclidean plane. We gather all this together in the following

**Definition (3.2).** Let \( a \) be any cardinal number. Then

(3.2.1) \( I_a \) denotes the bonded union of \( a \) copies of the closed unit interval \( I \) where the bonding point in each copy is one of the endpoints.

(3.2.2) \( J_a \) denotes the bonded union of \( a \) copies of \( J \) where the bonding point in each copy is the unique endpoint.

(3.2.3) \( C_a \) denotes the bonded union of \( a \) copies of \( C \) where the bonding points in each copy is arbitrary.

The following result summarizes some properties of these spaces.

**Proposition (3.3).**

(3.3.1) \( I_0 \) and \( I_1 \) are homeomorphic and this is the only case where \( I_a \) and \( I_b \) are homeomorphic with \( a \neq b \).

(3.3.2) \( J_0 \) is homeomorphic to the space \( R \) of real numbers.

(3.3.3) \( I_a, J_a, C_a \) are all arcswise connected for each cardinal number \( a \).

(3.3.4) The spaces \( I_a, J_a, C_a \) are all sprays, if and only if \( a \) is a finite cardinal.

**Proof.** With the possible exception of (3.3.4) the statements are rather evident. We verify (3.3.4) for the space \( I_a \), the remaining cases being similar. First of all, suppose \( a \) is a positive integer \( N \). We may take \( I_N \) to be the space \( \bigcup \{ I_n \}_{n=1}^{\infty} \) where
\[
I_n = \{(x, y) \in R \times R: y = x/n \text{ and } 0 \leq x < 1 \}.
\]

Let \( (A_\delta : \delta \in A) \) be an uncountable collection of retracts of \( I_\gamma \), each having more than one point. We consider two cases.

**Case 1.** An infinite number of the \( A_\delta \) contain the origin \( q \).

We assume that no boundary of any of the retracts intersects the interior, with respect to \( I_\gamma \), of any other retract and we obtain a contradiction. Let \( A' = \{ \delta \in A : q \not\in A_\delta \} \) and we consider the subspace \( I_{\gamma'} \) of \( I_\gamma \). There are two possibilities:

(1a) \( I_{\gamma'} \cap A_\delta \) is a nondegenerate subinterval of \( I_\gamma \) for some \( \gamma \in A' \).

(1b) \( I_{\gamma'} = \{ q \} \) for each \( \delta \in A' \).

If (1a) holds, then each \( I_{\gamma} \cap A_\delta \) must be either \( I_0 \cap A_\delta \) or \( \{ q \} \), otherwise the boundary of \( A_\delta \) would intersect the interior, with respect to \( I_\gamma \), of some \( A_\delta \), or conversely. It follows that regardless of which one of (1a) and (1b) holds, we may conclude than an infinite number of the \( A_\delta \) all intersect \( I_\gamma \) in an identical fashion. Denote the indices of all these \( A_\delta \) by \( A_\delta' \). In a similar manner, there is an infinite subset \( A'_\delta \) of \( A_\delta \) such that all of the retracts with indices in \( A'_\delta \) intersect \( I_\gamma \) and of course, \( I_{\gamma'} \) also in exactly the same way. Continuing in this manner, we conclude the existence of an infinite subset \( A''_\delta \) of indices such that any two retracts with indices in \( A''_\delta \) intersect \( I_{\gamma''} \), \( 1 \leq \gamma \leq \gamma'' \), in exactly the same way. This implies that all these retracts are identical which is desired contradiction since they are, in fact, all distinct.

**Case 2.** Only a finite number of the \( A_\delta \) contain the origin \( q \).

Let \( A' = \{ \delta \in A : q \not\in A_\delta \} \) and note that \( A' \) is uncountable. Then for each \( \delta \in A' \), the retract \( A_\delta \) is contained in \( I_\gamma \) which is just the free union of \( N \) copies of a half-open interval. Since \( A_\delta \) is connected it must be contained in one of these half-open intervals. It readily follows that some \( L_\gamma \) contains an uncountable number of these \( A_\delta \). Now each of these \( A_\delta \) is a nondegenerate subinterval of the half-open interval \( L_\gamma \) and it follows easily that the boundary of one of the \( A_\delta \) must intersect the interior (with respect to \( I_\gamma \)) of another. Thus, condition (2.10.3) is satisfied and since the remaining conditions in Definition (2.10) are also satisfied, we conclude that \( I_\gamma \) is a spray.

