Mutational retracts and extensions of mutations

by

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Abstract. Let \( X \) be a closed subset of a metrizable space \( X' \) considered as a closed subset of an ANR(\( \mathfrak{M} \))-space \( P \). A mutation \( [3] R: U(X', P) \rightarrow U(X, P) \) is called a mutational retraction if \( r(z) = z \) for every \( r \in R \) and \( z \in X \). If there exists a mutational retraction \( r: U(X', P) \rightarrow U(X, P) \) then the set \( X \) is called a mutational retract of the space \( X' \).

Every fundamental retract \([3]\) of a space \( X \) is a mutational retract of \( X' \), but not conversely. Every compact mutational retract of a space \( X' \) is a fundamental retract of \( X' \).

In the natural manner we define a mutational neighbourhood retract of a space \( X' \), mutational absolute retract \( (MAR) \) and mutational absolute neighbourhood retract \( (MANR) \).

Every PAR-space \( (PANR\)-space) \([3]\) is a MANR-space \( (MANR\)-space) \), but not conversely. Every compact MANR-space \( (MANR\)-space) \) is a PAR-space \( (PANR\)-space) \).

In order to extend some standard notions of the homotopy theory onto arbitrary compacts \( K \). Borsuk introduced in \([2]\) the notion of the fundamental sequence from a compactum \( X \) to a compactum \( Y \). Replacing maps by fundamental sequences one can obtain generalizations or modifications of many standard notions. In this manner \( K \). Borsuk introduced in \([3]\) the notions of the fundamental retract, the fundamental absolute retract and the fundamental absolute neighbourhood retract, which are generalizations of the notions of the retract, the absolute retract, and the absolute neighbourhood retract, respectively. Analogously in \([4]\) \( K \). Borsuk introduced the notion of shape, which is a modification of the homotopy type. In \([5]\) and \([6]\) these notions were extended to arbitrary metrizable spaces. Independently, in \([9]\) B. H. Fox extended the notion of shape to arbitrary metrizable spaces, introducing the notion of the mutation, as a modification of the notion of the fundamental sequence.

In this paper, replacing fundamental sequences by mutations, we introduce the notions of the mutational retract, the mutational absolute retract and the mutational absolute neighbourhood retract, as generalizations of the notions of the fundamental retract, the fundamental absolute retract, and the fundamental absolute neighbourhood retract, respectively.
§ 1. Basic notions. In this section we recall some definitions from [5], [6], and [9].

Consider closed subsets $X$ and $Y$ of metrizable spaces $P$ and $Q$, respectively. A fundamental sequence $f = (f_k, X, Y)_{P,Q}$ from $X$ to $Y$ in $P$, $Q$ is defined to be an ordered triple consisting of $X$, $Y$ and a sequence of maps $f_k: P \rightarrow Q$, $k = 1, 2, ...$, satisfying the following two conditions:

(1.1) For every neighbourhood $V$ of $Y$ (in $Q$) there exists a neighbourhood $U$ of $X$ (in $P$) such that $f_k U \supseteq f_{k+1} U$ in $V$ for almost all $k$.

(1.2) For every compactum $A \subseteq X$ there exists a compactum $B \subseteq Y$ such that for every neighbourhood $V$ of $B$ (in $Q$) there exists a neighbourhood $U$ of $A$ (in $P$) such that $f_k U \supseteq f_{k+1} U$ in $V$ for almost all $k$.

Two fundamental sequences $f = (f_k, X, Y)_{P,Q}$ and $g = (g_k, X, Y)_{P,Q}$ are called homotopic (notation: $f \simeq g$) if the following two conditions are satisfied:

(1.3) For every neighbourhood $V$ of $Y$ (in $Q$) there exists a neighbourhood $U$ of $X$ (in $P$) such that $f_k U \supseteq g_k U$ in $V$ for almost all $k$.

(1.4) For every compactum $A \subseteq X$ there exists a compactum $B \subseteq Y$ such that for every neighbourhood $V$ of $B$ (in $Q$) there exists a neighbourhood $U$ of $A$ (in $P$) such that $f_k U \supseteq g_k U$ in $V$ for almost all $k$.

A composition of fundamental sequences $f = (f_k, X, Y)_{P,Q}$ and $g = (g_k, Y, Z)_{Q,R}$ is defined to be the fundamental sequence $gf = (g_k f_k, X, Z)_{P,R}$, where $i_k: P \rightarrow P$, $k = 1, 2, ...$, is the identity map, is called the fundamental identity sequence.

