# A Boolean view of sequential compactness

by

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**Abstract.** Conditions for sequential compactness and strong sequential compactness of $D^\lambda$, where $\aleph_0 < \lambda < \mathfrak{c}$, are given. Sequential compactness of $D^{\aleph_1}$ is independent of the axioms of set theory.

Let $S(N)$ be the collection of all subsets of the natural numbers, $N$, and also the corresponding Boolean algebra; let $S_\omega(N)$ be the finite sets and also the corresponding Boolean ideal. A set $a$ in $S(N)$ determines an element $[a]$ of the quotient algebra. Without fearing confusion one may safely write "$a < b$" for the Boolean inequality $[a] < [b]$. Sometimes one finds that the investigation of some question turns in part on the properties of this ordering: this is the case, as we shall see, with the study of sequential compactness.

A topological space is sequentially compact if each sequence of points has a convergent subsequence. The Cantor set $D^{\aleph_0}$ is sequentially compact ($D$ stands for the two point Hausdorff space) but $D^{\mathfrak{c}}$, $\mathfrak{c}$ is the power of the reals, is not.

Definition. Call a *space strongly sequentially compact* if it is sequentially compact and, in addition, given any sequence of points and any limit of this sequence, there is a subsequence converging to this given limit point.

In a first countable space sequential compactness and strong sequential compactness are the same. The space $D^{\aleph_0}$ is therefore strongly sequentially compact; let us consider the property of $\lambda$, $\aleph_0 \leqslant \lambda < \mathfrak{c}$, that $D^\lambda$ is sequentially compact. Such a property is of interest only when the continuum hypothesis fails; for everything is known here if there are no cardinals between $\aleph_0$ and $\mathfrak{c}$. The continuum hypothesis is independent of the axioms of set theory: it holds in some mathematical universes but fails in others.

Theorem 1. *The following are equivalent:*

1. *$D^\lambda$ is strongly sequentially compact.*

2. *If $X_\alpha$ is strongly sequentially compact for $\alpha \in \lambda$, so is $\Pi X_\alpha$.*

3. *If $a_\alpha \subseteq N$, $\alpha \in \lambda$, and the collection $\{a_\alpha: \alpha \in \lambda\}$ has the finite intersection property then there is a set $b \subseteq N$ such that for each $\alpha$, $b < a_\alpha$.*

4. *Any compact Hausdorff space having weight at most $\lambda$ is strongly sequentially compact.*

Proof. Clearly 2 implies 1. Let us obtain 3 from 1. Define a sequence $\langle x_n \rangle$ in $D^\lambda$ by putting $x_n(\alpha) = 1$ if $n \in a_\alpha$ and $x_n(\alpha) = 0$ otherwise; let $x$ be the point which takes the value 1 on each coordinate. The point $x$ is a limit point of the sequence $\langle x_n \rangle$; here we are supposing that each $a_\alpha$ is infinite — this follows from the finite intersection property according to our conventions about dropping brackets. Now there will be a subsequence $\langle x_n \rangle$ $n \in b$ converging to $x$; this set $b$ satisfies 3.

Next, obtain 2 from 3. Let $\langle x_n \rangle$ be a sequence in $\Pi X_\alpha$ having a limit $x$. On each coordinate the sequence $\langle x_n(\alpha) \rangle$ has a subsequence $\langle x_n(\alpha) \rangle$ $n \in a_\alpha$ converging to the point $x(\alpha)$. The set $b$ given by 3 produces a sequence $\langle x_n \rangle$ $n \in b$ converging to $x$.

Finally, consider part 4; part 1 is an instance of 4. Conversely, a compact Hausdorff space, $X$, of weight $\leqslant \lambda$ must be a continuous image of some closed subset $Y$ of $D^\lambda$. Since 1 holds, $D^\lambda$ is strongly sequentially compact, thus $Y$ is as well; but a continuous map preserves strong sequential compactness.

The property of sequential compactness also corresponds to an algebraic property.

