# Closed retractions of Euclidean spaces

by

**Krzysztof Nowiński** (Warszawa)

**Abstract.** The problem of the characterization of the images of the Euclidean spaces under closed retractions is studied. The $c$-retract of the space $X$ is defined as the image of $X$ under some closed retraction. The following theorems are proved:

THEOREM 1. *Every compact retract of $E^n$ is the $c$-retract of $E^n$.*

THEOREM 2. *For every non-compact $c$-retract $R$ of $E^n$ $H^m(\omega R) = 0$ for $m = 1, \ldots, n-1$ and if $R \neq E^n$ then $H^n(\omega R) = 0$. ($H^m(X)$ denotes the $m$-th Čech cohomology group of $X$ and $\omega X$ denotes the one-point (Alexandrow) compactification of $X$).*

THEOREM 3. *The retract $R$ of the Euclidean plane $E^2$ is the $c$-retract of $E^2$ if and only if it does not disconnect $E^2$.*

The main purpose of this paper is to apply some methods investigated in [6] to the study of closed retractions. The paper gives some results about closed retractions of Euclidean spaces, particularly a complete characterization of all subsets $R$ of the Euclidean plane $E^2$ for which there exists a closed retraction $r: E^2 \to R$.

All notions and notations which are not defined here are taken from [1] and [2].

DEFINITION. The $c$-retract of the space $X$ is the *subset of $X$* which is the image of $X$ under some closed retraction.

PROPOSITION. *Let $R$ be a compact retract of $E^n$ for some $n$. Then $R$ is the $c$-retract of $E^n$.*

Proof. The set $R$ is an absolute retract in the sense of ([1], Sec. V. 1). On the other hand, $R \subset K(O, r)$ for some positive $r$. (We denote by $O$ the element $(0, \ldots, 0)$ of $E^n$). We denote by $S(r)$ the sphere obtained by matching to a point the set $E^n \setminus K(O, r)$. It is clear that $S(r)$ is a compact metric space and the quotient mapping $s: E^n \to S(r)$ is closed (see [6], Proposition 1). Simultaneously there exists a retraction $r_0: S(r) \to R$ which is closed since $S(r)$ is compact. The composition $r = r_0 \circ s$ is the desired closed retraction.

We denote by $H^m(X)$ the $m$th Čech cohomology group with integer coefficients of the space $X$. We can now prove the first main result of this paper.

THEOREM. *For every non-compact c-retract of* $E^n$, $H^i(\omega R) = 0$ *for* $i \leqslant n-1$ *and if* $R \neq E^n$ *then* $H^n(\omega R) = 0$.

Proof. If $n = 1$, then there is nothing to prove. Suppose now that $n > 1$, then, by ([6], Corollary to Theorem 7) we have $\gamma E^n = \omega E^n = S^n$. The closed retraction $r\colon E^n \to R$ can be extended to $\gamma r\colon \gamma E^n \to \gamma R$. (It is possible by ([5], Theorem 4).) The mapping $\gamma r$ is an epimorphism and hence $\gamma r(\gamma E^n \setminus E^n) = \gamma R \setminus R$, which means that $\gamma R = \omega R$. We can therefore obtain a mapping $s\colon S^n \to \omega R$ as the composition of the homeomorphism $i\colon S^n \to \gamma E^n$ and the mapping $\gamma r$. Let $j\colon \omega R \to S^n$ be the extension of the identity map of $R$ into $E^n$. Since $s \circ j = \mathrm{id}_R$, the mapping $(s \circ j)^m = j^m \circ s^m$ is the identity on $H^m(\omega R)$ for every $m$. So $s^m\colon H^m(\omega R) \to H^m(S^n)$ is a monomorphism, and since $H^m(S^n) = 0$ for $m \leqslant n-1$, the groups $H^m(\omega R)$ must vanish for $m \leqslant n-1$.

On the other hand, assuming that $R \neq E^n$, we can easily check that $\omega R$ is a proper subset of $S^n$ and hence $H^n(\omega R) = 0$.

COROLLARY. *If* $1 \leqslant k < n$, *then* $E^k$ *cannot be a c-retract of* $E^n$.

Proof. $\omega E^k = S^k$, hence $H^k(\omega E^k) \neq 0$ and it remains to apply Theorem 1.