Now suppose that \( a \) is an infinite cardinal. We show that \( I_\gamma \) is not a spray. Let \( A \) be any index set whose cardinal number is \( a \) and for each \( \delta \in A \), let \( h_\delta \) be a homeomorphism from the closed unit interval \( I \) onto a space \( I_\delta \). Then \( I_\delta \) is topologically the bounded union of the pairs \( \{ (x_+, h_\delta'(0)) \delta \in A \} \). Now for each proper nonvoid subset \( D \subseteq A \), we associate a retract of \( I_\delta \) as follows: let
\[
A_D = \bigcup \{ I_\delta : \delta \in D \}.
\]
It is evident that there are uncountably many such subspaces. To see
that they are retracts, choose any \( \gamma \in \Omega \) and define a function \( f \) as follows:
\[
\begin{align*}
  f(x) &= x \quad \text{for} \quad x \in \mathcal{A}_a, \\
  f(x) &= h_\delta(h^{-1}_\beta(x)) \quad \text{for} \quad x \in Y_\delta, \quad \delta \in \Omega.
\end{align*}
\]
The function \( f \) is continuous and it is the identity on its range which is
\( \mathcal{A}_a \). Thus, \( f \) is idempotent and, consequently, \( \mathcal{A}_a \) is a retract. Now in
each of these \( \mathcal{A}_a \) the boundary is the exceptional point which we denote by
\( g \); so if the boundary of one of these retracts was to intersect the
interior with respect to \( I_a \), say \( \mathcal{A}_a \), then \( g \) would be contained in
some open subset of \( I_a \) which would be contained in \( \mathcal{A}_a \). This, however,
is impossible since such an open subset would intersect each \( Y_\delta \), in more
than one point while \( \mathcal{A}_a \) intersects at least one \( Y_\delta \) at only one point,
the exceptional point \( g \). This follows since \( \Omega \) is a proper subset of \( \Delta \). Thus,
we have produced an uncountable family of nondegenerate retracts with
the property that given any one, its boundary will not intersect the
interior of any other. Hence condition (2.10.3) is not satisfied and, conse-
sequently, \( I_a \) cannot be a spray when \( a \) is infinite.

Remark. \( I_a \) satisfies every condition of Definition (2.10) with
the single exception of (2.10.3) so it was necessary to work with condition
(2.10.3) to show that \( I_a \) is a spray when \( a \) is an infinite cardinal.

Now we are in a position to verify a statement which we made in
the introduction.

**Proposition (3.4).** Given any collection of semigroups, there exists
an arcwise connected metric space \( X \) such that each semigroup in the
collection is isomorphic to a subsemigroup of \( S(X) \).

Proof. Since each semigroup in such a collection can be embedded in
\( S(\Delta) \) where \( \Delta \) is a sufficiently large discrete space, we need only show
that for any discrete space \( \Delta \), there exists an arcwise connected metric
space \( X \) such that \( S(\Delta) \) can be embedded in \( S(X) \). We use the same
notation as in the proof of Proposition (3.3). The symbol \( \Delta \) represents an
index set of cardinality \( a \) and we show that \( S(\Delta) \) (where \( \Delta \) is given
the discrete topology) can be embedded in \( S(I_a) \) where, as in the previous
proof, \( I_a \) is taken to be the bounded union of the pairs \( \{ (Y_\delta, h_\delta(0)) : \delta \in \Delta \} \).
For any \( f \in S(\Delta) \), we define a function \( \varphi(f) \) on \( I_a \) as follows:
\[
\begin{align*}
  \varphi(f)(x) &= h_\beta(h^{-1}_\delta(x)) \quad \text{for} \quad x \in Y_\delta - h_\delta(0), \\
  \varphi(f)(q) &= q \quad \text{where} \quad q \text{ is the exceptional point}.
\end{align*}
\]
One shows rather easily that \( \varphi(f) \) is a continuous selfmap of \( S(I_a) \) and
just as easily that \( \varphi \) is a monomorphism from \( S(\Delta) \) into \( S(I_a) \).

