if $S_1, S_2, S_3, \ldots$ and $S'_1, S'_2, S'_3, \ldots$ are two sequences in the collection U, there is a positive integer n such that $(\overline{S_n^*})$ does not intersect $(\overline{S'_n})^*$.

By Theorem 3, each sequence in this uncountable collection U determines a planar atriodic tree-like continuum with positive span. Further, if H and K are two continua each determined by a sequence in U, then H does not intersect K.

5. Remark. In 1939 Waraszkiewicz [3] published a paper in which he claimed that the plane contains no atriodic tree-like continuum which is not chainable. However, each continuum in the collection of plane continua described in this paper has positive span and, thus, is not chainable.

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On smooth continua

by

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Abstract. A metric continuum X is said to be *smooth at a point* p if for each subcontinuum K of X which contains p and for each open set V which contains K there exists an open connected set U such that $K \subset U \subset V$. If the continuum X is hereditarily unicoherent at p, then the definition of smoothness at p mentioned above is equivalent to the definition of smoothness at p introduced by G. R. Gordh in [7]. Moreover, it is proved that if the continuum X is hereditarily unicoherent at some point (or if X is an irreducible continuum), then X is hereditarily unicoherent at each point at which it is smooth; thus X is smooth in the sense of Gordh at each such point. Therefore, we conclude that the notion of smoothness at a point (introduced in this paper) is independent of hereditary unicoherence at a point, and smoothness at a point may be defined for all continua.

The set of all points of X at which X is smooth is called the *initial set* and is denoted by $I(X)$. It is proved that if a mapping f on a continuum X is monotone (or open, or quasi-interior), then $f(I(X)) \subset I(f(X))$. This is a generalization of Theorem 4.1 in [7] (Theorem 4 and Corollary 6 in [13]). Furthermore, we give a new relation between the initial set of the preimage and the initial set of the image for confluent mappings. Namely, if a mapping f on a continuum X is conflunet, then $f(X)\setminus I(f(X)) \subset f(X\setminus I(X))$.

§ 1. Introduction. Investigating smooth continua, defined by G. R. Gordh in [7], we have observed that the notion of smoothness of a continuum at some point at which it is hereditarily unicoherent, can be easily extended to the notion of smoothness of a continuum at some point at which it need not to be hereditarily unicoherent, i.e., that the notion of the smoothness of a continuum is independent of the notion of its hereditary unicoherence. Moreover, the idea of the smoothness of a continuum at a point is, as will be seen, a generalization of the idea of the local connectedness of the continuum at that point in some sense.

In this paper we study some properties of smooth continua, in particular we give some characterizations of them, and, incidentally, we investigate the invariability of smoothness under some classes of continuous mappings, and the co-existence of smoothness at some point and hereditary unicoherence at another point in an arbitrary continuum.

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§ 2. Definitions and preliminaries. The topological spaces under consideration will be assumed to be metric continua. If $A_1, A_2, \ldots$ is a sequence of subsets of a topological space X, then $\operatorname{Li}_{n\to\infty} A_n$ denotes the set of all points $x \in X$ for which every neighbourhood intersects A_n for almost all n, and $\operatorname{Ls}_{n\to\infty} A_n$ denotes the set of all points $x \in X$ for which every neighbourhood intersects A_n for arbitrarily large n. A sequence of subsets $A_1, A_2, \ldots$ is said to *converge* to a set (denoted by $\operatorname{Lim}_{n\to\infty} A_n = A$) in case $\operatorname{Li}_{n\to\infty} A_n = A = \operatorname{Ls}_{n\to\infty} A_n$. We have the following (see [9], § 47, II, Theorem 6, p. 171)

(2.1) PROPOSITION. *If $C_1, C_2, \ldots$ is a sequence of subcontinua of a compact metric space such that $\operatorname{Li}_{n\to\infty} C_n \neq \emptyset$, then the set $\operatorname{Ls}_{n\to\infty} C_n$ is a continuum.*

(2.2) LEMMA. *Let $C_1, C_2, \ldots$ be a sequence of subcontinua of a compact metric space X. If $x, y \in \operatorname{Li}_{n\to\infty} C_n$ and the continuum $\operatorname{Ls}_{n\to\infty} C_n$ is irreducible between points x and y, then the sequence $C_1, C_2, \ldots$ is convergent.*

Proof. Suppose, on the contrary, that the sequence $C_1, C_2, \ldots$ is not convergent, i.e. that $\operatorname{Ls}_{n\to\infty} C_n \setminus \operatorname{Li}_{n\to\infty} C_n \neq \emptyset$ (since we always have $\operatorname{Li}_{n\to\infty} C_n \subset \operatorname{Ls}_{n\to\infty} C_n$). Let $c \in \operatorname{Ls}_{n\to\infty} C_n \setminus \operatorname{Li}_{n\to\infty} C_n$. It follows from the definition of $\operatorname{Li}_{n\to\infty} C_n$ that there exist a neighbourhood U of the point c and a subsequence $C_{n_1}, C_{n_2}, \ldots$ of the sequence $C_1, C_2, \ldots$ such that none of the C_{n_m} for $m = 1, 2, \ldots$ intersects U. Let $L = \operatorname{Ls}_{m\to\infty} C_{n_m}$. Therefore we have $c \in U \subset X \setminus L$. Since $x, y \in \operatorname{Li}_{n\to\infty} C_n$, thus $x, y \in \operatorname{Li}_{n\to\infty} C_{n_m}$. Therefore, by Proposition (2.1), L is a continuum which contains the points x and y. Moreover, we have $L \subset \operatorname{Ls}_{n\to\infty} C_n$. The continuum $\operatorname{Ls}_{n\to\infty} C_n$ being irreducible between points x and y, we conclude that $L = \operatorname{Ls}_{n\to\infty} C_n$, whence $c \in L$, a contradiction.