In the case $n = 2$ we prove the following

LEMMA 1. *If* $R$ *is a c-retract of* $E^2$, *then* $R$ *does not disconnect* $E^2$.

Proof. If $R$ is compact, then our lemma follows from ([1], Theorem V.13.1). Suppose now that $R$ is not compact. The closure of $R$ in $S^n = \omega E^n$ is homeomorphic to $\omega R$ and $S^2 \setminus \omega R = E^2 \setminus R$. This means that if $E^2 \setminus R$ is not connected then so is $S^2 \setminus \omega R$. Applying to the pair $(S^2, \omega R)$ the Borsuk Theorem ([4], Theorem XI.3.t), we obtain an essential mapping $f\colon \omega R \to S^1$ (This means that $f$ is not homotopical to a constant map). Therefore we infer that the Brushlinski group $\pi^1(\omega R)$ is not trivial. It remains now to observe that $\pi^1(\omega R) = H^1(\omega R)$ (see [3], Theorem II.7.1), hence $H^1(\omega R) \neq 0$, which is impossible in connection with Theorem 1. This contradiction ends the proof.

We can now formulate the main theorem of this paper.

THEOREM 2. *The retract* $R$ *of the Euclidean plane* $E^2$ *is the c-retract of* $E^2$ *if and only if it does not disconnect* $E^2$.

Proof. The necessity of this condition is a consequence of Lemma 1 and, if $R$ is compact, then the condition is sufficient by Proposition 1. So it remains to prove that if $R$ is a non-compact retract of $E^2$ and $E^2 \setminus R$ is connected, then $R$ is a $c$-retract of the plane. To prove this, we formulate and prove four lemmas.

LEMMA 2. *Let* $f\colon X \to Y$ *be a continuous mapping and let* $Y$ *be a* $T_2$*-space. If there exists such a covering* $\mathcal{A} = \{A_s\}_{s \in S}$ *of* $X$ *by compact sets that* $f(\mathcal{A}) = \{f(A_s)\}_{s \in S}$ *is a locally finite collection, then the mapping* $f$ *is closed.*



Proof. Let $D$ be a closed subset of $X$. The sets $D_s = D \cap S_s$ are compact for every $s \in S$ and hence the sets $f(D_s)$ are closed in $Y$. So $f(D) = \bigcup_{s \in S} f(D_s)$ is the sum of a locally finite family of closed sets and hence $f(D)$ is closed.

We now introduce the following notations:

(i) $P_s^r = \overline{K}_r \setminus K_s$, where $s < r$ and $K_r = K\big((0,0), r\big) \subset E^2$,

(ii) if $A = \overline{A} \subset E^2$ then $K(A, r) = \bigcup_{x \in A} K(x, r)$,

(iii) if $(\varepsilon) = \varepsilon_1, \varepsilon_2, \ldots$ is a sequence of positive numbers, then we write $K\big(A, (\varepsilon)\big) = \bigcup_{n=1}^{\infty} K(A \cap P_{n-1}^n, \varepsilon_n)$,

(iv) if $A \subset E^2$, then $\mathrm{Fi}(A)$ (the filling of $A$) is the sum of $A$ and all the bounded components of the set $E^2 \setminus A$.

It is easy to check that

(i) $\mathrm{Fi}\big(\mathrm{Fi}(A)\big) = \mathrm{Fi}(A)$ for any $A \subset E^2$,

(ii) if $A = \overline{A}$ then $\mathrm{Fi}(A) = \overline{\mathrm{Fi}(A)}$,

(iii) if $A \subset B$, then $\mathrm{Fi}(A) \subset \mathrm{Fi}(B)$.

LEMMA 3. *If* $Z$ *is a compact subset of the plane not disconnecting* $E^2$, *then for every* $\varepsilon > 0$ *there exists an* $\eta\big(Z, (\varepsilon)\big) > 0$ *such that* $\overline{\mathrm{Fi}\Big(K\big(Z, \eta(Z, \varepsilon)\big)\Big)} \subset K(Z, \varepsilon)$.

The proof is an easy modification of the proof of ([1], Lemma V. 3.2) and will be omitted.