THEOREM 2. *The following are equivalent*:

1. *$D^\lambda$ is sequentially compact.*

2. *If $a_\alpha \subseteq N$, $\alpha \in \lambda$, are infinite, then there is a set $b \subseteq N$ such that for each $\alpha$ either $b < a_\alpha$ or $b < N \sim a_\alpha$.*

3. *Any compact Hausdorff space having weight at most $\lambda$ is sequentially compact.*

Proof. To obtain 2 from 1, one considers the sequence $\langle x_n \rangle$ where $x_n(\alpha) = 1$ if $n \in a_\alpha$ and $x_n(\alpha) = 0$ otherwise. This must have a convergent subsequence $\langle x_n \rangle$ $n \in b$; and $b < a_\alpha$ if $\langle x_n(\alpha) \rangle$ $n \in b$ converges to one, otherwise $b < N \sim a_\alpha$. Conversely, a sequence $\langle x_n \rangle$ determines a set $a_\alpha$ with $n \in a_\alpha$ when $x_n(\alpha) = 1$. Obtaining a set $b$ we find a limit point $x$ by putting $x(\alpha) = 1$ just where $\langle x_n(\alpha) \rangle$ $n \in b$ converges to 1.

The third property follows just as the corresponding property in Theorem 1.

The referee of this paper has provided another condition which is equivalent to these: the generalized Bolzano-Weierstrass condition. This condition is that every sequence of sets has a convergent subsequence. Hausdorff showed that this property holds in spaces of weight $\aleph_0$, Lubben [3] showed that it fails in spaces of weight $\mathfrak{c}$. One can find an account of the generalized Bolzano-Weierstrass property in [2], Chapter 2, Section 29.



PROPOSITION. *$D^\lambda$ is sequentially compact if and only if the generalized Bolzano-Weierstrass property holds in spaces of weight at most $\lambda$.*

Proof. A sequence of points $\langle x_n \rangle$ in $D^\lambda$ can be regarded as a sequence of sets $\langle \{x_n\} \rangle$ which, assuming the Bolzano-Weierstrass condition, will have a subsequence $\langle \{x_n\} \rangle$ $n \in a$ which converges. So $\langle x_n \rangle$ $n \in a$ must converge too, for $D^\lambda$ is compact.

Conversely, given a sequence of sets $\langle A_n \rangle$, in a space having an open basis $\{U_\alpha: \alpha \in \lambda\}$, one puts $B_n = \{\alpha: A_n \cap U_\alpha \neq 0\}$. Corresponding to $B_n$ there is its characteristic function $\beta_n$ which is a point of $D^\lambda$. The sequence $\langle \beta_n \rangle$ has a convergent subsequence $\langle \beta_n \rangle$ $n \in a$; one readily sees that $\langle A_n \rangle$ $n \in a$ is a convergent subsequence of $\langle A_n \rangle$.

THEOREM 3. *If $D^\lambda$ is sequentially compact then every set of reals of power $\lambda$ has Lebesgue measure zero.*

Proof. Let $d$ map infinite sets into the reals by putting $d(a) = \Sigma^{a(n)}/2^n$ where $a(n)$ is the value of the characteristic function of $a$. If $S$ is a subset of $[0, 1]$ of power $\lambda$ we have that $S \sim \text{range}(d)$ is countable; so one only has to show that $S \cap \text{range}(d)$ has measure zero. Say $S \cap \text{range}(d) = \{d(a_\alpha): \alpha \in \lambda\}$; by Theorem 2 choose $b$ so that $b < a_\alpha$ or $b < N \sim a_\alpha$. Now

$$S \cap \text{range}(d) \subseteq \{d(c): b < c\} \cup \{d(c): c < N \sim b\}.$$

But it is easily seen that both of these two sets have measure zero.

A somewhat similar argument serves to show that, under the same hypothesis, each set of reals of power $\lambda$ is first category.

DEFINITION. An ultrafilter $P$ in $\beta N \sim N$ (these terms are explained in [1]) is a *$P(\lambda)$ point* if it is in the interior of the intersection of any collection of less than $\lambda$ open sets containing it.

A $P(\aleph_1)$ point is called a *P-point*. If $X$ is a space then $T^\lambda X$ is the space generated by all intersections of less than $\lambda$ neighborhoods of $X$.

PROPOSITION. *The following are equivalent*:

1. *$D^\lambda$ is strongly sequentially compact for each $\lambda < \mathfrak{c}$.*

2. *The $P(\mathfrak{c})$ points in $\beta N \sim N$* are dense in $T^{\mathfrak{c}}(\beta N)$.