(2.3) THEOREM. *Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be sequences of points of a compact metric space X which converge to x and y respectively, and let $I(x, y)$ be a continuum irreducible between x and y. The following are equivalent:*

(i) *there exists a sequence $K_1, K_2, \ldots$ of continua such that $x_n, y_n \in K_n$ for each $n = 1, 2, \ldots$ and $\operatorname{Ls}_{n\to\infty} K_n = I(x, y)$;*

(ii) *there exists a sequence $K_1, K_2, \ldots$ of continua such that $x_n, y_n \in K_n$ for each $n = 1, 2, \ldots$ and $\operatorname{Lim}_{n\to\infty} K_n = I(x, y)$;*

(iii) *there exists a sequence of continua $I(x_1, y_1), I(x_2, y_2), \ldots$ which are irreducible between x_n and y_n, respectively, and such that $\operatorname{Ls}_{n\to\infty} I(x_n, y_n) = I(x, y)$;*

(iv) *there exists a sequence of continua $I(x_1, y_1), I(x_2, y_2), \ldots$ which are irreducible between x_n and y_n, respectively, and such that $\operatorname{Lim}_{n\to\infty} I(x_n, y_n) = I(x, y)$.*

Proof. (i) implies (ii) by Lemma (2.2). We prove that (ii) implies (iii). Let x_n, y_n and K_n be like in (ii) and let $I(x_n, y_n)$ be a continuum irreducible between x_n and y_n, which lies in K_n for $n = 1, 2, \ldots$ Therefore $\operatorname{Ls}_{n\to\infty} I(x_n, y_n) \subset \operatorname{Ls}_{n\to\infty} K_n = \operatorname{Lim}_{n\to\infty} K_n = I(x, y)$. Since $x, y \in \operatorname{Li}_{n\to\infty} I(x_n, y_n) \subset \operatorname{Ls}_{n\to\infty} I(x_n, y_n)$, thus, by Proposition (2.1), $\operatorname{Ls}_{n\to\infty} I(x_n, y_n)$ is a continuum containing the points x and y. So we have $\operatorname{Ls}_{n\to\infty} I(x_n, y_n) = I(x, y)$ by the irreducibility of $I(x, y)$ between x and y.

Further, (iii) implies (iv) by Lemma (2.2), and obviously (iv) implies (i). The proof of Theorem (2.3) is complete.

We say that a continuum X is *smooth at the point* $p \in X$ if for each convergent sequence $x_1, x_2, \ldots$ of points of X and for each subcontinuum K of X such that $p, x \in K$, where $x = \lim_{n\to\infty} x_n$, there exists a sequence $K_1, K_2, \ldots$ of subcontinua of X such that $p, x_n \in K_n$ for each $n = 1, 2, \ldots$, and $\operatorname{Lim}_{n\to\infty} K_n = K$.

The next theorem gives some characterizations of continua which are smooth at some point.

(2.4) THEOREM. *Let $x_1, x_2, \ldots$ be a sequence of points of a compact metric space X which is convergent to a point $x \in X$, and let $p \in X$. The following are equivalent:*

(i) *for each continuum K such that $p, x \in K$, there exists a sequence $K_1, K_2, \ldots$ of subcontinua of X such that $p, x_n \in K_n$ for each $n = 1, 2, \ldots$ and $\operatorname{Lim}_{n\to\infty} K_n = K$;*

(ii) *for each irreducible continuum $I(p, x)$ between p and x, there exists a sequence $K_1, K_2, \ldots$ of subcontinua of X such that $p, x_n \in K_n$ for each $n = 1, 2, \ldots$ and $\operatorname{Ls}_{n\to\infty} K_n = I(p, x)$;*

(iii) *for each irreducible continuum $I(p, x)$ between p, and x, there exists a sequence $K_1, K_2, \ldots$ of subcontinua of X such that $p, x_n \in K_n$ for each $n = 1, 2, \ldots$ and $\operatorname{Lim}_{n\to\infty} K_n = I(p, x)$,*

(iv) *for each irreducible continuum $I(p, x)$ between p and x, there exists a sequence $I(p, x_1), I(p, x_2), \ldots$ of irreducible subcontinua between p and x_n, respectively, such that $\operatorname{Ls}_{n\to\infty} I(p, x_n) = I(p, x)$;*

(v) *for each irreducible continuum $I(p, x)$ between p and x, there exists a sequence $I(p, x_1), I(p, x_2), \ldots$ of irreducible continua between p and x, respectively, such that $\operatorname{Lim}_{n\to\infty} I(p, x_n) = I(p, x)$.*

Proof. (i) obviously implies (ii), (ii) implies (iii) by Lemma (2.2), (iii) implies (iv), and (iv) implies (v) by Theorem (2.3) (taking $y_n = p$ for each $n = 1, 2, \ldots$). Therefore it remains to prove that (v) implies (i). Let a continuum K be an arbitrary subcontinuum of X containing p and x. K contains a continuum $I(p, x)$ irreducible between p and x. It follows from (v) that there exists a sequence $I(p, x_1)$, $I(p, x_2)$, ... of subcontinua of X which are irreducible between p and x_n, respectively, and such that $\operatorname{Lim}_{n\to\infty} I(p, x_n) = I(p, x)$. Putting $K_n = K \cup I(p, x_n)$ for each $n = 1, 2, \ldots$ we see that $p, x_n \in K_n$ for each $n = 1, 2, \ldots$ whence K_n are continua. Further, $\operatorname{Lim}_{n\to\infty} K_n = K \cup \operatorname{Lim}_{n\to\infty} I(p, x_n) = K \cup I(p, x) = K$, because $I(p, x) \subset K$. The proof of Theorem (2.5) is complete.

Recall (see [7], p. 52) that the continuum X is hereditarily unicoherent at the point p if the intersection of any two subcontinua each of which contains p is connected. An equivalent condition runs as follows (see [7], Theorem 1.3, p. 52): the continuum X is hereditarily unicoherent at p if and only if, given any point $x \in X$, there exists a unique subcontinuum of X which is irreducible between p and x. If the continuum X is hereditarily unicoherent at p and $q \in X$, then pq will denote the unique subcontinuum which is irreducible between p and q.

Gordh in [7], p. 52, introduced the concept of the smoothness of a continuum at a point as follows. A continuum X is said to be *smooth at a point* p of X is hereditarily unicoherent at p and, for each sequence of points $a_1, a_2, \ldots$ of X which is convergent to a point a, the sequence of irreducible continua $pa_1, pa_2, \ldots$ is convergent to the continuum pa, i.e., $\operatorname{Lim}_{n\to\infty} pa_n = pa$.

In this paper we define the smoothness of a continuum X in such a way that if a continuum X is smooth at the point p, then X need not be hereditarily unicoherent at p, i.e., it is not as in Gordh's definition of smoothness; but condition (v) of Theorem (2.4) implies that, if X is hereditarily unicoherent at the point p, then both the definitions of the smoothness of a continuum X at the point p are equivalent. Moreover, as we will prove in Theorem (4.3), if a continuum X is hereditarily unicoherent at some point and X is smooth at the point p, then X is hereditarily unicoherent at p. Therefore the notion of smoothness at a point (introduced in this paper) is independent of the notion of hereditary unicoherence at a point, and smoothness at a point may be defined for all continua which are or are not hereditarily unicoherent at some point.

Similarly to what was done by Gordh, the set of all points of an arbitrary continuum X at which X is smooth is called the *initial set* of X and is denoted by $I(X)$. If $I(X) \neq \emptyset$, then X is said to be *smooth*.

