On the inequivalence of the Borsuk and the H-shape theories for arbitrary metric spaces

by

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Abstract. Among the methods that have been given to extend K. Borsuk's shape theory are the ones given by Borsuk for metrizable spaces and by L. Rubin and the author for Hausdorff spaces. These two approaches cannot be equivalent on metrizable spaces as the first doesn't preserve sums and products while the second does. In this paper some conditions are given under which the second concept is courser than the first.

1. Introduction. Since K. Borsuk first introduced the notion of the shape of a compactum there have been several methods given to extend this notion to non-compact spaces. Both Borsuk [3] and R. H. Fox [5] have given extensions to arbitrary metric spaces. The method given by Borsuk seems to be "internal" in nature; whereas, Fox's method seems to be "external". L. Rubin and the author [10] have given a method of extending the notion of shape to arbitrary Hausdorff spaces. This method also extends the notion of shape of compact Hausdorff spaces given by S. Mardešić and J. Segal [8], [9]. The last method, called "H-Shape", seems also to be "internal" in nature.

It is known [1], [6] that Fox's definition of shape is strictly courser than Borsuk's; i.e., if two metric spaces have the same Borsuk shape then they have the same Fox shape, but not conversely. In particular, Borsuk's shape is not preserved under sums and products. Since the H-shape does preserve sums and products [10], it follows that the Borsuk shape and the H-shape are not equivalent on arbitrary metric spaces. In this paper we give some conditions under which H-shape is courser than Borsuk shape.

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2. Borsuk's shape for arbitrary metric spaces. Suppose M and N are absolute retracts (for metric spaces) and X and Y are closed subsets of M and N, respectively. A *fundamental sequence* from X to Y in (M, N) [3] $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$, is a sequence of maps $\varphi_k\colon M \to N$ that satisfies the following condition:

(2.1) For every set $A \subset X$ there is a set $B \subset Y$ (being compact, if A if compact) such that for every neighborhood V of B (in N) there is a neighborhood U of A (in M) and an index k_0 such that is $k \geqslant k_0$ then $\varphi_k|_U \simeq \varphi_{k+1}|_U$ in V.

Given a set $A \subset X$, a set $B \subset Y$ is said to be $\underline{\varphi}$*-assigned* to A if B satisfies (2.1) for the set A. Note that if B is $\underline{\varphi}$-assigned to A, $A' \subset A$ and $B' \subset Y$ is a set such that B' is compact if A' is compact and $B \subset B'$, then B' is $\underline{\varphi}$-assigned to A'.

If Z is a closed subset of the absolute retract P and $\underline{\theta} = \{\theta_k, Y, Z\}_{N,P}$ is another fundamental sequence then the *composition* $\underline{\theta\varphi} = \{\theta_k\varphi_k, X, Z\}_{M,P}$ is also a fundamental sequence. Note that if B is $\underline{\varphi}$-assigned to A and C is $\underline{\theta}$-assigned to B, then C is $\underline{\theta\varphi}$-assigned to A. If $M = N$ and $X = Y$ then setting each $\varphi_k = 1_M$: $M \to M$ (the identity map) one has the *identity fundamental sequence* $\underline{1}_{X,M} = \{1_M, X, X\}_{M,M}$.

Two fundamental sequences $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ and $\underline{\theta} = \{\theta_k, X, Y\}_{M,N}$ are said to be *homotopic*, $\underline{\varphi} \simeq \underline{\theta}$, if they satisfy the following condition:

(2.2) For every set $A \subset X$ there is a set $B \subset Y$ (being compact, if A is compact) such that for every neighborhood V of B (in N) there is a neighborhood U of A (in M) and an index k_0 such that if $k \geqslant k_0$ then $\varphi_k|_U \simeq \theta_k|_U$ in V.

It is known [3] that the relation $\simeq$ on fundamental sequences is an equivalence relation that is compositive. Note that if $\underline{\varphi} \simeq \underline{\theta}$, $A \subset X$ and $B \subset Y$ (compact if A is compact) is a set that satisfies (2.2), then B is $\underline{\theta}$-assigned to A if it is $\underline{\varphi}$-assigned to A.