**The embedding theorem.** In this section, we prove the main result
of the paper.

**Main Theorem.** Let \( X \) be a strongly conformable, quasi-homogeneous,
completely regular space which is not totally disconnected and let \( Y \) be a spray.
Then for each monomorphism \( \varphi \) from \( S(X) \) into \( S(Y) \), there exists a unique
idempotent \( v \) of \( S(Y) \) and a unique homeomorphism \( h \) from \( X \) onto the range
of \( v \) such that
\[
\varphi(f) = h \circ f \circ h^{-1} \circ v
\]
for each \( f \in S(X) \).

Proof. Let \( \varphi \) be a monomorphism from \( S(X) \) into \( S(Y) \) and let \( L \)
denote the collection of all constant functions on \( X \). Then \( L \) is a left zero
semigroup. Moreover, since \( X \) is completely regular and \( T \), and is not
totally disconnected, it must have uncountably many elements. Thus,
\( L \) is an uncountable left zero semigroup and consequently, \( L' = \varphi(L) \)
is also. We first show, by contradiction, that at least one function in \( L' \)
is constant. Assume that the contrary is true. Since each function in \( L' \)
is idempotent, the range of each such function is a nondegenerate retract
of \( Y \) and it now follows from (3.10.5) that there exist two functions \( v \)
and \( w \) of \( L' \) such that
\[
(1) \quad \text{bd} V \cap \text{int} W \neq \emptyset
\]
where \( V \) is the range of \( \varphi \), \( W \) is the range of \( \varphi \), \( \text{bd} V \) is the boundary of \( V \).
and \( \text{int} W \) is the interior (both with respect to \( Y \)) of \( W \). Let \( r \) be any point
in \( \text{bd} V \cap \text{int} W \). Since retracts are closed in Hausdorff spaces, \( r \in V \)
and hence \( \varphi(r) = r \). It follows that
\[
(2) \quad r \in \text{int} W \cap \varphi^{-1}(\text{int} W) \cap \text{cl}(Y - V).
\]
Consequently,
\[
(3) \quad \text{int} W \cap \varphi^{-1}(\text{int} W) \cap (Y - V) \neq \emptyset
\]
and we choose any point \( t \) which belongs to the latter set. Then,

\[
(4) \quad t \in W - V
\]

and

\[
(5) \quad w(t) \in W.
\]

Since \( w \) is the identity on its range, it follows from (5) that \( w(w(t)) = w(t) \). However, it follows from (4) that \( w(t) \neq t = w(t) \). All this implies that \( w = e \neq w \) which is a contradiction since \( L^* \) is a left zero semigroup. Therefore \( L^* \) must contain a function which maps every point of \( Y \) into some point \( p \) in \( X \). We denote this function by \( \langle y \rangle \). But then for any \( f \in L^* \), we have \( f \cdot \langle y \rangle = \langle f(y) \rangle \). That is, \( L^* \) consists entirely of constant functions. This fact allows us to define a mapping \( h \) from \( X \) into \( Y \). Let \( a \in X \) be given. Then \( \langle a \rangle \in L^* \) and \( \langle a \rangle \in L^* \) is a constant function. Thus,

\[
\varphi (a) = \langle y \rangle \quad \text{for some } y \in Y.
\]

We define \( h(a) = y \) and we note that

\[
(6) \quad \varphi (a) = h(a) \quad \text{for each } a \in X.
\]