§ 3. The initial set. Recall that, given a subset G of a continuum X and a point $p \in G$, the set of all points of G which can be joined with p by a continuum contained in G is called the *constituent* of the point p in the set G (see [9], p. 188). Denote the constituent of the point p in the set G by $C(G, p)$. We have the following

(3.1) THEOREM. *Let p be an arbitrary point of a continuum X. The following are equivalent*:

(i) *X is smooth at p*,

(ii) *for each open set G such that $p \in G$, the set $C(G, p)$ is open*,

(iii) *for each subcontinuum N of X such that $p \in N$ and for each open set V which contains N, there exists an open connected set U such that $N \subset U \subset V$*,

(iv) *for each subcontinuum N of X such that $p \in N$ and for each open set V which contains N there exists a continuum K such that $N \subset \operatorname{Int} K \subset K \subset V$.*

Proof. (i) implies (ii). Let X be smooth at p and let G be an open set such that $p \in G$. We have $C(G, p) = \{x \in X$: there exists a continuum R such that $p, x \in R$ and $R \cap (X \setminus G) = \emptyset\}$. We will prove that $C(G, p)$ is open. Let $x_1, x_2, \ldots$ be a sequence of points of $X \setminus C(G, p)$ which converges to x. Let R be an arbitrary subcontinuum of X which contains the points p and x. Since X is smooth at p, there exists a sequence $R_1, R_2, \ldots$ of subcontinua of X such that $p, x_n \in R_n$ for each $n = 1, 2, \ldots$ and $\operatorname{Lim}_{n\to\infty} R_n = R$. The points x_n are in $X \setminus C(G, p)$ for each $n = 1, 2, \ldots$; thus $R \cap (X \setminus G) \neq \emptyset$ by the definition of $C(G, p)$. Therefore $R \cap (X \setminus G) = \operatorname{Lim}_{n\to\infty} R \cap (X \setminus G) \neq \emptyset$. Hence, for each continuum R which contains p and x, we have $R \cap (X \setminus G) \neq \emptyset$, and thus $x \in X \setminus C(G, p)$. Consequently, $X \setminus C(G, p)$ is closed.

(ii) implies (iii). It is sufficient to take $U = C(V, p)$.

(iii) implies (iv). Let N be an arbitrary subcontinuum of X such that $p \in N$ and let V be an arbitrary open set such that $N \subset V$. Then there exists an open set G such that $N \subset G \subset \bar{G} \subset V$. It follows from (iii) that there exists an open connected set U such that $N \subset U \subset G$. Put $K = \bar{U}$. Obviously K is a continuum and $N \subset U \subset \operatorname{Int} K \subset K = \bar{U} \subset \bar{G} \subset V$, i.e., (iv) is satisfied.

(iv) implies (i). Let a sequence $x_1, x_2, \ldots$ of points of X be convergent to a point x, and let N be an arbitrary subcontinuum of X such that $p, x \in N$. Let $Q(N, \varepsilon)$ denote the union of open metric balls with the centres in the set N and with the radii $\varepsilon > 0$. According to (iv), for each $\varepsilon > 0$ there exists a continuum K_ε such that $N \subset \operatorname{Int} K_\varepsilon \subset K_\varepsilon \subset Q(N, \varepsilon)$. The sequence $x_1, x_2, \ldots$ is convergent to $x \in N$, and thus there exists a positive integer m such that $x_n \in \operatorname{Int} K_\varepsilon$ if $n > m$. This implies that we can choose a sequence $K_1, K_2, \ldots$ of subcontinua of X in such a way that p, $x_n \in K_n$ and $N \subset \operatorname{Int} K_n \subset K_n \subset Q(N, \varepsilon_n)$ for each $n = 1, 2, \ldots$, where the sequence $\varepsilon_1, \varepsilon_2, \ldots$ of positive numbers is convergent to zero. Then $\operatorname{Lim}_{n\to\infty} K_n = N$. Thus X is smooth at p. The proof of (3.1) is complete.

(3.2) COROLLARY. *A continuum X is locally connected at each point of $I(X)$.*

(3.3) COROLLARY. *A continuum X is locally connected if and only if it is smooth at each of its points*, i.e., *if $I(X) = X$.*

We recall here the notion of aposyndeticity of F. B. Jones (see [8], p. 404, and see also [4]). Namely, let X be a continuum and let x and y be distinct points of X; if X contains a continuum K such that $x \in \operatorname{Int} K \subset K \subset X \setminus \{y\}$, then X is said to be *aposyndetic at x with respect to y.*

We have, by Theorem (3.1) (iv), the following

(3.4) COROLLARY. *Let a continuum X be smooth at the point p, and let y be a point of X. If K is a subcontinuum of X such that $p \in K \subset X \setminus \{y\}$, then X is aposyndetic at each point $x \in K$ with respect to y.*

The condition of Corollary (3.4) does not characterize continua which are smooth at some point: there are continua which are not smooth at any point but which satisfy this condition for each point. This can be seen from the following

(3.5) EXAMPLE. Consider in the Euclidean plane endowed with the ordinary rectangular coordinate system Oxy the union A of all straight line intervals joining the point $(0, 1)$ with the points $(1/n, 0)$ for $n = 1, 2, \ldots$ and the point $(0, 1)$ with the points $(0, 0)$ and $(-1, 0)$. Let B be the image of A under the reflection through the line $y = 0$. Let $M = A \cup B \cup C$, where C is a straight line interval joining points $(0, 0)$ and $(-1, 0)$. Then M is a continuum aposyndetic at each point with respect to another point, but M is not smooth at any point. Observe also that the continuum $A \cup B$ is aposyndetic at each point with respect to another point and it is smooth at each point of its local connectivity.

It is proved that if a continuum X is hereditarily unicoherent at the point p and simultaneously smooth at p, then every indecomposable subcontinuum of X has a void interior (see [7], Corollary 3.3, p. 55). But this is not true if X is only smooth at p, this can be seen from the following.

(3.6) EXAMPLE. The continuum X will be considered as a subset of the Euclidean plane endowed with the ordinary rectangular coordinate system Oxy. The continuum X consists of

(i) all semi-circles with ordinates ≥ 0, with centre $(1/2, 0)$ and passing through every point of the Cantor ternary set C,

(ii) all semi-circles with ordinates ≤ 0, which have for $n \geq 1$ the centre at $(5/(2\cdot 3^n), 0)$ and pass through each point of the set C lying in the interval $2/3^n \leq x \leq 1/3^{n-1}$,

(iii) all points of the form $(x, 0)$ where $0 \leq x \leq 1$.

The set described by conditions (i) and (ii) is a well-known indecomposable continuum (see [9], § 48, V, Example 1, p. 204) and has a non-void interior in the continuum X which is smooth at each point $(x, 0)$, where $0 \leq x \leq 1$.

§ 4. Hereditary unicoherence at some point. In this section we consider continua which are hereditarily unicoherent at some point and which are smooth at another point. We will prove that those continua are also hereditarily unicoherent at the point at which they are smooth.