If X, Y are closed subsets of absolute retracts M, N, respectively, and $\underline{\varphi} = \{\varphi_k, X, Y\}_{N,M}$ is a fundamental sequence then $\underline{\varphi}$ is called a *fundamental equivalence* if there is a fundamental sequence $\underline{\theta} = \{\theta_k, Y, X\}_{M,N}$ such that $\underline{\theta\varphi} \simeq \underline{1}_{X,M}$ and $\underline{\varphi\theta} \simeq \underline{1}_{Y,N}$. The fundamental sequence $\underline{\theta}$ is called the *homotopy inverse* of $\underline{\varphi}$. If there is a fundamental equivalence $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ then X and Y are said to have the *same* (*Borsuk*) *shape*, $\mathrm{Sh}_B(X) = \mathrm{Sh}_B(Y)$. It is known [3] that this relation depends neither on the absolute retracts M and N in which X and Y are respectively embedded as closed subspaces, nor on the respective embeddings. Since any metrizable space can be embedded as a closed subset of an absolute retract, the notion of Borsuk shape gives an equivalence relation on the family of all metrizable spaces [3].

If X and Y are compact then [3] (2.1) is equivalent to:

(2.3) For every neighborhood V of Y (in N) there is a neighborhood U of X (in M) and an index k_0 such that if $k \geqslant k_0$ then $\varphi_k|_U \simeq \varphi_{k+1}|_U$ in V.

This is the usual definition for a fundamental sequence between compacta, whenever $M = N = Q$ (Hilbert cube). In this case, we drop the subscript Q, Q and write $\underline{\varphi} = \{\varphi_k, X, Y\}$ is a fundamental sequence (in Q).

In [9], Mardešić and Segal showed that if X and Y are compact subsets of Q, then a fundamental sequence $\underline{\varphi} = \{\varphi_k, X, Y\}$ has a related map of ANR-sequences $\underline{f}$: $\underline{X} \to \underline{Y}$ for $\underline{X}$ and $\underline{Y}$ inclusion ANR-sequences associated with X and Y, respectively.

(2.4) This relation is such that:

(1) If $X \subset Y$ then an "inclusion map of systems" $\underline{i} = (i, i_n)$: $\underline{X} \to \underline{Y}$ (where i_n: $X_{i(n)} \to Y_n$ is an inclusion map for each n) is related to the "inclusion fundamental sequence" $\underline{i} = \{1_Q, X, Y\}$.

(2) Suppose $Z \subset Q$ is compact and $\underline{Z}$ is an inclusion ANR-system associated with Z. If $\underline{f}$: $\underline{X} \to \underline{Y}$ is related to $\underline{\varphi} = \{\varphi_k, X, Y\}$, $\underline{g} = \underline{Y} \to \underline{Z}$ is related to $\underline{\theta} = \{\theta_k, Y, Z\}$ and $\underline{h}$: $\underline{X} \to \underline{Z}$ is related to $\underline{\theta\varphi} = \{\theta_k\varphi_k, X, Z\}$ then (Lemma 6 of [9]) $\underline{h} \simeq \underline{gf}$.

(3) Suppose $\underline{f}$: $\underline{X} \to \underline{Y}$ is related to $\underline{\varphi} = \{\varphi_k, X, Y\}$ and $\underline{g}$: $\underline{X} \to \underline{Y}$ is related to $\underline{\theta} = \{\theta_k, X, Y\}$. Then (Lemma 5 of [9]) $\underline{\varphi} \simeq \underline{\theta}$ is equivalent to $\underline{f} \simeq \underline{g}$.

If X and Y are compact subsets of absolute retracts M and N, respectively, then there exist embeddings σ: $X \to Q$ and τ: $Y \to Q$. Since Q and M are absolute retracts, there exist extensions Σ: $Q \to M$ and T: $N \to Q$ of σ^{-1}: $\sigma(X) \cong X \subset M$ and τ: $Y \to Q$, respectively. If $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ is a fundamental sequence, then $\hat{\underline{\varphi}} = \{T\varphi_k\Sigma, \sigma(X), \tau(Y)\}$ is a fundamental sequence (see [3]).

Let us note several things about the relationships between $\underline{\varphi}$ and $\hat{\underline{\varphi}}$:

(2.5) (1) The homotopy class of $\hat{\underline{\varphi}}$ is independent of the extensions Σ and T.

(2) If $X \subset Y \subset M$ and $\sigma = \tau|_X$: $X \to Q$, then $\hat{\underline{i}} = \{T1_M\Sigma, \tau(X), \tau(Y)\}$ is homotopic to $\underline{i} = \{1_Q, \tau(X), \tau(Y)\}$.

