A normal form for some semigroups generated by idempotents

by

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Abstract. Finite semigroups generated by closure operators on classes of universal algebras are considered. A normal form representation for the elements of these semigroups which are positively ordered, is derived. In addition, an upper bound for the length of the words in these semigroups is computed.

Semigroups of operators on classes of universal algebras have been studied by various authors [G1, G2, H2, N, P]. Most of the attention has been given the finite semigroups of operators which arise in theories of equational classes of partial or full algebras. One very useful sufficient criterion for the finiteness of such semigroups was proved by E. Nelson (see Theorem 1 below) in [N]. In connection with a theory of equations for partial algebras (see [H1]), the author studied the semigroup of operators which determine the model classes in this theory [H2, Proposition 5 ff.]. Because of the size of that semigroup it seemed desirable to have a computer program that would calculate the elements of the semigroup. In this note we want to prove a normal form representation for the elements of finite semigroups of operators. With this normal form it will be easy to write a computer program.

A triple $(H, \cdot, \leqslant)$ consisting of a semigroup $(H, \cdot)$ and a partial order on the set H will be called a *positively ordered semigroup* if, for all $a, b, c, d \in H$:

(1) $$a \leqslant b \text{ and } c \leqslant d \quad \text{imply} \quad a\cdot c \leqslant b\cdot d,$$

(2) $$a\cdot b \geqslant b \quad \text{and} \quad a\cdot b \geqslant a.$$

THEOREM 1 (E. Nelson). *If $(H, \cdot, \leqslant)$ is a positively ordered semigroup generated by n idempotent elements $x_1, \ldots, x_n$, $n \in N$, such that $x_j x_i \leqslant x_i x_j$, for all $1 \leqslant i \leqslant j \leqslant n$, then H is finite.*

In the following, $(*)$ stands for the hypothesis of Theorem 1, and, for any set $K \subset H$, $[K]$ shall denote the subsemigroup of H generated by K. The next two Lemmas list some simple, but useful properties of positively ordered semigroups.

Lemma 1. *Let* $(H,\cdot,\leqslant)$ *be a positively ordered semigroup generated by a (not necessarily finite) set* M.

(a) *If* h *is a minimal element of* $(H,\leqslant)$, *then* $h\in M$.

(b) *If* $h\in H$ *is idempotent, then* $\{x|\ x\leqslant h\}$ *is a positively ordered subsemigroup of* H.

(c) *If the set of maximal elements in* $(H,\leqslant)$ *is non-empty, then* H *has a largest element which is the zero of* $(H,\cdot)$.

(d) *If* $(H,\cdot,\leqslant)$ *satisfies* $(*)$, *then*

$$x_i h x_i = \begin{cases} x_i h, & \text{if} \quad h\in[x_i,\dots,x_n], \\ h x_i, & \text{if} \quad h\in[x_1,\dots,x_i], \end{cases} \quad \text{for all } 1\leqslant i\leqslant n.$$

Lemma 2. *Let* $(H,\cdot,\leqslant)$ *be a positively ordered semigroup generated by* n *idempotent elements* $x_1,\dots,x_n$. *Then the following statements are equivalent*:

(i) $$x_j x_i \leqslant x_i x_j, \quad \text{for all } 1\leqslant i\leqslant j\leqslant n,$$

(ii) $$x_i x_j x_i = \begin{cases} x_i x_j, & \text{if} \quad 1\leqslant i\leqslant j\leqslant n, \\ x_j x_i, & \text{if} \quad 1\leqslant j\leqslant i\leqslant n. \end{cases}$$

In order to obtain a short formulation for the normal form of elements of H, we need to introduce the following auxiliary numbers. Suppose that $h\in H$ has the representation $h = x_{i_1}\cdot\ldots\cdot x_{i_k}$, where all x_{i_j}, $1\leqslant j\leqslant k$, are elements of a generating set M of H. Assume further that, for some indices $1\leqslant a<\beta\leqslant k$, $x_{i_a}=x_{i_\beta}$. Then we define $m(a,\beta):=\min\{i_\gamma|\ a\leqslant\gamma\leqslant\beta\}$ and $M(a,\beta):=\max\{i_\gamma|\ a\leqslant\gamma\leqslant\beta\}$.