(4.1) LEMMA. *Let a continuum X be hereditarily unicoherent at a point p. For every subcontinuum K of X and for every point $x \in K$, the set $px \setminus K$ is connected.*

Proof. Let K be an arbitrary subcontinuum of X and let a point x belong to K. Then the set $px \cup K$ is a continuum. Suppose, on the contrary, that the set $px \setminus K$ is not connected. Therefore $px \setminus K$ is the union of two non-empty separated sets P and Q. Observe that $px \setminus K = (px \cup K) \setminus K$. Thus the continuum K separates the continuum $px \cup K$. It follows from Theorem 3 in [9], § 47, I, p. 168 that the sets $K \cup P$ and $K \cup Q$ are continua. The continuum K fails to contain the point p because otherwise $px \subset K$ and hence $px \setminus K$ would be empty; thus either $p \in P$ or $p \in Q$. If $p \in P$, then $K \cup P$ contains an irreducible continuum between p and x, because $x \in K$. Consequently, the continuum X being hereditarily unicoherent at the point p, we have $px \subset K \cup P$. Thereby, we see that $Q \subset px \subset K \cup P$, which contradicts the fact that $\emptyset \neq Q \subset (X \setminus K) \cap (X \setminus P)$. If $p \in Q$, we obtain a contradiction in a similar way. The proof of Lemma (4.1) is complete.

We also have (see [9], § 48, II, Theorem 3, p. 193) the following

(4.2) PROPOSITION. *Let X be an irreducible space between the points a and b, and let C be a closed connected set. If $a \in C$, then the set $X \setminus C$ is connected.*

(4.3) THEOREM. *Let a continuum X be hereditarily unicoherent at some point. If X is smooth at a point q, then X is hereditarily unicoherent at q.*

Proof. Let a continuum X be hereditarily unicoherent at a point p and let q be an arbitrary point of $I(X)$. We can obviously assume that $p \neq q$. We will show that for an arbitrary point x of X there exists a unique subcontinuum of X which is irreducible between q and x (cf. [7], Theorem 1.3, p. 52). Firstly we will prove that

(a) if $x \in pq$, then there exists in pq a unique subcontinuum irreducible between x and q.

Suppose, on the contrary, that $I(x,q)$ and $I'(x,q)$ are different continua irreducible between points x and q, and each of them is contained in pq. Since $I(x,q)\neq I'(x,q)$, there exist points z and z' such that $z\in I(x,q)\setminus I'(x,q)$ and $z'\in I'(x,q)\setminus I(x,q)$. Obviously $p\notin I(x,q)\cup \cup I'(x,q)$. Therefore there exists an open connected set V such that $I(x,q)\subset V\subset \overline{V}\subset X\setminus\{z',p\}$ by the smoothness of X at q and Theorem (3.1) (iii). The set $\overline{V}$ is a continuum, and thus the set $pq\setminus\overline{V}$ is connected by Lemma (4.1), and $p, z'\in pq\setminus\overline{V}$. Thereby $pz'\subset\overline{pq\setminus\overline{V}}$. Further, since $z\in I(x,q)\subset V$, we have $\overline{pq\setminus\overline{V}}\subset X\setminus\{z\}$. Consequently, $pz'\cup I'(x,q)$ is a continuum which joins p and q and it is contained in $pq\setminus\{z\}$, contrary to the irreducibility of pq between p and q.

(b) if $q\in px$, then there exists in px a unique subcontinuum which is irreducible between q and x.

Suppose, on the contrary, that $I(q,x)$ and $I'(q,x)$ are different continua both irreducible between points q and x, and both contained in px. Since $I(q,x)\neq I'(q,x)$, there exist points z and z' such that $z\in I(q,x)\setminus I'(q,x)$ and $z'\in I'(q,x)\setminus I(q,x)$. Obviously $p\notin I(q,x)\cup I'(q,x)$. Therefore there exists an open connected set V such that $I(q,x)\subset V\subset\overline{V}\subset X\setminus\{z',p\}$ by the smoothness of X at q and Theorem (3.1) (iii). The set $\overline{V}$ is a continuum; thus the set $px\setminus\overline{V}$ is connected by Lemma (4.1), and then $p, z'\in px\setminus\overline{V}$. Thereby $pz'\subset\overline{px\setminus\overline{V}}\subset X\setminus\{z\}$ as before. The set $pz'\cup I'(q,x)$ is a continuum which joins p and x, and lies outside z, contrary to the irreducibility of px between p and x.

Now consider three cases.

1'. $x\in pq$. We shall prove that there exists a unique subcontinuum of X which is irreducible between x and q. It follows from (a) that there exists a unique subcontinuum xq of pq which is irreducible between x and q. Suppose, on the contrary, that $I(x,q)$ is an irreducible subcontinuum of X between x and q which is different from xq. Therefore there exist a point z such that $z\in xq\setminus I(x,q)$. Obviously $p\notin I(x,q)$. Since X is smooth at q, we have, by Theorem (3.1) (iii), an open connected set V such that $I(x,q)\subset V\subset\overline{V}\subset X\setminus\{z,p\}$. It follows from Lemma (4.1) that the set $pq\setminus\overline{V}$ is connected; thus $\overline{pq\setminus\overline{V}}$ is a continuum. Moreover, $p\in\overline{pq\setminus\overline{V}}\subset pq$, and thus by Proposition (4.2) the set $\overline{pq\setminus\overline{pq\setminus\overline{V}}}$ is a continuum. Since $x,q\in\overline{pq\setminus\overline{pq\setminus\overline{V}}}\subset\overline{V}\cap pq$, we have $xq\subset\overline{V}$ by (a). But $z\in xq$, a contradiction.

2'. $q\in px$. We shall prove that there exists a unique subcontinuum of X which is irreducible between q and x. It follows from (b) that there exists a unique subcontinuum qx of px which is irreducible between q and x. Suppose, on the contrary, that $I(q,x)$ is an irreducible subcontinuum of X between q and x which is different from qx. Therefore there exists a point z such that $z\in qx\setminus I(q,x)$. Obviously $p\notin I(q,x)$. Since X is smooth at q, we infer, by Theorem (3.1) (iii), that there exists an open connected set V such that $I(q,x)\subset V\subset\overline{V}\subset X\setminus\{z,p\}$. It follows from Lemma (4.1) that the set $px\setminus\overline{V}$ is connected, thus $\overline{px\setminus\overline{V}}$ is a continuum. Moreover, $p\in\overline{px\setminus\overline{V}}\subset px$, and thus, by Proposition (4.2), the set $\overline{px\setminus\overline{px\setminus\overline{V}}}$ is a continuum. Since $q,x\in\overline{px\setminus\overline{px\setminus\overline{V}}}\subset\overline{V}\cap px$, we have $qx\subset\overline{V}$ by (b). But $z\in qx$, a contradiction.