(3) Suppose Z is a compact subset of an absolute retract P, ψ: $Z \to Q$, an embedding, Ψ: $P \to Q$ an extension of ψ and T': $Q \to N$ and extension of τ^{-1}: $\tau(Y) \cong Y \subset N$. If $\underline{\theta} = \{\theta_k, Y, Z\}_{N,P}$ is a fundamental sequence then $\hat{\underline{\theta}}\hat{\underline{\varphi}} = \{\Psi\theta_k T' T\varphi_k\Sigma, \sigma(X), \psi(Z)\}$ is homotopic to $\widehat{\underline{\theta\varphi}} = \{\Psi\theta_k\varphi_k\Sigma, \sigma(X), \psi(Z)\}$.

(4) If $\underline{\theta} = \{\theta_k, X, Y\}_{M,N}$ is a fundamental sequence and $\underline{\theta} \simeq \underline{\varphi}$ then $\hat{\underline{\theta}} = \{T\theta_k\Sigma, \sigma(X), \tau(Y)\}$ is homotopic to $\hat{\underline{\varphi}} = \{T\varphi_k\Sigma, \sigma(X), \tau(Y)\}$. All of these are immediate consequences of Theorem 2.1 of [11].

In the context of [7], one can combine (2.4) and (2.5) to obtain the following proposition.

(2.6) PROPOSITION. *A fundamental sequence* $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ *between compact subsets of absolute retracts has a related shape map* $\underline{f}$: $X \to Y$. *Furthermore, this relationship is such that the following hold:*

(1) *If $X \subset Y \subset M$ then the "inclusion shape map" $\underline{i}\colon X \to Y$ is related to the "inclusion fundamental sequence" $\underline{i} = \{1_M, X, Y\}_{M,M}$.*

(2) *If $\underline{f}\colon X \to Y$ and $\underline{g}\colon Y \to Z$ are related to $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ and $\underline{\theta} = \{\theta_k, Y, Z\}_{N,P}$, respectively, then $\underline{gf}\colon X \to Z$ is related to $\underline{\theta\varphi} = \{g_k f_k, X, Z\}_{N,P}$.*

(3) *If $\underline{f}\colon X \to Y$ and $\underline{g}\colon X \to Y$ are related to $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ and $\underline{\theta} = \{\theta_k, X, Y\}_{M,N}$, respectively, and $\underline{\varphi} \simeq \underline{\theta}$ then $\underline{f} = \underline{g}$.*

Statement (1) of (2.6) implies, in particular, that the identity shape map $\underline{1}_X\colon X \to X$ is related to the identity fundamental sequence $\underline{1}_{X,M} = \{1_M, X, X\}_{M,M}$.

3. Shape for Hausdorff spaces. A CS-*system* [10] is a direct system $X^* = \{X_\omega, \underline{p}_{\omega\omega'}, \Omega\}$ in the (compact) shape category of [7]. A CS-morphism $F = (f, \underline{f}_\omega)\colon X^* \to Y^* = \{Y_\lambda, \underline{q}_{\lambda\lambda'}, \Lambda\}$ consists of an increasing function $f\colon \Omega \to \Lambda$ together with a collection of shape maps $\underline{f}_\omega\colon X_\omega \to Y_{f(\omega)}$ such that if $\omega \leqslant \omega'$ then $\underline{q}_{f(\omega)f(\omega')}\underline{f}_\omega = \underline{f}_{\omega'}\underline{p}_{\omega\omega'}$. If the identity $I_{X^*} = (1, \underline{1}_\omega)\colon X^* \to X^*$ and composition are defined in the usual fashion, one has a category, denoted CS.

Two CS-morphisms $F = (f, \underline{f}_\omega)$, $G = (g, \underline{g}_\omega)\colon X^* \to Y^*$ are *homotopic*, $F \simeq G$, provided for each index $\omega \in \Omega$ there is an index $\lambda \in \Lambda$, $\lambda \geqslant f(\omega)$, $g(\omega)$, such that $\underline{q}_{f(\omega)\lambda}\underline{f}_\omega = \underline{q}_{g(\omega)\lambda}\underline{g}_\omega$. It is known [10] that the relation $\simeq$ is a morphism equivalence relation in CS.

