words; i.e. in particular, $x_{i_{\alpha+1}}$ and $x_{j_{\alpha+1}}$ are different generators. Now put $h' = x_{i_1}\cdot\ldots\cdot x_{i_{\alpha+1}}$. But h' is a left divisor of $x_{i_1}\cdot\ldots\cdot x_{i_k}$, so that it also is a left divisor of $x_{j_1}\cdot\ldots\cdot x_{j_m} = x_{i_1}\cdot\ldots\cdot x_{i_\alpha}\cdot x_{j_{\alpha+1}}\cdot\ldots\cdot x_{j_m}$. Since the defining relations of $H^*(n)$ only "reshuffle", but do not eliminate generators, $x_{j_{\alpha+1}}$ as well as $x_{i_{\alpha+1}}$ have to occur in both representations. Therefore, the two representations of h have to be of the form:

$$x_{i_1}\cdot\ldots\cdot x_{i_\alpha}\cdot x_{i_{\alpha+1}}\cdot\ldots\cdot x_{j_{\alpha+1}}\cdot\ldots\cdot x_{i_k} = x_{i_1}\cdot\ldots\cdot x_{i_\alpha}\cdot x_{j_{\alpha+1}}\cdot\ldots\cdot x_{i_{\alpha+1}}\cdot\ldots\cdot x_{j_m}.$$

Hence, there is a shortest finite chain $h_0, \ldots, h_p$, $p > 0$, of words such that $h_0 = x_{i_1}\cdot\ldots\cdot x_{i_k}$ and $h_p = x_{j_1}\cdot\ldots\cdot x_{j_m}$ and such that h_i and h_{i+1}, for $0 \leqslant i < p$, differ by one application of the identities in the presentation of $H^*(n)$ — i.e. they are different representations of h. This chain then has to contain the subsequence

$$\ldots\cdot x_{i_{\alpha+1}}\cdot x_{j_{\alpha+1}}\cdot\ldots = \ldots\cdot x_{j_{\alpha+1}}\cdot x_{i_{\alpha+1}}\cdot x_{j_{\alpha+1}}\cdot\ldots = \ldots\cdot x_{j_{\alpha+1}}\cdot x_{i_{\alpha+1}}\cdot\ldots$$

Consequently, $j_{\alpha+1} = i_{\alpha+1}$, which contradicts our choice of the two generators $x_{i_{\alpha+1}} \neq x_{j_{\alpha+1}}$. This argument shows

$$x_{i_1}\cdot\ldots\cdot x_{i_k} = x_{i_1}\cdot\ldots\cdot x_{i_k}\cdot x_{j_{k+1}}\cdot\ldots\cdot x_{j_m}.$$

Again, because of the form of the defining relations for $H^*(n)$, we have $\{j_{k+1}, \ldots, j_m\} \subset \{i_1, \ldots, i_k\}$, so that $x_{j_1}\cdot\ldots\cdot x_{j_m}$ is not reduced if $m > k$. Therefore, both representations have to be identical.

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A remark on a paper of H. Höft

by

A. Iwanik (Wrocław)

Abstract. In this note we obtain the least upper bound of the length of words in the class of semigroups generated by n idempotents $x_1, \ldots, x_n$ and satisfying $x_i x_j x_i = x_i x_j = x_j x_i x_j$ for $1 \leqslant i \leqslant j \leqslant n$. These semigroups were examined in [1].

In [1] there is examined a length of reduced words in a semigroup H generated by n idempotents $x_1, \ldots, x_n$, such that every $h \in H$ has the following representation:

(POS 1) $h = x_{i_1}\cdot\ldots\cdot x_{i_k}$, $k \geqslant 1$; $x_{i_j} \in \{x_1, \ldots, x_n\}$ for all $1 \leqslant j \leqslant k$,

(POS 2) if $x_{i_\alpha} = x_{i_\beta}$, for some $1 \leqslant \alpha < \beta \leqslant k$, then $\beta \geqslant \alpha+3$ and $\min\{i_\gamma:\ \alpha \leqslant \gamma \leqslant \beta\} < i_\alpha < \max\{i_\gamma:\ \alpha \leqslant \gamma \leqslant \beta\}$

and an upper bound $a(n)$ o the length is found. In this note we obtain the least upper bound $L^*(n)$ of the length of reduced words in the class of semigroups generated by n idempotents $x_1, \ldots, x_n$ and satisfying (POS 1)-(POS 2). We shall obs rve that $L^*(n) < a(n)$ for $n \geqslant 5$.

It follows from Theorem 4 of [1] that $L^*(n)$ is equal to the maximal length of reduced words in $H^*(n)$, where $H^*(n)$ is the semigroup described by the presentation $\langle x_1, \ldots, x_n |\ x_i x_i = x_i,\ 1 \leqslant i \leqslant n;\ x_i x_j x_i = x_i x_j = x_j x_i x_j,\ 1 \leqslant i \leqslant j \leqslant n\rangle$. We shall show that

$$L^*(n) = \lambda_n,$$

where the sequence λ_n is defined by $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_n = 2\lambda_{n-2}+2$ for $n > 2$, i.e. $\lambda_n = \varepsilon_n 2^{[n/2]} - 2$, where $\varepsilon_n = 2$ or 3 according as n is even or odd.

First we show by induction that in $H^*(n)$ there is an element with a reduced word of the length λ_n. For $n = 1, 2$ it is trivial. Let an element $a \in H^*(n-2)$ has the reduced word of the length λ_{n-2}, $n > 2$. Without any loss of generality we can assume that $H^*(n-2) = [x_2, \ldots, x_{n-1}]$. Observe, that $a x_1 x_n a \in H^*(n)$ has the reduced word of the length $2\lambda_{n-2} + 2 = \lambda_n$. Therefore, $\lambda_n \leqslant L^*(n)$. The equality for $n = 1, 2$ is trivial.

Now, let $n > 2$ and $b \in H^*(n)$. Observe, that x_n occurs at most once in the reduced representation of b. Moreover, if x_r occurs at most m_r times, then x_{r-1} occurs at most $m_{r-1} = m_n + \ldots + m_r + 1$ times. Therefore, $m_r = 2^{n-r}$. Analogously, x_1 occurs at most once and x_r at most 2^{r-1} times in the reduced representation of b; $r = 1, \ldots, n$. Hence

$$L^*(n) \leqslant \Lambda_n\,, \qquad \text{where} \qquad \Lambda_n = \sum_{r=1}^{n} \min(2^{n-r}, 2^{r-1})\,.$$

If $n > 2$, we obtain

$$2\Lambda_{n-2} + 2 = \sum_{r=1}^{n-2} \min(2^{n-(r+1)}, 2^r) + 2 = \sum_{r=2}^{n-1} \min(2^{n-r}, 2^{r-1}) + 2 = \Lambda_n\,.$$

Thus

$$\Lambda_n = \lambda_n = L^*(n) \qquad \text{for} \qquad n = 1, 2, \ldots$$

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INSTITUTE OF MATHEMATICS AND THEORETICAL PHYSICS,
TECHNICAL UNIVERSITY,
Wrocław

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ERRATA

Page, ligne	Au lieu de	Lire
53_{15}	V'_4	V'_3
62^{14}	$\overline{U}_{i-1}$	$\overline{U}''_{i-1}$
63^{16}	U	$\boldsymbol{U}$