3'. $x\in X\setminus pq$ and $q\in X\setminus px$. Let $I(q,x)$ be an arbitrary irreducible subcontinuum of X between points q and x. It follows from the hereditary unicoherence of X at p that, firstly, the intersection $px\cap pq$ is a continuum, and, secondly, the inclusion $px\subset I(q,x)\cup pq$ holds true. Therefore we have $px\setminus(px\cap pq)\subset I(q,x)$. The set $\overline{px\setminus(px\cap pq)}$ is a continuum by Proposition (4.2). Moreover, $x\in\overline{px\setminus(px\cap pq)}\subset I(q,x)$ and, by the connectedness of $px\cup pq$, we have $\overline{px\setminus(px\cap pq)}\cap pq\neq\emptyset$. Take a point $z\in\overline{px\setminus(px\cap pq)}\cap pq$. It follows from 1' that there exists exactly one subcontinuum zq of X which is irreducible between z and q; thus $zq\subset I(q,x)$. Therefore the set $\overline{px\setminus(px\cap pq)}\cup zq$ is a continuum containing points q and x and contained in $I(q,x)$. Then $\overline{px\setminus(px\cap pq)}\cup zq = I(q,x)$ by the irreducibility of $I(q,x)$ between q and x. This shows the uniqueness of $I(q,x)$. The proof of Theorem (4.3) is complete.

§ 5. Irreducible smooth continua. In this section we consider irreducible continua which are smooth at some point. We will prove that these continua are also hereditarily unicoherent at the point at which they are smooth. Some properties of irreducible smooth continua were studied by J. J. Charatonik in [2].

(5.1) LEMMA. *Let a continuum X be irreducible between a and b,* i.e., *$X=I(a,b)$, and let X be locally connected at the point p. If $a\neq p\neq b$, then there exist subcontinua $I(a,p)$ and $I(p,b)$ of X irreducible between a and p, p and b, respectively, and such that*

(i) *$I(a,p)\cap I(p,b)=X$, and*

(ii) *$I(a,p)\cap I(p,b)=\{p\}$.*

Proof. If we take a positive number ε such that

$$0<\varepsilon<\min\{\operatorname{dist}(a,p),\ \operatorname{dist}(p,b)\},$$

then, by the local connectedness of X at p, there exists an open connected set V_ε with diameter less than ε. Take the closure $\overline{V}_\varepsilon$ of V_ε. Then $\overline{V}_\varepsilon$ separates X in such a way that $X=A_\varepsilon\cup B_\varepsilon$, A_ε and B_ε are continua, $a\in A_\varepsilon$, $b\in B_\varepsilon$ and $A_\varepsilon\cap B_\varepsilon=\overline{V}_\varepsilon$. Observe that, if $0<\varepsilon<\varepsilon'$, then $A_\varepsilon\subset A_{\varepsilon'}$ and $B_\varepsilon\subset B_{\varepsilon'}$. We define $A=\bigcap_{\varepsilon>0}A_\varepsilon$ and $B=\bigcap_{\varepsilon>0}B_\varepsilon$. It is easily to check that $A\cap B=\{p\}$, $a\in A$ and $b\in B$, and A and B are continua. Therefore, if we take an irreducible continuum $I(a,p)$ between a and p which is contained in A, and an irreducible continuum $I(p,b)$ between p and b which is contained in B, then $I(a,p)$ and $I(p,b)$ satisfy (i) and (ii).

(5.2) Lemma. *Let a continuum $I(a,b)$ irreducible between a and b be the union of two subcontinua $I(a,p)$ and $I(p,b)$ irreducible between a and p and p and b, respectively, and such that $I(a,p)\cap I(p,b)=\{p\}$. Then*

(i) *if $x\in I(a,p)$, then each subcontinuum of $I(a,b)$ irreducible between x and p is contained in $I(a,p)$.*

(ii) *if $x\in I(p,b)$, then each subcontinuum of $I(a,b)$ irreducible between p and x is contained in $I(p,b)$.*

Proof. Let $x\in I(a,p)$, and let $I(x,p)$ be an arbitrary irreducible subcontinuum of $I(a,b)$ between x and p. Then $I(x,p)\setminus\{p\}$ is connected by Proposition (4.2), and it is contained in $I(a,p)\setminus\{p\}$, because p separates $I(a,b)$ by assumption. Therefore $I(x,p)=\overline{I(x,p)\setminus\{p\}}\subset I(a,p)$. If $x\in I(p,b)$ the proof is the same.

(5.3) Theorem. *Let a continuum X be irreducible between points a and b. If X is smooth at the point p, then X is hereditarily unicoherent at p.*

Proof. Let x be an arbitrary point of X. We will show that there exists a unique subcontinuum of X which is irreducible between p and x. Consider three cases.

1′. $a=p$. Thus we have $X=I(p,b)$. Suppose, on the contrary, that $I(p,x)$ and $I'(p,x)$ are different continua, both irreducible between points p and x. Since $I(p,x)\neq I'(p,x)$, there exist points z and z' such that $z\in I(p,x)\setminus I'(p,x)$ and $z'\in I'(p,x)\setminus I(p,x)$. Obviously $b\notin I(p,x)\cup I'(p,x)$. Therefore, by the smoothness of X at p and by Theorem (3.1) (iii), there exists an open connected set V such that $I(p,x)\subset V\subset\overline{V}\subset X\setminus\{z',b\}$. The set $\overline{V}$ is a subcontinuum of X and $p\in\overline{V}$; thus, by Proposition (4.2), the set $X\setminus\overline{V}$ is connected. Moreover, $z',b\in X\setminus\overline{V}$; hence the set $\overline{X\setminus\overline{V}}\cup I'(p,x)$ is a subcontinuum of X containing points p and b, and thus $X=I'(p,x)\cup\overline{X\setminus\overline{V}}$. Therefore $z\in I'(p,x)\cup\overline{X\setminus\overline{V}}$, a contradiction.

2′. $p=b$. The proof is the same as in 1′.