LEMMA 4. *If* $R$ *is a retract of* $E^2$, *then for every* $r, s$ *such that* $0 < r < s$ *only finitely many components* $W_1, \ldots, W_n$ *of* $P_r^s \setminus R$ *intersect simultaneously* $S_r$ *and* $S_s$.

Proof. Let us fix one of such components, $W_0$. It contains an arc $l_0$ joining $S_r$ and $S_s$. Now, if $W_k \neq W_0$ then $l_0 \cup W_k$ disconnects $P_r^s$ and the components $V_1$ and $V_2$ of $P_r^s \setminus (l_0 \cup W_k)$ both contain points of $R$. Denoting by $A_k$ the set $\overline{W}_k \cap S_{(r+s)/2}$, we can observe that if $r\colon E^2 \to R$ is the retraction, then $r(A_k) \setminus P_r^s \neq \emptyset$. In fact, if $r(A_k) \subset P_r^s$ then, since $A_k \cap V_1 \neq \emptyset \neq A_k \cap V_2$, there exists some arc joining $V_1$ and $V_2$ and contained in $R \cap P_r^s$, which contradicts the definition of $V_1$ and $V_2$. So there exists a point $a_k \in W_k \cap r^{-1}(E^2 \setminus P_r^s) \cap S_{(r+s)/2}$.

Assume now that there exist infinitely many components $W_n$ of $P_r^s$ joining $S_s$ and $S_r$. We can easily repeat the construction of the point $a_k$, and denoting by $a$ the accumulation point of the set $A = \{a_1, a_2, \ldots\}$ we obtain:

(i) $a \in R$, since $S_{(r+s)/2} \setminus R$ is open in $S_{(r+s)/2}$ and the points $a_n$ are from disjoint components of the set $S_{(r+s)/2} \setminus R$.

(ii) $r(a) \notin \mathrm{Int} P_r^s$, since $r(a) \subset \overline{r(A)}$ and $r(A) \cap P_r^s = \emptyset$.

The contradiction between (i) and (ii) establishes our lemma.

LEMMA 5. *Let $R$ be a non-compact retract of $E^2$ which does not disconnect the plane and let $(\varepsilon) = \varepsilon_1, \varepsilon_2, \ldots$ be a strictly decreasing sequence of positive real numbers such that $\varepsilon_1 < 1/16$. Then there exists a sequence $(\eta) = \eta_1, \eta_2, \ldots$ of positive reals such that $\overline{\mathrm{Fi}\big(K(R, (\eta))\big)} \subset K\big(R, (\varepsilon)\big)$.*

Proof. We define inductively the sequence of sets $X_1 \subset X_2 \subset \ldots \subset E^2$ and the sequence of positive real numbers $(\eta) = \eta_1, \eta_2, \ldots$ satisfying the following conditions:

(i) $R \subset R \cup \bigcup_{m=1}^{n} K(R \cap K_m, \eta_m) \subset X_n \subset K\big(R, (\varepsilon)\big)$,

(ii) $X_n = \mathrm{Fi}(X_n)$,

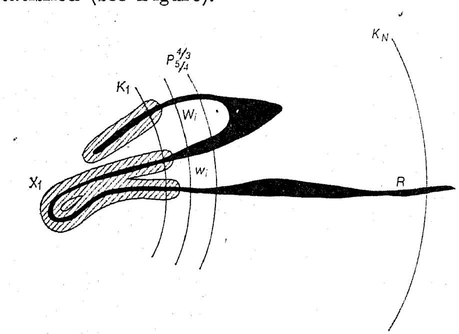
(iii) $X_n \setminus X_{n-1} \subset \mathrm{Int}\, P^{n+1/3}_{n-4/3}$,