We use this latter fact to get

\[
\langle \varphi (f) \cdot h(a) \rangle = \langle \varphi (f) \cdot \varphi (a) \rangle = \langle \varphi (f) \rangle = \varphi (\varphi (a)) = \langle h(a) \rangle = \langle \varphi (f) \rangle
\]

which implies that

\[
(7) \quad \varphi (f) \cdot h = h \cdot f \quad \text{for each } f \in S(X).
\]

One uses this fact to prove that

\[
(8) \quad h(f^{-1}(a)) = h(X) \cap \langle \varphi (f) \rangle^{-1}(h(a))
\]

for each \( a \in X \) and \( f \in S(X) \). We verify only one inclusion since the other follows in a similar manner. Suppose that

\[
y \in h(X) \cap \langle \varphi (f) \rangle^{-1}(h(a)).
\]

Then \( y = h(a) \) for some \( a \in X \) and \( \langle f(y) \rangle = h(a) \) or, equivalently,

\[
\varphi (f) \cdot h = h \cdot f
\]

which is the same as

\[
\varphi (f) = \varphi (a).
\]

Thus,

\[
\langle f(y) \rangle = \langle a \rangle \quad \text{since } \varphi \text{ is injective and this implies that } a \in f^{-1}(y).
\]

In turn, implies that \( y = h(a) = h(f^{-1}(a)) \).

Now, since \( X \) is an \( S^* \)-space, the sets of the form \( f^{-1}(a) \), \( a \in X \) and \( f \in S(X) \) form a basis for the closed subsets of \( X \) and it is now immediate from (8) that \( h^{-1} \) is a continuous map from \( h(X) \) onto \( X \). Next, we want to show that \( h \) is also continuous and since \( X \) is first countable, we can use sequences to do this. Suppose that \( \langle a_n \rangle_{n=1}^{\infty} \) is a sequence of distinct points of \( X \) which converges to a point \( p \) in \( X \) which is distinct from all the points in the sequence. We must show that the sequence \( \langle h(a_n) \rangle_{n=1}^{\infty} \) converges to \( h(p) \). Since \( X \) is uncountable and \( h \) is injective, \( h(X) \) is also uncountable and, in view of (2.10.1), has a nonisolated point \( q \). Moreover, \( h(X) \) is first countable since \( Y \), so there exists an infinite sequence of distinct points \( h(X) \) which converges to \( q \). Denote the set consisting of the points of the sequence together with the limit point \( q \) by \( A \). Then \( A \) is a compact, countable subset of \( h(X) \) which has exactly one limit point. Now \( h^{-1} \) is injective and continuous and since \( X \) is Hausdorff, the restriction of \( h^{-1} \) to \( A \) is a homeomorphism. Consequently, \( h^{-1} \) is also a compact, countable subset of \( X \) with exactly one limit point. Now let

\[
B = \{ \cup (a_n)_{n=1}^{\infty} \} \cup \{ p \}.
\]

Then \( B \) is also a compact, countable subset of \( X \) and since \( X \) is strongly conformable there exists a continuous selfmap \( f \) of \( X \) which maps \( h^{-1}(A) \) into \( B \) in such a way that \( B - f(h^{-1}(A)) \) is finite. Thus, there is a positive integer \( N \) such that \( a_N \) belongs to \( f(h^{-1}(A)) \) for all \( N \). It also follows that

\[
(9) \quad f(h^{-1}(q)) = p.
\]

Now for each \( n > N \), choose a point in \( A \) which \( f \cdot h^{-1} \) maps into \( a_n \) and denote the point by \( y_n \). In this way, we get a sequence \( \langle y_n \rangle_{n=1}^{\infty} \) of distinct points in \( A \) which must necessarily converge to the point \( q \). Then, for \( n > N \), it follows from (7) that

\[
h(a_n) = h(f(h^{-1}(y_n))) = \varphi (f) \cdot h\langle h^{-1}(y_n) \rangle = \varphi (f) \langle y_n \rangle
\]

and from (7) and (9), we get

\[
h(p) = h(f(h^{-1}(q))) = \varphi (f) \cdot h\langle h^{-1}(q) \rangle = \varphi (f) \langle q \rangle.
\]

Since \( \varphi (f) \) is continuous on \( Y \) and \( \lim y_n = q \), it now becomes apparent that \( \lim h(a_n) = h(p) \). Thus, \( h \) is continuous and we have now established that \( h \) is a homeomorphism from \( X \) into \( Y \).