3′. $a\neq p\neq b$. Since X is smooth at p, thus, by Corollary (3.2), X is locally connected at p. Therefore, by Lemma (5.1) there exist two subcontinua $I(a,p)$ and $I(p,b)$ of X irreducible between a and p and between p and b, respectively, and such that $X=I(a,p)\cup I(p,b)$ and $I(a,p)\cap I(p,b)=\{p\}$. We can assume that the point x belongs to $I(a,p)$ (if $x\in I(p,b)$, the proof is the same). Then each subcontinuum of X irreducible between x and p is contained in $I(a,p)$ by Lemma (5.2). Suppose, on the contrary, that $I(x,p)$ and $I'(x,p)$ are different subcontinua of $I(a,p)$, both irreducible between points x and p. Since $I(x,p)\neq I'(x,p)$, there exist points z and z' such that $z\in I(x,p)\setminus I'(x,p)$ and $z'\in I'(x,p)\setminus I(x,p)$. Obviously $\{a,b\}\cap\big(I(x,p)\cup I'(x,p)\big)=\emptyset$. Therefore, by the smoothness of X at p and by Theorem (3.1) (iii), there exists an open connected set V such that $I(x,p)\subset V\subset\overline{V}\subset X\setminus\{a,b,z'\}$. The set $\overline{V}$ is a continuum, and $\{a,b\}\cap\overline{V}=\emptyset$; thus $X\setminus\overline{V}=G\cup H$, where G and H are open and connected, $a\in G$ and $b\in H$ (see [9], § 48, II, Theorem 3, p. 193). Moreover, $H\subset I(p,b)$ and $a,z'\in\overline{G}$. Therefore the set $I(p,b)\cup\overline{G}\cup I'(x,p)$ is a continuum containing points a and b, thus $I(p,b)\cup\overline{G}\cup I'(x,p)=X$ by the irreducibility of X between a and b. Consequently, $z\in I(p,b)\cup\overline{G}\cup I'(x,p)$, a contradiction. The proof of (5.3) is complete.

If an irreducible continuum X is locally connected at a point p only, without being smooth at p, then X need not be hereditarily unicoherent at p, i.e., the assumption of the smoothness of X at p in Theorem (5.3) is essential. This can be seen from the following

(5.4) Example. Let (x,y) denote a point of the Euclidean plane having x and y as its rectangular coordinate. Put $a_n=(1/n,1)$, $b_n=(1/n,-1)$ and $c_n=\big((n+1)/n,0\big)$, where $n=1,2,\ldots$ Consider the union A of straight line intervals joining sequentially points $c_1,a_1,a_2,c_2,b_1,b_2,c_3,a_3,a_4,c_4,b_3,b_4,c_5,\ldots$ and so on. Let B be the image of A under the reflection through the line $x=0$. The closure of $A\cup B$ is an irreducible continuum such that it is locally connected at each point of $A\cup B$ and it is not hereditarily unicoherent at any point.

§ 6. Mappings and the smoothness of continua.

Recall that a continuous mapping f of a topological space X onto a topological space Y is called *monotone* if, for any continuum Q in Y, the set $f^{-1}(Q)$ has only one component, i.e. if it is connected (see [9], p. 131)

(6.1) Lemma. *If G is an arbitrary subset of a continuum X, a point p is in G and a mapping $f\colon X\to f(X)$ is monotone, then*

$$f^{-1}\big(C(G,f(p))\big)=C\big(f^{-1}(G),p\big).$$

Proof. Let $x\in f^{-1}\big(C(G,f(p))\big)$, i.e., $f(x)\in C\big(G,f(p)\big)$. Thus there is a subcontinuum K of $f(x)$ which contains points $f(x)$ and $f(p)$ and is contained in G. The mapping f being monotone, the set $f^{-1}(K)$ is a continuum which contains the points x and p and is contained in $f^{-1}(G)$. This shows that $x\in C\big(f^{-1}(G),p\big)$. Conversely, let $x\in C\big(f^{-1}(G),p\big)$. This means that there is a subcontinuum L of X which contains the points x and p and is contained in $f^{-1}(G)$. Its image under f, the set $f(L)$, is a continuum which contains points $f(x)$ and $f(p)$ and is contained in $ff^{-1}(G)=G$. Thus $f(x)\in C\big(G,f(p)\big)$, whence $x\in f^{-1}\big(C(G,f(p))\big)$. The proof of the lemma is finished.

Note that the inclusion $C(f^{-1}(G,p))\subset f^{-1}(C(G,f(p)))$ holds for an arbitrary continuous mapping f.

(6.2) THEOREM. *If $f\colon X\to f(X)$ is a monotone mapping of a continuum X, then $f(I(X))\subset I(f(X))$.*

Proof. Let a continuum X be smooth at the point p. We will show that $f(p)$ is an initial point of $f(X)$. Suppose, on the contrary, that $f(X)$ is not smooth at $f(p)$. Then, by Theorem (3.1) (ii), there is an open subset G of $f(X)$ such that $f(p)\in G$ and the constituent $C(G,f(p))$ of the point $f(p)$ in the set G is not open. Therefore there exists a convergent sequence $y_1,y_2,\ldots$ of points of $f(X)\setminus C(G,f(p))$ such that its limit-point y belongs to $C(G,f(p))$. Since the mapping f is continuous, the set $f^{-1}(G)$ is open; moreover, $p\in f^{-1}(G)$, and thus the constituent $C(f^{-1}(G),p)$ of the point p in the set $f^{-1}(G)$ is open by the smoothness of X at p and by Theorem (3.1) (ii). Since the mapping f is monotone, we have $f^{-1}(C(G,f(p)))=C(f^{-1}(G),p)$, by Lemma (6.1). Thus $f^{-1}(y)\subset C(f^{-1}(G),p)$ and $f^{-1}(y_n)\cap C(f^{-1}(G),p)=\emptyset$ for each $n=1,2,\ldots$ Take for each $n=1,2,\ldots$ a point x_n of X such that $f(x_n)=y_n$. Let $x_{n_1},x_{n_2},\ldots$ be a convergent subsequence of the sequence $x_1,x_2,\ldots$ and let a point x be its limit. It follows from the continuity of f that $f(x)=y$, i.e., $x\in f^{-1}(y)$; thus $x\in C(f^{-1}(G),p)$. Further, $x_{n_1},x_{n_2},\ldots$ is a sequence of points in $X\setminus C(f^{-1}(G),p)$, and $\lim_{k\to\infty}x_{n_k}=x$. Therefore $C(f^{-1}(G),p)$ is not open, a contradiction. The proof of Theorem (6.2) is complete.

We say that a continuous mapping is *open* provided it transforms open sets into open sets (see [16], p. 348).

(6.3) THEOREM. *If $f\colon X\to f(X)$ is an open mapping of a continuum X, then $f(I(X))\subset I(f(X))$.*

Proof. Let a continuum X be smooth at a point p. We will show that $f(p)$ is an initial point of $f(X)$. It suffices to show, by Theorem (3.1) (iii) that for each continuum K such that $f(p)\in K$ and for each open set V containing K there exists an open connected set U such that $K\subset U\subset V$. Let K be an arbitrary subcontinuum of $f(X)$ such that $f(p)\in K$ and let V be an arbitrary open set such that $K\subset V$. Since the mapping f is open, we infer, by Whyburn's theorem (7.5) in [18], p. 148, that each component of the inverse image $f^{-1}(K)$ is mapped onto K under f. In particular, if we take a component C_0 of $f^{-1}(K)$ such that $p\in C_0$, then $f(C_0)=K$. Since $p\in C_0$ and C_0 is contained in the set $f^{-1}(V)$ which is open by the continuity of f, we infer, by the smoothness of X at p and Theorem (3.1) (iii), that there exists an open connected set G such that $C_0\subset G\subset f^{-1}(V)$. Since the mapping f is open, $f(G)$ is an open connected set, and $K=f(C_0)\subset f(G)\subset V$. Putting $f(G)=U$, we see that the proof of (6.3) is complete.