Suppose X is a Hausdorff space. Let $c(X) = \{A \subset X \mid A \text{ is compact}\}$ be directed by inclusion. Then there is a CS-system $C(X) = \{A, \underline{i}_{AA'}, c(X)\}$ where if $A \subset A'$ then $\underline{i}_{AA'}\colon A \to A'$ is the inclusion shape map. Two Hausdorff spaces X and Y are said to have the *same H-shape* [10], $\mathrm{Sh}_H(X) = \mathrm{Sh}_H(Y)$, if there exist CS-morphisms $F\colon C(X) \to C(Y)$ and $G\colon C(Y) \to C(X)$ such that $GF \simeq I_{C(X)}$ and $FG \simeq I_{C(Y)}$. The notion of H-shape gives an equivalence relation on the family of all Hausdorff spaces [10].

A cover $\mathcal{F}$ of a Hausdorff space X is said to be CS-*cofinal* if there is a function $g\colon c(X) \to \mathcal{F}$ satisfying

(1) if $A \in c(X)$ then $A \subset g(A)$,

(2) if $A, A' \in c(X)$ and $A \subset A'$ then $g(A) \subset g(A')$.

If $\mathcal{F}$ is a compact cover of X that is CS-cofinal, $\mathcal{F}$ defines a CS-system $X^* = \{A, \underline{i}_{AA'}, \mathcal{F}\}$ where if $A, A' \in \mathcal{F}$ and $A \subset A'$ then $\underline{i}_{AA'}\colon A \to A'$ is the inclusion shape map. It is known [10] that the shape of a Hausdorff space X is completely determined by any compact cover $\mathcal{F}$ of X that is CS-cofinal. That is, if X and Y are Hausdorff spaces and $\mathcal{F}$ and $\mathcal{G}$ are compact covers of X and Y, respectively, that are CS-cofinal then $\mathrm{Sh}_H(X) = \mathrm{Sh}_H(Y)$ iff there exist CS-morphisms $F\colon X^* = \{A, \underline{i}_{AA'}, \mathcal{F}\} \to Y = \{B, \underline{j}_{BB'}, \mathcal{G}\}$ and $G\colon Y^* \to X^*$ such that $GF \simeq I_{X^*}$ and $FG \simeq I_{Y^*}$.

4. Observations and examples. Suppose X and Y are closed subsets of absolute retracts M and N, respectively, and $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ is a fundamental sequence. If $\mathcal{F}$ and $\mathcal{G}$ are compact covers of X and Y, respectively, that are CS-cofinal then for each $A \in \mathcal{F}$ there is a $B \in \mathcal{G}$ such that B is $\underline{\varphi}$-assigned to A. This means that $\underline{\varphi}_{A,B} = \{\varphi_k, A, B\}_{M,N}$ is a fundamental sequence between compacta; and hence, there is a related shape map $\underline{f}_{A,B}\colon A \to B$. If $A' \subset A$ and $B \subset B'$, let $\underline{\xi} = \{1_M, A', A\}_{M,M}$ and $\underline{\eta} = \{1_N, B, B'\}_{N,N}$ denote the inclusion fundamental sequences and $\underline{i}\colon A' \to A$ and $\underline{j}\colon B \to B'$ the inclusion shape maps. Then $\underline{\varphi}_{A,B'}\underline{\xi} = \underline{\eta}\underline{\varphi}_{A',B}$ and thus $\underline{f}_{A,B'}\underline{i} = \underline{j}\underline{f}_{A',B}$. If one can find an increasing function $f\colon \mathcal{F} \to \mathcal{G}$ such that $f(A)$ is φ-assigned to A then there is a CS-morphism $F = (f, \underline{f}_A)\colon X^* \to Y^*$ where $\underline{f}_A = \underline{f}_{A,f(A)}\colon A \to f(A)$ is related to $\underline{\varphi}_{A,f(A)} = \{\varphi_k, A, f(A)\}_{M,N}$. We say the pair $(\mathcal{F}, \mathcal{G})$ is $\underline{\varphi}$-*compatible* whenever such a function $f\colon \mathcal{F} \to \mathcal{G}$ exists and the function $f\colon \mathcal{F} \to \mathcal{G}$ is called a $\underline{\varphi}$-*compatibility function*. The CS-morphism $F = (f, \underline{f}_A)\colon X^* \to Y^*$ is said to be *induced* by $\underline{\varphi}$.