Next, we show that \( h(X) \) is a closed subset of \( Y \). Suppose, to the contrary, that this is not so. Then there exists a sequence \( \langle a_n \rangle_{n=1}^{\infty} \) of distinct points of \( h(X) \) which converges to some point \( r \in Y \) but \( h(X) \). The set \( \langle a_n \rangle_{n=1}^{\infty} \) has no limit points in \( h(X) \) and since \( h \) is a homeomorphism, the set \( \langle h^{-1}(a_n) \rangle_{n=1}^{\infty} \) has no limit points in \( X \). Thus \( \langle h^{-1}(a_n) \rangle_{n=1}^{\infty} \) is disjoint closed subsets of \( X \) and since \( X \) is a strong
S*-space, there exist distinct points p and q in X and a continuous self-map f of X such that
\[ f^{-1}(a_{n-1}) = p \quad \text{and} \quad f^{-1}(a_n) = q \]
for each positive integer n. We again appeal to (7) to conclude that
\[ \varphi(f)(a_{n-1}) = \lambda(p) \quad \text{and} \quad \varphi(f)(a_n) = \lambda(q). \]

Since \( h(p) \) and \( h(q) \) are distinct, the sequence \( (\varphi(f)(a_n))_{n=1}^\infty \) does not converge. This, however, is a contradiction since \( (a_{n-1})_{n=1}^\infty \) does converge. This establishes the fact that \( h(X) \) is a closed subset of \( Y \).

Now let \( \iota_X \) denote the identity mapping on X. Then \( \varphi(\iota_X) \) is an idempotent of \( S(Y) \). We denote it by \( v \) and its range by \( V \). We next want to show that
\[ \lambda(X) = V. \]

One inclusion is rather easy to get. For any \( x \in X \), we have
\[ \langle \iota_X(x) \rangle = \varphi(\iota_X)(x) = \varphi(\varphi(\iota_X)(x)) = \varphi(\varphi(x)) = \langle \iota_X(x) \rangle. \]

Thus, \( \iota_X(x) = \lambda(x) \) which implies that \( h(X) \subset V \). Suppose, however, that (10) does not hold. Then \( h(X) \) is a proper closed subset of \( V \) which is connected since it is a continuous image of the connected space \( X \). It follows that there exists a point \( p \) such that
\[ p \in h(X) \cap \partial V = \partial V = \partial h(X). \]

Since \( X \) is not totally disconnected, neither is \( h(X) \) and condition (2.10.2) assures the existence of a nonempty open subset \( G \) of \( Y \) such that \( G \subset h(X) \). Then \( h^{-1}(G) \) is a nonempty open subset of \( X \) and since \( X \) is quasi-homogeneous there exist continuous selfmaps \( f \) and \( g \) of \( X \) such that \( h^{-1}(G) \subset \varphi(f)(G) \subset \varphi(g)(G) \subset G \). It follows from this and (7) that
\[ \varphi(g)(p) = \varphi(g)(h^{-1}(p)) = \lambda(g(h^{-1}(p))) \in G \]
and, hence, there exists an open subset \( Y \) of \( Y \) such that
\[ \lambda(X) = V - h(X). \]

From (11), we conclude that there is a point \( q \) such that
\[ q \in h(X) \cap \partial V. \]

The statements (12) and (13) together with the fact that \( G \subset h(X) \) result in
\[ \varphi(g)(q) \in \lambda(X). \]

We use this and the fact that \( g \in V \) to get
\[ g = \varphi(g)(q) = \varphi(f)(q) = \varphi(f)((q) = \varphi(f)(\varphi(g)(q)) = \varphi(f)(h(X)). \]