A continuous mapping $f\colon X\to f(X)$ is called *quasi-interior at a point* $y\in f(X)$ if, for each open set $U\subset X$ such that a component of $f^{-1}(y)$ is contained in U, we have $y\in \operatorname{Int}f(U)$. A continuous mapping $f\colon X\to f(X)$ is called *quasi-interior* if f is quasi-interior at each point of $f(X)$ (see [19], p. 9). It is proved in [10], Corollary 3.1, that a continuous mapping h is quasi-interior if and only if h is an OM-mapping, i.e. h can be represented as a composition of mappings f and g, by writing $h=gf$ (which means $h(x)=g(f(x))$ for each $x\in X$), where f is monotone and g is open. Therefore, by Theorem (6.2) and Theorem (6.3) we have the following

(6.4) COROLLARY. *If a mapping $f\colon X\to f(X)$ of a continuum X is quasi-interior, then $f(I(X))\subset I(f(X))$.*

Observe that Corollary (6.4) is a generalization of Corollary 6′ in [13]. Moreover, we obtain, by Theorem (4.3), the following

(6.5) COROLLARY. *If a continuum X is smooth at a point p, if a mapping $f\colon X\to f(X)$ is quasi-interior and if the continuum $f(X)$ is hereditarily unicoherent at some point, then $f(X)$ is smooth at $f(p)$ in the sense of Gordh.*

Recall that a continuous mapping $f\colon X\to f(X)$ of a topological space X is called a *local homeomorphism* if for each point x of X there exist an open neighbourhood U of x such that $f(U)$ is a neighbourhood of $f(x)$ and that $f|U$ is a homeomorphism (see e.g. [18], p. 199). We have the following

(6.6) COROLLARY. *If $f\colon X\to f(X)$ is a local homeomorphism of an irreducible smooth continuum X, then the continuum $f(X)$ is irreducible and smooth, and $f(I(X))\subset I(f(X))$.*

In fact, if the continuum X is irreducible and smooth at p, then, by Theorem (5.3), X is hereditarily unicoherent at p. Thus it is smooth in the sense used in [2] and therefore it is of type λ (see [2], Proposition 1). Hence Theorem 6 in [15], p. 73 can be applied, and thereby we see that if f is a local homeomorphism of X, then $f(X)$ is an irreducible continuum. Since each local homeomorphism is an open mapping (see [15], Remark 1, p. 70), we infer, by Theorem (6.3), that $f(X)$ is smooth at $f(p)$.

Corollary (6.6) is a positive solution of the problem set by J. J. Charatonik in the Proc. International Symposium in Topology, Budva (Yugoslavia) 1972.

A continuous mapping f of a topological space X onto a topological space $f(X)$ is *confluent* if, for every subcontinuum Q of $f(X)$, each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q (see [1], p. 213). The class of confluent mappings comprises open mappings (see [1], VI, p. 214), monotone mappings (see [1], V, p. 214) and quasi-interior mappings (see [10], Corollary 2.7). As we can see by Theorem (6.2), Theorem (6.3) and Corollary (6.4), a monotone (or open, or quasi-interior) image of a smooth continuum is a smooth continuum. Moreover, a monotone (or open, or quasi-interior) image of an initial point of a continuum X is an initial point of its image. It is known (see [3], p. 309) that if a continuum X is smooth at p, and $f\colon X\to f(X)$ is a confluent mapping, then $f(p)$ need not be an initial point of $f(X)$. In spite of this, one can ask whether a confluent image of a smooth continuum is also a smooth continuum. The answer is negative. This can be seen from the following

(6.7) Example. Consider the following subsets of the Euclidean plane endowed with the ordinary rectangular coordinate system Oxy.

$$A=\left\{\left(x,-\sin\frac{\pi}{x}\right)\colon -1\leqslant x<0\right\}\cup\{(0,y)\colon -1\leqslant y\leqslant 1\}\,,$$

$$B=\left\{\left(x,-\sin\frac{\pi}{x+2}\right)\colon -2<x\leqslant -1\right\}\cup\{(-2,y)\colon -1\leqslant y\leqslant 1\}\,,$$

$$C_n=\left\{\left(x,\sin\frac{\pi}{2-2^{n+1}x}\right)\colon \frac{1}{2^{n+1}}\leqslant x<\frac{1}{2^n}\right\}\cup\left\{\left(\frac{1}{2^{n+1}},y\right)\colon -1\leqslant y\leqslant 1\right\},$$

where $n=1,2,\ldots$

Put $C=\bigcup_{n=1}^{\infty}C_n$. The union $X=A\cup B\cup C$ is a continuum and it is smooth at each point $(x,y)\in X$ such that $-2<x<0$. We define a confluent mapping f of X as follows

$$f(x,y)=\begin{cases}(0,y) & \text{if}\quad x<0\,,\\ (x,y) & \text{if}\quad x\geqslant 0\,.\end{cases}$$

The image of X under f is the continuum C, i.e., $f(X)=C$, which is not smooth at any point.

This example shows that a confluent image of a continuum which is hereditarily unicoherent at each point, hereditarily decomposable, irreducible and smooth need not be smooth. One can observe that the mapping f in this example is quasi-monotone (for the definition see [17], p. 136, cf. also [18], p. 151). Therefore, however quasi-monotone mappings preserve the irreducibility of continua of type λ (see [6]), they do not preserve the smoothness of such continua.

Recall that a dendroid means a continuum which is hereditarily unicoherent at each point and is arcwise connected. It is asked in [3], p. 310, whether a confluent image of a smooth dendroid is a smooth dendroid. Since confluent mappings do preserve the property to be a dendroid ([1], Corollary 1, p. 219), the question concerns the preserving of smoothness only. The answer is affirmative, as it was recently proved (see [14], Corollary 3.4).