(4.1) Proposition. *If $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ is a fundamental sequence and $f, g\colon \mathcal{F} \to \mathcal{G}$ are $\underline{\varphi}$-compatibility functions between compact covers $\mathcal{F}$, $\mathcal{G}$ of X, Y, respectively, that are CS-cofinal, then the respective induced maps $F = (f, \underline{f}_A)\colon X^* \to Y^*$ and $G = (g, \underline{g}_A)\colon X^* \to Y^*$ are homotopic.*

Proof. If $A \in \mathcal{F}$, choose $B \in \mathcal{G}$ such that $B \supset f(A) \cup g(A)$. Then B is $\underline{\varphi}$-assigned to A and both $\underline{j}_{f(A)B}\underline{f}_A$ and $\underline{j}_{g(A)B}\underline{g}_A$ are related to $\underline{\varphi}_{A,B} = \{\varphi_k, A, B\}_{M,N}$. Thus $\underline{j}_{f(A)B}\underline{f}_A = \underline{j}_{g(A)B}\underline{g}_A$ and $F \simeq G$.

(4.2) Proposition. *Suppose X is a closed subset of an absolute retract N and $\mathcal{F}, \mathcal{F}'$ are compact covers of X that are CS-cofinal. Then the pair $(\mathcal{F}, \mathcal{F}')$ is $\underline{1}_{X,M}$-compatible, where $\underline{1}_{X,M} = \{1_M, X, X\}_{M,M}$ is the identity fundamental sequence.*

Proof. Let $f'\colon c(M) \to \mathcal{F}'$ be the function guaranteed by $\mathcal{F}'$ being CS-cofinal. The function $f = f'|_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F}'$ is a $\underline{1}_{X,M}$-compatibility function.

The proof of the following is clear.

(4.3) Proposition. *The identity CS-morphism $I_{X^*} = (1, \underline{1}_A)\colon X^* \to X^*$ is induced by the identity fundamental sequence.*

(4.4) Proposition. *Suppose $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ and $\underline{\theta} = \{\theta_k, Y, Z\}_{N,P}$ are fundamental sequence and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are compact covers of X, Y, Z, respectively, that are CS-cofinal. If $(\mathcal{F}, \mathcal{G})$ is $\underline{\varphi}$-compatible and $(\mathcal{G}, \mathcal{H})$ is $\underline{\theta}$-compatible, then $(\mathcal{F}, \mathcal{H})$ is $\underline{\theta\varphi}$-compatible.*

Proof. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a $\underline{\varphi}$-compatibility function and $g\colon \mathcal{G} \to \mathcal{H}$ be a $\underline{\theta}$-compatibility function. It is routine to verify that $gf\colon \mathcal{F} \to \mathcal{H}$ is a $\underline{\varphi\theta}$-compatibility function.

(4.5) COROLLARY. *Suppose* $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ *is a fundamental sequence and* $(\mathcal{F}, \mathcal{G})$ *is a* $\underline{\varphi}$*-compatible pair. If* $\mathcal{F}', \mathcal{G}'$ *are compact covers of* X, Y, *respectively, that are* CS*-cofinal then* $(\mathcal{F}', \mathcal{G}')$ *is a* $\underline{\varphi}$*-compatible pair.*

Proof. By (4.2), $(\mathcal{F}', \mathcal{F})$ is $\underline{1}_{X,M}$-compatible and $(\mathcal{G}, \mathcal{G}')$ is $\underline{1}_{Y,N}$-compatible. Applying (4.4), $(\mathcal{F}', \mathcal{G}')$ is $\underline{\varphi}$-compatible.

(4.6) COROLLARY. *If* $F = (f, f_A)$: $X^* \to Y^*$ *is induced by* $\underline{\varphi}$ *and* $G = (g, g_B)$: $Y^* \to Z^*$ *is induced by* $\underline{\theta}$, *then* $GF = (gf, g_{f(A)}f_A)$: $X^* \to Z^*$ *is induced by* $\underline{\theta\varphi}$.

(4.7) PROPOSITION. *Suppose* $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ *and* $\underline{\theta} = \{\theta_k, X, Y\}_{M,N}$ *are homotopic fundamental sequences and that* $(\mathcal{F}, \mathcal{G})$ *is both a* $\underline{\varphi}$ *and* $\underline{\theta}$*-compatible pair. If* $F = (f, f_A)$, $G = (g, g_A)$: $X^* \to Y^*$ *are induced by* $\underline{\varphi}, \underline{\theta}$, *respectively, then* $F \simeq G$.