This can be proved rather in the manner of Theorem 4.14 in [1]. Let us see now something of the difference in strength between the properties of Theorem 1 and those of Theorem 2.

THEOREM 4. *If $D^\lambda$ is strongly sequentially compact then $\mathfrak{c} = 2^\lambda$ but if we assume instead that $D^\lambda$ is only sequentially compact this need not be the case.*

Proof. The first part seems widely known in one form or other; one can construct a tree of distinct sets ordered by $<$, such that if $f: \lambda \to 2$, then $a(f)$ is a terminal point of the branch $\{a(f \cap (\alpha \times 2)): \alpha \in \lambda\}$. Property 3 of Theorem 1 allows the construction to continue at limit ordinals.

For the rest of the theorem — assuming set theory is consistent — one begins with a model $M$ of set theory and $A_{\aleph_2}$ (this is defined in [4]).

One can check that $D^{\aleph_1}$ is strongly sequentially compact in $M$ (see, for example, Theorem 4.10 of [1]) therefore $M$ contains a $P(\mathfrak{c})$ point. Carry out a Cohen extension of $M$ obtaining a new model $A$ in which $2^{\aleph_1} = \aleph_3$ and $S(N)^M = S(N)^A$. The $P(\mathfrak{c})$ point $P$ of $M$ is still an ultrafilter and a $P(\mathfrak{c})$ point. Let $\{a_\alpha: A \in \aleph_1\}$ be a collection of infinite sets, in order to meet the conditions of Theorem 2 we may as well suppose that $N \sim a_\alpha$ is infinite too. Either $a_\alpha$ or $N \sim a_\alpha$ is in $P$; let $a'_\alpha$ be $a_\alpha$ if $a_\alpha \in P$ and $N \sim a_\alpha$ otherwise; since $P$ is a $P(\mathfrak{c})$ point there is a $b \subseteq N$ such that for each $\alpha$, $b < a'_\alpha$. This $b$ satisfies condition 2 of Theorem 2.

**References**

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[3] R. G. Lubben, *Limiting sets in abstract spaces*, Trans. Am. Math. Soc. 29 (1928), pp. 668–685.
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*Reçu par la Rédaction le 27. 9. 1971*

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# Completely regular proximities and $RC$-proximities

by

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**1.** The author recently introduced the theory of $RC$-proximities to characterize the spaces that can be embedded in a regular-closed space. The present results are concerned with the manner in which $RC$-proximities are made up from other types of proximities.

The theory of $RC$-proximities was developed in our paper [HS]; this paper is a continuation of [HS], and terms, notations, and techniques introduced therein will be used herein without further reference.

**2.** ***LO*-proximities and *R*-proximities.** A proximity $\delta$ such that the induced closure operator is topological and such that from $A\ non\delta\ B$ there follows $\mathrm{cl}A\ non\delta\ \mathrm{cl}B$ is called a *$LO$-proximity*. An *$R$-proximity* is defined in [HS] as a proximity satisfying Axioms P1-P5 of [HS]; a *$LR$-proximity* is a proximity that is simultaneously a $LO$-proximity and an $R$-proximity.

There are three proximities that can be defined on any $T_1$ space and that will be useful later in forming examples. These proximities are considered in [CH]; it is appropriate here to observe that the proximities considered in [CH] are more general than those that we consider, since Axiom P4, which requires that distinct points be far, need not be satisfied by the proximities of [CH].

The proximities considered below do satisfy P4, however, since the associated topologies are $T_1$.

2.1 [CH, 25A. 18(a)]. *For any $T_1$ space $X$ the relation $A\ \delta_s\ B$ if $(A \cap \mathrm{cl}_X B) \cup (\mathrm{cl}_X A \cap B) \neq \emptyset$ is the finest proximity that induces the topology of $X$.*

2.2 [CH, 25A. 18(b)]. *For any $T_1$ space $X$ the relation $A\ \delta_c\ B$ if $A\ \delta_s\ B$ or both $A$ and $B$ are infinite is the coarsest proximity that induces the topology of $X$.*

2.3 [CH, 25A. 18(c)]. *For any $T_1$ space $X$ the relation $A\ \delta_w\ B$ if $\mathrm{cl}_X A \cap \mathrm{cl}_X B \neq \emptyset$ is a proximity that induces the topology of $X$.*

The following results are readily established from the definitions.