Theorem 2. *Let* $(H,\cdot,\leqslant)$ *satisfy* $(*)$. *Then every* $h\in H$ *has the following representation*:

(POS 1) $h = x_{i_1},\dots,x_{i_k}$, $k\in N$, $k\geqslant 1$; $x_{i_j}\in\{x_1,\dots,x_n\}$, *for all* $1\leqslant j\leqslant k$.

(POS 2) *If* $x_{i_a}=x_{i_\beta}$, *for some* $1\leqslant a<\beta\leqslant k$, *then* $\beta\geqslant a+3$ *and* $m(a,\beta)<i_a<M(a,\beta)$.

Proof. Obviously, (POS 1) holds since $(H,\cdot)$ is a semigroup generated by $\{x_1,\dots,x_n\}$; and since the generating elements are idempotent, we must have $\beta>a+1$ in (POS 2). Let now $\beta=a+2$, then the subword $x_{i_a}\cdot x_{i_{a+1}}\cdot x_{i_\beta}$ of h can be shortened by Lemma 2. Therefore, $\beta\geqslant a+3$ must hold. Finally, assume $i_a\leqslant m(a,\beta)$ — or, dually, $M(a,\beta)\leqslant i_a$ — then we may apply Lemma 1(d) to the subword $h' = x_{i_{a+1}}\cdot\ldots\cdot x_{i_{\beta-1}}$, and again the subword $x_{i_a}\cdot h'\cdot x_{i_\beta}$ can be shortened to $x_{i_a}\cdot h'$ — or, dually, to $h'\cdot x_{i_\beta}$.

In the following, for any $h\in H$, h^* shall denote a representation of element h in the form (POS 1)-(POS 2); we also will refer to h^* as a *reduct of* h. With Theorem 2 we can improve on the upper bound of the length of reducts of elements in H given by E. Nelson in [N] as 2^n-1, if H is generated by n elements. Suppose $(H,\cdot,\leqslant)$ satisfies $(*)$. Consider the subsemigroup $K:=[x_{i_1},\dots,x_{i_k}]$ of H generated by k different generating elements of H. Let $L(k)$ be the maximal length of any reduct of a word $h\in K$.

Theorem 3. *Let* $(H,\cdot,\leqslant)$ *satisfy* $(*)$.

(1) *If* $h\in H$, *then* x_1 *and* x_n *occur at most once in a reduct* h^* *of* h.

(2) $L(k)\leqslant a(k)$, *for all* $k\leqslant n$, *where* $a(k)$ *is given by the recursion*:

$$a(1)=1,\ a(2)=2,\ a(3)=4,\ a(4)=6,$$
$$a(k)=2\cdot a(k-2)+a(k-4)+2, \quad \text{for} \quad k>4.$$

Proof. The first part follows from (POS 2). In order to prove the inequality for $L(k)$, let $h\in[x_1,\dots,x_k]\subset H$. Without loss of generality, its reduct h^* may be written as $h^* = h_1\cdot x_1\cdot h_2\cdot x_k\cdot h_3$, where h_i, for $i=1,2,3$, may be empty and $h_i\in[x_2,\dots,x_{k-1}]$ are reducts. Therefore, x_{k-1} occurs in h_1 or h_2 at most once, but not in both, and similarly, x_2 occurs in h_2 or h_3 at most once, but not in both. We get then the following two possibilities: (i) two of the words h_i contain at most $k-3$ and the third at most $k-2$ generators, or (ii) two of the words h_i contain at most $k-2$ and the third at most $k-4$ generators. These possibilities, considered as maximum values for the length of a reduct, yield the following recursion formulas: for the number in (i): $b(k)=b(k-2)+2\cdot b(k-3)+2$ with $b(1)=1$, $b(2)=2$, and $b(3)=4$; and for the numbers in (ii): $a(k)=2\cdot a(k-2)+a(k-4)+2$ with $a(1)=1$, $a(2)=2$, $a(3)=4$, $a(4)=6$. By induction on n, one first shows that $a(k)-a(k-1)\geqslant a(k-1)-a(k-2)$, and with this inequality that $a(n)\geqslant b(n)$, for all n. Naturally, the initial values of $a(k)=b(k)=L(k)$, $k\leqslant 4$, are calculated directly.