That is, \( g = \varphi(f)(h(x)) \) for some \( x \in X \). But (7) implies that \( \varphi(f)(h(x)) = h(f(x)) \) which, in turn, implies that \( g \in h(X) \). This, of course, contradicts (13) so we conclude that statement (10) is, indeed, valid. Because of (10), we have \( v = h \cdot h^{-1} \cdot v \). We use this and (7) to show that
\[ \varphi(f) = h \cdot f \cdot h^{-1} \cdot v \quad \text{for each} \quad f \in S(X). \]

Let any \( f \in S(X) \) be given. Then,
\[ \varphi(f) = \varphi(f \cdot \iota_X) = \varphi(f) \cdot \varphi(\iota_X) = \varphi(f) \cdot v \]
\[ = \varphi(f) \cdot h \cdot h^{-1} \cdot v = h \cdot f \cdot h^{-1} \cdot v \]
which verifies (15).

In order to complete the proof of the theorem we need only to show that the function \( k \) and \( v \) are unique. Suppose that there also exist functions \( k \) and \( v \) such that
\[ \varphi(f) = h \cdot f \cdot h^{-1} \cdot v \quad \text{for each} \quad f \in S(X). \]

Then (15) and (16) together yield
\[ v = \varphi(\iota_X) = w \]
and for any \( x \in X \),
\[ \langle \iota_X(x) \rangle = \varphi(\iota_X) = \langle \iota_X(x) \rangle \]
which implies that \( h = k \).

5. Applications of the main theorem. The results in this section show that, in a certain sense, the semigroups of selfmaps on a number of sprays contain very few subsemigroups which are isomorphic to semigroups of continuous selfmaps.

**Theorem (6.1).** Let \( X \) be a strongly conformable, quasi-homogeneous, completely regular space which is not totally disconnected. Then for each positive integer \( N \), \( S(X) \) can be embedded in \( S(\mathbb{I}) \) if and only if \( X \) is homeomorphic to the closed unit interval \( I \).

**Proof.** We use the same notation as in the proof of Proposition (3.3). In particular, we take \( I_N \) to be \( \bigcup \{ I_n \}_{n=1}^{\infty} \) where
\[ I_n = \{ (x, y) \in \mathbb{R} \times \mathbb{R} : y = n|x| \quad \text{and} \quad 0 < x < 1 \}. \]

First, suppose that \( X \) is homeomorphic to \( I \). Then there exists a homeomorphism \( h \) from \( X \) onto \( I_n \) and an idempotent \( v \) of \( S(\mathbb{I}) \) whose range
is $L_0$. The mapping which sends $f$ in $S(X)$ into $h \cdot f \cdot h^{-1} \cdot v$ is an embedding of $S(X)$ into $S(I_X)$.

Conversely, suppose $S(X)$ can be embedded into $S(I_X)$ with some monomorphism $\varphi$. Then, according to the main theorem, there exists an idempotent $e$ of $S(I_X)$ and a homomorphism $h$ from $X$ onto the range $V$ of $e$ such that $\varphi(f) = h \cdot f \cdot h^{-1} \cdot v$ for each $f \in S(X)$. Thus, $X$ is homeomorphic to the retract $V$. We must show that $V$ is homeomorphic to $I$. First of all, $V$ has more than one point since $X$ does. Furthermore, any non-degenerate retract of $I_X$ is homeomorphic to some $I_M$ where $1 < M < N$ so $V$ is homeomorphic to such an $I_M$. We only need to show that $M$ cannot exceed $2$ since otherwise there could not possibly exist two continuous selfmaps $f$ and $g$ of $I_M$ so that $f \circ g$ would be the identity on $I_M$ while $g$ would map the origin into the open subset

$$\{(x,y) \in R \times R : y = x \text{ and } \frac{1}{3} < x < \frac{2}{3}\}.$$ 

This latter theorem was essentially proven by appealing to the main theorem and then determining which retracts of $I$ were quasi-homeomorphic. The same technique yields analogous results for the spaces $J_Y$ and $C_Y$ and we state these results without proof.