(iv) $X_n \setminus R \subset K_{n+1/3}$,

(v) $X_n \setminus K_{n-1/3} \subset K(R, \varepsilon_{n+1}/2)$

for $n = 1, 2, \ldots$

In the first step we construct the set $X_1$. Let $W_1, \ldots, W_k$ be the components of $P^{4/3}_{5/4} \setminus R$ intersecting both $S_{4/3}$ and $S_{5/4}$ (see Lemma 4). We select for $i = 1, \ldots, k$ points $w_i \in W_i$ and we put $\xi_1 = \min\big(d(w_i, R)\big)$. Now, let $V_i$ be the component of $E^2 \setminus (K_{5/4} \cup R)$ containing $W_i$. All bounded sets from the family $V_1, \ldots, V_k$ are contained in some $K_N$ ($N \geqslant 3$). We put $\eta_1 = \eta(R \cap K_N, \min(\xi_1, \varepsilon_2/2))$, where $\eta$ is defined as in Lemma 3. The set $X = \overline{\mathrm{Fi}\big(K(R \cap K_1, \eta_1) \cup R\big)}$ satisfies the conditions (i), (ii), (iv) and (v) given above, and if we put $X_0 = \emptyset$ then condition (iii) is also satisfied (see Figure).




To verify this, we notice first that $\eta_1 < \varepsilon_2 < 1/16$. Hence

$$K(R \cap K_1, \eta_1) \cup R \subset K_{5/4} \cup R;$$

on the other hand, if $A$ is a bounded component of $E^2 \setminus \overline{K(R \cap K_1, \eta_1) \cup R}$ then $A$ cannot intersect both $S_{4/3}$ and $S_{5/4}$, which follows from the definition of $\eta_1$. Thus conditions (iii) and (iv) are satisfied. Condition (ii) is satisfied by the property (i) of the operation Fi. Conditions (i) and (v) are also easy to check.



We now assume that we have defined the sets $X_1, \ldots, X_{n-1}$ and the numbers $\eta_1, \ldots, \eta_{n-1}$ satisfying conditions (i)-(v). We define $X_n$ and $\eta_n$ as follows:

Let $W_1, \ldots, W_k$ be the components of $P^{n+1/3}_{n+1/4} \setminus R$ joining $S_{n+1/3}$ and $S_{n+1/4}$. As above, we select $w_i \in W_i$ and we put $\xi_1 = \min\big(d(w_i, X_{n-1})\big)$. Now, let $N \geqslant n+2$ be such a number that all bounded components of $E^2 \setminus (R \cup K_{n+1/4})$ are contained in $K_N$. Observe now that only a finite number of the components $G_i$ of $P^{n-5/4}_{n-4/3} \setminus X_{n-1}$ intersect both $S_{n-5/4}$ and $S_{n-4/3}$. Using similar arguments as in the proof of Lemma 4 we can check that the number of such components is not greater than the number of the components of $P^{n-5/4}_{n-4/3} \cap X_{n-1}$ joining $S_{n-5/4}$ and $S_{n-4/3}$. But every such component contains at least one of the points of $R$, and so it contains some ball of diameter $\eta_{n-1}/2$, and those balls are mutually disjoint.

We select the points $g_i \in G_i$ for every $i$ and we put $\xi_2 = \min\big(d(g_i, X_{n-1})\big)$. We define $\eta_n = \eta\big(X_{n-1} \cap K_N, \min(\xi_1, \xi_2, \varepsilon_{n+1}/2)\big)$ and we denote $X_n = \overline{\mathrm{Fi}\big(X_{n-1} \cup K(P^n_{n-1} \cap X_{n-1}, \eta_n)\big)}$.

We now prove that the set $X_n$ satisfies conditions (i)-(v). The first part of (i) and (ii) follow immediately from the definition of $X_n$. Notice now that all of the bounded components of $E^2 \setminus \big(X_{n-1} \cup K(P^n_{n-1} \cap X_{n-1}, \eta_n)\big)$ are contained in $P^{n+1/3}_{n-4/3}$, which follows from the definitions of $N$, $\xi_1$ and $\xi_2$. Thus conditions (iii) and (iv) are satisfied. Condition (v) and the second inclusion of (i) are now easy to verify. This finishes the description of the inductive step.

We define $X = \bigcup_{n=1}^{\infty} X_n$. The set $X$ satisfies the following conditions:

(vi) $K\big(R, (\eta)\big) \subset X \subset K\big(R, (\varepsilon)\big)$,

(vii) $\mathrm{Fi}(X) = \overline{X} = X$.