Theorem 2 states the existence, but not the uniqueness of the representation (POS 1)-(POS 2). Obviously, uniqueness can be realized only in a "free" object. Therefore, we shall pass from positively ordered semigroups to semigroups (without order) satisfying the relations of Lemma 2. Specifically, let $H^*(n)$, $n\in N$, be the semigroup described by the presentation

$$\langle x_1,\dots,x_n|\ x_ix_i=x_i,\ 1\leqslant i\leqslant n;\ x_ix_jx_i=x_ix_j=x_jx_ix_j,\ 1\leqslant i\leqslant j\leqslant n\rangle.$$

Then one verifies Lemma 1(d) for the semigroup $H^*(n)$. Consequently, Theorem 2 is true for $H^*(n)$.

Theorem 4. *Every element* $h\in H^*(n)$ *has a unique representation of the form* (POS 1)-(POS 2).

Proof. By the preceding remarks, every element of $H^*(n)$ admits a representation of the form (POS 1)-(POS 2). Suppose now that $x_{i_1}\cdot\ldots\cdot x_{i_k} = x_{j_1}\cdot\ldots\cdot x_{j_m}$, $k\leqslant m$, are two such representations of element $h\in H^*(n)$. Let $x_{i_1}\cdot\ldots\cdot x_{i_a}$, $0\leqslant a<k$, be the largest common initial part of both

words; i.e. in particular, $x_{i_{\alpha+1}}$ and $x_{j_{\alpha+1}}$ are different generators. Now put $h' = x_{i_1}\cdot\ldots\cdot x_{i_{\alpha+1}}$. But h' is a left divisor of $x_{i_1}\cdot\ldots\cdot x_{i_k}$, so that it also is a left divisor of $x_{j_1}\cdot\ldots\cdot x_{j_m} = x_{i_1}\cdot\ldots\cdot x_{i_\alpha}\cdot x_{j_{\alpha+1}}\cdot\ldots\cdot x_{j_m}$. Since the defining relations of $H^*(n)$ only "reshuffle", but do not eliminate generators, $x_{j_{\alpha+1}}$ as well as $x_{i_{\alpha+1}}$ have to occur in both representations. Therefore, the two representations of h have to be of the form:

$$x_{i_1}\cdot\ldots\cdot x_{i_\alpha}\cdot x_{i_{\alpha+1}}\cdot\ldots\cdot x_{j_{\alpha+1}}\cdot\ldots\cdot x_{i_k} = x_{i_1}\cdot\ldots\cdot x_{i_\alpha}\cdot x_{j_{\alpha+1}}\cdot\ldots\cdot\cdot x_{i_{\alpha+1}}\cdot\ldots\cdot x_{j_m}.$$

Hence, there is a shortest finite chain $h_0, \ldots, h_p$, $p > 0$, of words such that $h_0 = x_{i_1}\cdot\ldots\cdot x_{i_k}$ and $h_p = x_{j_1}\cdot\ldots\cdot x_{j_m}$ and such that h_i and h_{i+1}, for $0 \leqslant i < p$, differ by one application of the identities in the presentation of $H^*(n)$ — i.e. they are different representations of h. This chain then has to contain the subsequence

$$\ldots\cdot x_{i_{\alpha+1}}\cdot x_{j_{\alpha+1}}\cdot\ldots = \ldots\cdot x_{j_{\alpha+1}}\cdot x_{i_{\alpha+1}}\cdot x_{j_{\alpha+1}}\cdot\ldots = \ldots\cdot x_{j_{\alpha+1}}\cdot x_{i_{\alpha+1}}\cdot\ldots$$