**Theorem (5.2).** Let $X$ be a strongly conformable quasi-homeomorphic completely regular space which is not totally disconnected. Then, for each positive integer $N$, $S(X)$ can be embedded in $S(I_N)$ ($N \geq 2$) if and only if $X$ is homeomorphic to either $I$, $B$ or a half-open interval.

**Theorem (5.3).** Let $X$ be a strongly conformable quasi-homeomorphic, completely regular space which is not totally disconnected. Then, for each positive integer $N$, $S(X)$ can be embedded in $S(U_N)$ if and only if $X$ is homeomorphic to either $I$ or the unit circle $C$.

In each of the previous three theorems, we required that $X$ not be totally disconnected. It turns out that when one is considering embedding $S(X)$ into either $S(I)$ or $S(B)$, this requirement can be dropped without significantly increasing the number of possibilities for $X$. The precise statements are given in the next two results.

**Theorem (5.4).** Let $X$ be a strongly conformable, quasi-homeomorphic, completely regular space. Then $S(X)$ can be embedded into $S(I)$ if and only if $X$ is homeomorphic to either $I$, the two-point discrete space or the one-point space.

**Proof.** (Necessity). Suppose that $S(X)$ can be embedded in $S(I)$. It is immediate from Theorem (5.1) that if $X$ is not totally disconnected, then $X$ is homeomorphic to $I$. We consider the case where $X$ is totally disconnected and we show that $X$ cannot have more than two points.

Assume to the contrary that it does. Since it is not connected, it is the union of two nonempty disjoint clopen (both closed and open) subsets $A$ and $A'$. One of these, say $A'$, has more than one point and since it is not connected, it must be the union of two nonempty disjoint clopen subsets $B$ and $C$. Then $A$, $B$ and $C$ are mutually disjoint nonempty clopen subsets of $X$ whose union is all of $X$. Choose points $a \in A$, $b \in B$ and $c \in C$ and define a function $f$ by

$$f(x) = b \text{ for } x \in A,$$
$$f(x) = c \text{ for } x \in B,$$
$$f(x) = a \text{ for } x \in C.$$ 

Then $f$ is continuous and $\{f, f', f''\}$ is a subset of $S(X)$ with three elements. This implies that $S(I)$ has a subgroup of order three and we have reached a contradiction since Theorem (3.6) of [3, p. 145] assures us that the only finite subgroups of $S(I)$ (and $S(B)$ as well) have order either one or two. Thus $X$, in this case, has no more than two points.

(Sufficiency). It is immediately evident that if $X$ is homeomorphic to either $I$ or the one-point space, then $S(X)$ can be embedded in $S(I)$. It is slightly less immediate when $X$ is the two-point discrete space so we give an argument for this case. Choose any continuous selfmap of $I$ so that $f \neq \text{id}_{I}$, the identity map on $I$, but $f \cdot f = \text{id}_{I}$. For example, the map which sends a point $x$ into $1-x$ will do. Then choose two points $a$ and $b$ such that $f(a) = b$ and $f(b) = a$. One readily shows that $S(X)$ is isomorphic to the subsemigroup of $S(I)$ consisting of the functions $\langle a, b, f \rangle$ and $\text{id}_{I}$.

The analogous theorem about $S(R)$ is proven in much the same way so we will be content with merely stating it.

**Theorem (5.5).** Let $X$ be a strongly conformable, quasi-homeomorphic, completely regular space. Then $S(X)$ can be embedded in $S(R)$ if and only if $X$ is homeomorphic to either $R$, $I$, a half-open interval, the two-point discrete space or the one-point space.