We have the following

(6.8) Theorem. *If a mapping $f\colon X\to f(X)$ of a continuum X is confluent, then $f(X)\setminus I(f(X))\subset f(X\setminus I(X))$.*

Proof. Assume, on the contrary, that $q\in f(X)\setminus I(f(X))$ and $f^{-1}(q)\subset I(X)$. Therefore, by Theorem (3.1) (iii), there exist a continuum K such that $q\in K\subset f(X)$ and an open set V containing K and such that each connected set U with property $K\subset U\subset V$ is not open. Take $p\in f^{-1}(q)\subset X$. Then $p\in f^{-1}(V)$. Since the image $C(f^{-1}(V),p)$ under f contains the point $q=f(p)$, the union $U=\bigcup\{f(C(f^{-1}(V),p))\colon p\in f^{-1}(q)\}$ is connected. Further, let M be a component of $f^{-1}(K)$ which contains the point p. Thus M is contained in $C(f^{-1}(V),p)$, whence, by the confluence of f, we have $K=f(M)\subset f(C(f^{-1}(V),p))\subset U\subset V$. By the assumption the set U is not open. Hence, there exists a sequence $y_1,y_2,\ldots$ of points of $f(X)$ such that $y_n\in f(X)\setminus U$ for each $n=1,2,\ldots$ which is convergent to a point $y\in U$. Write $A=\bigcup\{C(f^{-1}(V),p)\colon p\in f^{-1}(q)\}$. This implies that $f^{-1}(y)\subset A$. Indeed, it follows from $y\in U$ that there is a point $x'\in C(f^{-1}(V),p')$ for some $p'\in f^{-1}(q)$ with $f(x')=y$. Therefore there is a continuum L such that $x',p'\in L\subset C(f^{-1}(V),p')$. Putting $N=f(L)$ we have a continuum N with $q,y\in N\subset U$. Let $x\in X$ be such that $f(x)=y$. Since $y\in N\subset U\subset V$, we have $f^{-1}(y)\subset f^{-1}(N)\subset f^{-1}(V)$, i.e. $x\in f^{-1}(N)\subset f^{-1}(V)$. Take a component C of the set $f^{-1}(N)$ such that $x\in C$. The component C is a continuum contained in V; moreover, by the confluence of f, there exists a point $p\in C$ such that $f(p)=q$, and thus $x\in C(f^{-1}(V),p)$.

It follows from $y_n\in f(X)\setminus U$ that $f^{-1}(y_n)\subset X\setminus f^{-1}(U)\subset X\setminus A$. Take a convergent subsequence $x_1,x_2,\ldots$ of points of X such that $f(x_k)=y_{n_k}$ and put $x=\lim_{k\to\infty}x_k$. By the continuity of f we have $x\in f^{-1}(y)\subset A$. The sets $C(f^{-1}(V),p)$ are open for each $p\in f^{-1}(q)\subset I(X)$ by the smoothness of X at p, see Theorem (3.1) (ii). Thus the set $X\setminus A$ is closed. Since this set contains points x_k which converge to the point x, we conclude that $x\in X\setminus A$, a contradiction.

(6.9) Example. Put in the Cartesian coordinates in the plane $a_n=(2^{-n},-1)$, $b_n=(2^{-n},0)$, $c_n=(3\cdot 2^{-(n+1)},0)$ and $d_n=(3\cdot 2^{-(n+1)},1)$. Join consecutively a_n,b_n,c_n,d_n, and the point $(0,2)$ by straight line segment and take the closure of the union of polygonal lines obtained in this way. The resulting continuum M (homoemorphic with the well-known harmonic fan) is not smooth only at the point of the straight segment joining points $(-1,0)$ and $(2,0)$. Define $f(x,y)=(x,|y|)$ for each point $(x,y)\in M$. The image $f(M)$ is a continuum which is not smooth at any point.

Observe that the conclusion of Theorem (6.8) is not satisfied for this example; therefore the assumption of the confluence of f in this theorem is essential.

A continuous mapping f of a topological space X onto a topological space Y is said to be *semi-confluent* if for every subcontinuum Q of Y and for each two components C_1 and C_2 of the inverse image $f^{-1}(q)$ either $f(C_1)\subset f(C_2)$ or $f(C_2)\subset f(C_1)$ (see [12]). Observe further that the mapping f defined in Example (6.9) is semi-confluent. Thus, if $f\colon X\to f(X)$ is semi-confluent, then the condition $f(X)\setminus I(f(X))\subset f(X\setminus I(X))$ need not be satisfied.

Recall that, if we denote by $L(X)$ the set of all points of a continuum X at which X is locally connected, then for each continuous mapping $f\colon X\to f(X)$ we have $f(X)\setminus L(f(X))\subset f(X\setminus L(X))$ (see [5], (3), p. 28). This inclusion resembles that of Theorem (6.8).

§ 7. Some remarks on the heredity of smoothness. It is easy to see that smoothness is not hereditary. Moreover, if a continuum X is smooth at p, then X can contain a subcontinuum K such that $p\in K$ and K is not smooth at any point. However, if X is hereditarily unicoherent at some point and it is smooth at the point p, then each subcontinuum containing p is smooth at p.

We can introduce the following definition. A continuum X is said to be *hereditarily smooth* at p if each subcontinuum of X which contains p is smooth at p.

(7.1) Corollary. *If a continuum X is hereditarily unicoherent at some point and X is smooth at p, then X is hereditarily smooth at p.*

In fact, according to the assumption, we infer that X is hereditarily unicoherent at the point p by Theorem (4.3). Therefore the continuum X is smooth at p in the sense of Gordh (see [7], p. 52, cf. § 2 here); thus each subcontinuum of X which contains the point p is smooth at p by (2.8) in [11], i.e., X is hereditarily smooth at p.

Recall that a continuum is hereditarily locally connected if each subcontinuum of it is locally connected (see [9], p. 268). It follows from Corollary (3.3) that

(7.2) Corollary. *A continuum X is hereditarily smooth at each point if and only if X is hereditarily locally connected.*

We also have

(7.3) Theorem. *If an arcwise connected continuum X is hereditarily smooth at some point, then each subcontinuum of X is smooth.*

Proof. Let an arcwise connected continuum X be hereditarily smooth at p and let K be an arbitrary subcontinuum of X. Take an arc pz such that $pz\cap K=\{z\}$. We will show that K is smooth at z. Let $x_1, x_2, \ldots$ be a convergent sequence of points of K and let $x=\lim_{n\to\infty} x_n$. Let Q be a subcontinuum of K such that $x, z\in Q$. X being hereditarily smooth at p, the continuum $pz\cup K$ is smooth at p. Therefore there is a sequence $K_1, K_2, \ldots$ of subcontinua of $pz\cup K$ such that $x_n, p\in K_n$ for each $n=1,2,\ldots$ and $\operatorname{Lim}_{n\to\infty} K_n=Q\cup pz$ by the definition of smoothness of $pz\cup K$ at p. We define $Q_n=K\cap K_n$. Obviously Q_n is a continuum for each $n=1,2,\ldots$, and $x_n, z\in Q_n$ for each $n=1,2,\ldots$, and, moreover, $\operatorname{Lim}_{n\to\infty} Q_n=Q$. The proof of (7.3) is complete.

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