Consequently, $j_{\alpha+1} = i_{\alpha+1}$, which contradicts our choice of the two generators $x_{i_{\alpha+1}} \neq x_{j_{\alpha+1}}$. This argument shows

$$x_{i_1}\cdot\ldots\cdot x_{i_k} = x_{i_1}\cdot\ldots\cdot x_{i_k}\cdot x_{j_{k+1}}\cdot\ldots\cdot x_{j_m}.$$

Again, because of the form of the defining relations for $H^*(n)$, we have $\{j_{k+1}, \ldots, j_m\} \subset \{i_1, \ldots, i_k\}$, so that $x_{j_1}\cdot\ldots\cdot x_{j_m}$ is not reduced if $m > k$. Therefore, both representations have to be identical.

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Reçu par la Rédaction le 29. 11. 1972

A remark on a paper of H. Höft

by

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Abstract. In this note we obtain the least upper bound of the length of words in the class of semigroups generated by n idempotents $x_1, \ldots, x_n$ and satisfying $x_ix_jx_i = x_ix_j = x_jx_ix_j$ for $1 \leqslant i \leqslant j \leqslant n$. These semigroups were examined in [1].

In [1] there is examined a length of reduced words in a semigroup H generated by n idempotents $x_1, \ldots, x_n$, such that every $h \in H$ has the following representation:

(POS 1) $h = x_{i_1}\cdot\ldots\cdot x_{i_k}$, $k \geqslant 1$; $x_{i_j} \in \{x_1, \ldots, x_n\}$ for all $1 \leqslant j \leqslant k$,

(POS 2) if $x_{i_\alpha} = x_{i_\beta}$, for some $1 \leqslant \alpha < \beta \leqslant k$, then $\beta \geqslant \alpha+3$ and $\min\{i_\gamma\colon \alpha \leqslant \gamma \leqslant \beta\} < i_\alpha < \max\{i_\gamma\colon \alpha \leqslant \gamma \leqslant \beta\}$

and an upper bound $a(n)$ o the length is found. In this note we obtain the least upper bound $L^*(n)$ of the length of reduced words in the class of semigroups generated by n idempotents $x_1, \ldots, x_n$ and satisfying (POS 1)-(POS 2). We shall obs rve that $L^*(n) < a(n)$ for $n \geqslant 5$.

It follows from Theorem 4 of [1] that $L^*(n)$ is equal to the maximal length of reduced words in $H^*(n)$, where $H^*(n)$ is the semigroup described by the presentation $\langle x_1, \ldots, x_n |\ x_ix_i = x_i,\ 1 \leqslant i \leqslant n;\ x_ix_jx_i = x_ix_j = x_jx_ix_j,\ 1 \leqslant i \leqslant j \leqslant n\rangle$. We shall show that

$$L^*(n) = \lambda_n,$$

where the sequence λ_n is defined by $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_n = 2\lambda_{n-2}+2$ for $n > 2$, i.e. $\lambda_n = \varepsilon_n 2^{[n/2]} - 2$, where $\varepsilon_n = 2$ or 3 according as n is even or odd.

First we show by induction that in $H^*(n)$ there is an element with a reduced word of the length λ_n. For $n = 1, 2$ it is trivial. Let an element $a \in H^*(n-2)$ has the reduced word of the length λ_{n-2}, $n > 2$. Without any loss of generality we can assume that $H^*(n-2) = [x_2, \ldots, x_{n-1}]$. Observe, that $ax_1x_na \in H^*(n)$ has the reduced word of the length $2\lambda_{n-2} + 2 = \lambda_n$. Therefore, $\lambda_n \leqslant L^*(n)$. The equality for $n = 1, 2$ is trivial.

Now, let $n > 2$ and $b \in H^*(n)$. Observe, that x_n occurs at most once in the reduced representation of b. Moreover, if x_r occurs at most m_r times, then x_{r-1} occurs at most $m_{r-1} = m_n + \ldots + m_r + 1$ times. There-