Proof. Let $A \in \mathcal{F}$. Since $\underline{\varphi} \simeq \underline{\theta}$ there is a $B \in c(Y)$ satisfying (2.2). Choose $B' \in \mathcal{G}$ such that $B' \supset B, f(A), g(A)$. Then $\underline{\varphi}_{A,B'} = \{\varphi_k, A, B'\}_{M,N}$ and $\underline{\theta}_{A,B'} = \{\theta_k, A, B'\}_{M,N}$ are homotopic fundamental sequences. Let $\underline{\xi} = \{1_N, f(A), B'\}_{N,N}$ and $\underline{\eta} = \{1_N, g(A), B'\}_{N,N}$ be the inclusion fundamental sequences. Then $\underline{\xi\varphi}_{A,f(A)} = \underline{\varphi}_{A,B'} \simeq \underline{\theta}_{A,B'} = \underline{\eta\theta}_{A,g(A)}$. This implies $j_{f(A)B'}f_A = j_{g(A)B'}g_A$; and hence, $F \simeq G$.

(4.8) COROLLARY. *If* $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ *and* $\underline{\theta} = \{\theta_k, Y, X\}_{N,M}$ *are homotopy inverse fundamental sequences and there exist compact covers* $\mathcal{F}, \mathcal{G}$ *of* X, Y, *respectively, that are* CS*-cofinal and such that* $(\mathcal{F}, c(Y))$ *is* $\underline{\varphi}$*-compatible and* $(\mathcal{G}, c(X))$ *is* $\underline{\theta}$*-compatible then* $\mathrm{Sh}_H(X) = \mathrm{Sh}_H(Y)$.

Proof. Since $(\mathcal{F}, c(Y))$ is $\underline{\varphi}$-compatible and $(\mathcal{G}, c(X))$ is $\underline{\theta}$-compatible, it follows from (4.5) that $(\mathcal{F}, \mathcal{G})$ is $\underline{\varphi}$-compatible and $(\mathcal{G}, \mathcal{F})$ is $\underline{\theta}$-compatible. Let $F = (f, f_A)$: $X^* \to Y^*$ and $G = (g, g_B)$: $Y^* \to X^*$ be CS-morphisms induced by $\underline{\varphi}$ and $\underline{\theta}$, respectively. By 4.4, $(\mathcal{F}, \mathcal{F})$ is $\underline{\theta\varphi}$-compatible and $(\mathcal{G}, \mathcal{G})$ is $\underline{\varphi\theta}$-compatible. Applying 4.2, 4.3, 4.6, and 4.7, it follows that $GF \simeq I_{X^*}$ and $FG \simeq I_{Y^*}$.

(4.9) EXAMPLE. Suppose there is a compact cover $\mathcal{F} = \{A_\omega | \ \omega \in \Omega\}$ of X that is CS-cofinal and such that each $A \in \mathcal{F}$ has only finitely many $A' \in \mathcal{F}$ for which $A' \in A$. This will occur, for example, whenever X is paracompact and locally compact (see Example 3 of [10]). If $A \in \mathcal{F}$, let $\eta(A) = \mathrm{card}\{A' \in \mathcal{F} | \ A' \subset A\}$. By induction on $\eta(A)$, it is routine to construct a $\underline{\varphi}$-compatibility function f: $\mathcal{F} \to c(Y)$.

(4.10) COROLLARY. *If* X *and* Y *are locally compact metrizable spaces, then* $\mathrm{Sh}_B(X) = \mathrm{Sh}_B(Y)$ *implies* $\mathrm{Sh}_H(X) = \mathrm{Sh}_H(Y)$.

(4.11) Note. Corollary 4.10 implies, in particular, that if X and Y are closed subsets of E^n (Euclidean n-space) and $\mathrm{Sh}_B(X) = \mathrm{Sh}_B(Y)$ then $\mathrm{Sh}_H(X) = \mathrm{Sh}_H(Y)$. Example 3.4 of [1] gives an example for which X and Y may be embedded as closed subsets of E^2, $\mathrm{Sh}_H(X) = \mathrm{Sh}_H(Y)$, but $\mathrm{Sh}_B(X) \neq \mathrm{Sh}_B(Y)$.

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