Condition (vi) follows from (i). The set $X$ is closed as the sum of the locally finite family of closed sets $X_n \cap P^{n+1}_{n-2}$. Now, let $U$ be a bounded component of the set $E^2 \setminus X$. So $U \subset K_N$ for some $N$ and, since $X \cap K_N = X_{N+1} \cap K_N$, we have $U \subset \mathrm{Fi}(X_{N+1})$, which is impossible as $X_{N+1} = \mathrm{Fi}(X_{N+1})$.

This finishes the proof of Lemma 5.

We can now return to the proof of Theorem 2.

Let $r\colon E^2 \to R$ be some retraction. If $R = E^2$, then there is nothing to prove; hence we can assume that $R \neq E^2$. We prove that there exists a closed set $P$ satisfying the following conditions:

(i) $R \subset P$,

(ii) $r|P$ is closed,

(iii) there exists a homeomorphism $h\colon E^2 \to E^2$ such that $h(P)$ is a closed half-plane.

Let $(\varepsilon) = \varepsilon_1, \varepsilon_2, \dots$ be a sequence of positive numbers such that

(iv) $\varepsilon_1 < 1/16$,

(v) $\varepsilon_n < \varepsilon_{n-1}$,

(vi) if $x, y \in K_{n+1}$ and $\varrho(x, y) \leqslant \varepsilon_n$ then $\varrho(r(x), r(y)) < 1$.

Such a sequence exists because of the uniform continuity of the mapping $r$ on every closed ball. The sequence satisfies the assumptions of Lemma 5 and hence there exists a sequence $(\eta) = \eta_1, \eta_2, \dots$ such that $\mathrm{Fi}(K(R, (\eta))) \subset K(R, (\varepsilon))$.

Let $\mathfrak{T}$ be such a locally finite triangulation of the plane that

(vii) if $\sigma \in \mathfrak{T}$ and $\sigma \subset E^2 \setminus K_{n-1}$, then $\operatorname{diam} \sigma < \eta_n$,

(viii) if $\sigma \in \mathfrak{T}$ is a two-dimensional closed simplex and $\sigma \cap R \neq \emptyset$, then $\operatorname{Int}\sigma \cap R \neq \emptyset$.

We put as $P'$ the sum of all two-dimensional closed simplexes from $\mathfrak{T}$ intersecting $R$ and let $P = \mathrm{Fi}(P')$. It follows from the definition of $(\eta)$ and $\mathfrak{T}$ that $P \subset K(R, (\varepsilon))$. Let us observe now that $\mathrm{Fr}(P)$ is a simple broken line. In fact, let us assume that $a$ is a point of self-intersection of $\mathrm{Fr}(P)$. This means that $a$ is a common vertex of at least two two-dimensional simplexes of $\mathfrak{T}$ and $(P \cap \mathrm{st}(a, \mathfrak{T})) \setminus \{a\}$ is not connected. It can easily be checked, by using (viii) that both components of $(P \cap \mathrm{st}(a, \mathfrak{T})) \setminus \{a\}$ must contain points from $R$. Let $p, q$ be nearlest to $a$ such points. It is clear that $p \neq a \neq q$. Let $k = \overline{a, p} \cup \overline{a, q}$ and $l = k \cup r(k) \subset P$. It is clear that the set $l$ disconnects the plane and that one of the components of $E^2 \setminus l$ is bounded and contains some non-void component of $E^2 \setminus P$, which is impossible, since $P = \mathrm{Fi}(P)$.

We now prove that $P$ does not disconnect the plane. Assuming the contrary, we denote by $U, V$ the components of $E^2 \setminus P$. Since $E^2 \setminus R$ is connected, there exists an arc $l$ joining $U$ and $V$, disjoint with $R$. Let $\varepsilon > 0$ be such a number that $K(l, \varepsilon) \cap R = \emptyset$. We can easily check, using similar arguments as above, that $l$ disconnects $P$ and both components of $P \setminus l$ are unbounded. Let $N$ be such an integer that $\eta_N < \varepsilon/2$ and $l \subset K_N$. Denoting by $K, L$ the two unbounded components of $P \setminus l$ we obtain that $K \setminus K_N \neq \emptyset \neq L \setminus K_N$. Moreover, $P' \cap (K \setminus K_N) \neq 0 \neq P' \cap (L \setminus K_N)$. Let $p \in P' \cap (K \setminus K_N)$ and $q \in P' \cap (L \setminus K_N)$. It follows from the definition of $P'$ that there exist $x, y \in R$ such that $\varrho(p, x) < \eta_N$, $\varrho(q, y) < \eta_N$, $\overline{p, x} \cup \overline{q, y} \subset P'$. Hence $R \cap K \neq \emptyset \neq R \cap L$ and, since $R \cap l = \emptyset$, $R$ is not connected. This contradiction finishes the proof of the fact that $E^2 \setminus P$ is connected.