Now let us observe that there are many monomorphisms from $S(I)$ into $S(I)$ which are not automorphisms. We need only choose an idempotent $e$ different from the identity map and a homomorphism $h$ from $I$ onto the range of $e$ and define

$$\varphi(f) = h \cdot f \cdot h^{-1} \cdot e.$$ 

Then $\varphi$ is a monomorphism from $S(I)$ into $S(I)$ which is not an automorphism. The concluding result of this paper shows that the situation is quite different for $S(R)$.
THEOREM (5.6). Every monomorphism from $S(I)$ into $S(I)$ is, in fact, an automorphism.

Proof. Let $\varphi$ be a monomorphism from $S(I)$ into $S(I)$. By the main theorem, there exists an idempotent $e$ on $S(I)$ and a homeomorphism $h$ from $I$ onto the range $\nu$ of $e$ such that $\varphi(f) = h \cdot f \cdot h^{-1} \cdot e$ for each $f$ in $S(I)$. Now $\nu$ is closed in $I$ and since $I$ is homeomorphic to $I$, we must have $\nu = I$. This forces $e$ to be the identity map which means that $\varphi$ is an automorphism.

6. Some concluding remarks. Let $S$ be a semigroup with identity $e$ and let $T$ be an arbitrary semigroup. The notion of embedding $S$ into $T$ with an $a$-monomorphism was introduced in [2] and we recall the definition now. A monomorphism $\varphi$ from $S$ into $T$ is an $a$-monomorphism if for each left zero $z \in S$, $\varphi(z) = z$ implies $z \in \varphi(S)$. Now for any two spaces $X$ and $Y$, if one chooses an idempotent continuous selfmap $\varphi$ of $X$ and if there exists a homeomorphism $h$ from $X$ onto the range of $\varphi$, then the mapping $\varphi$ defined by

\begin{equation}
\varphi(f) = h \cdot f \cdot h^{-1} \cdot \varphi \quad \text{for each } f \in S(I)
\end{equation}

is an $a$-monomorphism from $S(I)$ into $S(I)$. It was shown in [6, Theorems (5.6) and (5.7)] that for a great many spaces $X$ and $Y$ all the $a$-monomorphisms are contained in exactly this manner. For example, it follows from Theorem (5.6) of [6] that if $X$ is any Hausdorff $S^*$-space then any $a$-monomorphism from $S(I)$ into $S(I)$ (we recall that $S(I)$ is the Stone-Čech compactification of the closed union $I$) must take the form (2). This places a considerable restriction on $X$ since it must then be homeomorphic to a retract of $S(I)$ which, among other things, forces it to be compact, connected and to contain a dense arcwise connected subspace. And yet, we observe in the proof of Proposition (3.5) of this paper that any semigroup may be embedded in $S(I)$ if one chooses the cardinal number $\gamma$ to be sufficiently large. This means that for a large cardinal number $\gamma$, there are many monomorphisms from semigroups of continuous selfmaps into $S(I)$ which are not $a$-monomorphisms. The main theorem of this paper shows, among other things, that quite the opposite is true about sprays. That is, for a great many $X$. If there is a monomorphism from $S(I)$ into $S(I)$, $X$ a spray, then it must be an $a$-monomorphism.

We conclude this paper with one more observation and that is that various semigroups of relations are a great deal more lenient in allowing embeddings than are the corresponding semigroups of continuous self-maps. For a specific example, let $X$ be any Hausdorff space and $S(X)$ denote the semigroup of all compact relations (compact subsets of $X \times X$) under ordinary composition of relations. If $X$ happens to be compact, then $S(X)$ is a subsemigroup of $S(I)$. At any rate, it follows from Theorem (5.2) of [7, p. 52] that if $X$ is any compact metric space, then $S(X)$ can be embedded in $S(I)$ while Theorem (5.4) of this paper tell us that among all the semigroups $S(X)$ ($X$ strongly conformable, quasi-homogeneous and completely regular) only three can be embedded in $S(I)$. In particular, it follows that for the closed unit ball $B^N$ in Euclidean $N$-space, $N > 1$, there is a monomorphism from $S(I)$ into $S(I)$ (in fact, there are many) but it must map some functions in $S(I)$ into closed relations on $I$ which are not continuous functions on $I$.

References


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