We now fix a one-dimensional simplex $\sigma_0 = \overline{a_0, a} \in \mathfrak{T}$ and we define a homeomorphical embedding $f\colon E \to \mathrm{Fr}(P)$ as follows:

Let $f|[0, 1]$ be a linear mapping onto $\sigma_0$. Assume now that we have defined the mapping $f$ on the segment $[k, 1]$ and let $a_k = f(k)$ and $a_1 = f(l)$ be the endpoints of the broken line $f([k, l])$. It is clear that there exist two one-dimensional simplexes $\sigma_k$ and $\sigma_l$ from $\mathrm{Fr}(P) \setminus f((k, l))$ such that $a_k \in \sigma_k$, $a_l \in \sigma_l$. We can extend $f$ over $[k-1, l+1]$, putting on the segments $[k-1, k]$ and $[l, l+1]$ the uniquely defined linear maps onto $\sigma_k$ and $\sigma_l$, respectively. Since the triangulation $\mathfrak{T}$ is locally finite, the mapping $f$ is closed by Lemma 2 and hence it can be extended to $\gamma f\colon \gamma E^1 \to \gamma E^2 = \omega E^2 = S^2$ (see [6], Theorem 4). It is easy to check that $\gamma f(x) = \omega$, where $\{\omega\} = \omega E^2 \setminus E^2$, for $x \in \gamma E^1 \setminus E^1$. Hence we can define the mapping $F\colon \omega E^1 = S^1 \to S^2 = \omega E^2$, which is an extension of $f$.



We now prove that $F(E) = \mathrm{Fr}P$. In fact, if that is not so, then we can repeat this construction for some $\sigma_0' \subset \mathrm{Fr}(P) \setminus f(E)$ obtaining the embedding $f'\colon E \times \{0, 1\} \to \mathrm{Fr}P$. It is then easy to verify that $\mathrm{Fr}P$ disconnects the plane into at least three components, which is impossible, since both $P$ and $E^2 \setminus P$ are connected. We can now apply ([5], Theorem 61.V.1) to obtain a homeomorphism $G\colon S^2 \to S^2$ such that $G(\omega) = \omega$ and $G(\mathrm{Fr}P)$ is the equator of the sphere. It is clear that $G(P)$ is the closed half-plane $E \times [0, \infty)$. We now denote by $s$ the retraction of $E^2$ onto $P$ obtained by the composition $G^{-1} \circ s_0 \circ G$, where $s_0\colon E^2 \to E \times [0, \infty)$ is a closed retraction defined by $s_0(x, y) = (x, |y|)$. We now prove that $\bar{r} = r \circ s$ is the desired closed retraction. In fact, let $\mathcal{A} = \{P \cap P^n_{n-1}\}_{n=1}^{\infty}$ (we assume $P^r_s = K_r$ if $s \leqslant 0$). It follows from the definition of the sequence $(\varepsilon)$, that $r(P \cap P^n_{n-1}) \subset P^{n+1}_{n-2}$ and hence the family $\mathcal{A}$ and the mapping $r$ satisfy the assumptions of Lemma 2 and thus the mapping $r|P$ is closed. So $\bar{r}$ is a closed retraction as the superposition of two closed retractions and the proof of Theorem 2 is finished.

## References


[1] K. Borsuk, *Theory of Retracts*, Warszawa 1967.

[2] R. Engelking, *Outline of General Topology*, Amsterdam 1968.

[3] Sze-Tsen Hu, *Homotopy Theory*, New York–London 1959.

[4] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton 1952.

[5] K. Kuratowski, *Topologija*, Moskwa 1968.

[6] K. Nowiński, *Closed mappings and the Freudenthal compactification*, Fund. Math. 76 (1972), pp. 71–83.


*Reçu par la Redaction le 10. 7. 1972*