By the Kuratowski-Wojdyłowski theorem ([1], p. 78) any metrizable space $X$ may be considered as a closed subset of an $\mathbb{R}^\infty$-space $P$. Two metrizable spaces $X$ and $Y$ are said to be of the same shape (in the sense of Borsuk (notation: $Sh(X) \equiv Sh(Y)$) if there exist two fundamental sequences $f = (f_k, X, Y)_{P,Q}$ and $g = (g_k, Y, X)_{Q,P}$, where $P$ and $Q$ are $\mathbb{R}^\infty$-spaces containing $X$ and $Y$, respectively, as closed subsets, such that

(1.5) $f_k \simeq i_Y$ and $g_k \simeq i_X$.

If the fundamental sequences $f$ and $g$ satisfy the first condition of (1.5), then the shape of $X$ (in the sense of Borsuk) is said to dominate the shape of $Y$ (notation: $Sh(X) \succ Sh(Y)$).

If $X$ is a closed subset of a metrizable space $X'$ and $X'$ is a closed subset of an $\mathbb{R}^\infty$-space $P$, then a fundamental sequence $r = (r_\alpha, X', X)_{P,Q}$ such that $r_\alpha(x) = x$ for every $x \in X$ and $\alpha = 1, 2, ...$, is called a fundamental retraction of $X'$ to $X$. If there exists a fundamental retraction of $X'$ to $X$, then $X$ is said to be a fundamental retract of $X'$.

A closed subset $X$ of a metrizable space $X'$ is said to be a fundamental neighbourhood retraction of $X'$ if there exists a closed neighbourhood $W$ of $X$ in $X'$ such that $X$ is a fundamental retraction of $W$.

A metrizable space is called a fundamental absolute retract (notation: $X \in F(A)$) if for every metrizable space $X'$, containing $X$ as a closed subset, the set $X$ is a fundamental retract of $X'$. A metrizable space $X$ is said to be a fundamental absolute neighbourhood retract (notation: $X \in F(A)$) if every metrizable space $X'$ containing $X$ as a closed subset, the set $X$ is a fundamental neighbourhood retract of $X'$.

Let $X$ be a closed subset of an $\mathbb{R}^\infty$-space $P$. The family $U(X, P, Q)$ of all open neighbourhoods of $X$ in $P$ is called a complete neighbourhood system of $X$ in $P$.

Consider two arbitrary complete neighbourhood systems $U(X, P, Q)$ and $U(Y, Q, Q)$. A mutation $f: U(X, P, P) \rightarrow V(Y, Q, Q)$ is defined as a collection of maps $f: U \rightarrow V$, where $U \in U(X, P, P)$ and $V \in U(Y, Q, Q)$, such that

(1.6) If $f \circ f: U \rightarrow V$, $U' \subseteq U$, $U' \in U(X, P, P)$, $V \subseteq V'$, $V \in U(Y, Q, Q)$ and $f': U' \rightarrow V'$ is defined by $f'(x) = f(x')$ for $x \in U'$, then $f \circ f' = f$.

(1.7) Every neighbourhood $V \subseteq V(Y, Q, Q)$ is the range of a map $f: U \rightarrow V$.

(1.8) If $f_1, f_2: U \rightarrow V$ and $f_1 \circ f_2 : U \rightarrow V$, then there exists a $U' \subseteq U(X, P, P)$ such that $U' \subseteq U$ and $f_1 \circ f_2 \subseteq f_1 U'$.

If $U(X, P, P)$ is a complete neighbourhood system, then the collection of all inclusions $U: U' \rightarrow U$, where $U' \subseteq U(X, P, P)$ and $U' \subseteq U$, is a mutation from $U(X, P, P)$ to itself.

Consider two mutations $f: U(X, P, P) \rightarrow V(Y, Q, Q)$ and $g: V(Y, Q, Q) \rightarrow W(Z, R)$. The composition $gf: U(X, P, P) \rightarrow W(Z, R)$ of the mutations $f$ and $g$ is the mutation constituting the collection of all compositions $gf$ such that $f \circ f, g \circ g$ and $gf$ is defined.

Two mutations $f, g: U(X, P, P) \rightarrow V(Y, Q, Q)$ are homotopic (notation: $f \simeq g$) if

(1.9) For every $f \circ f$ and $g \circ g$ such that $f \circ f: U \rightarrow V$ there exists a $U' \subseteq U(X, P, P)$ such that $U' \subseteq U$ and $f \circ f \subseteq g U'$. 

By the Kuratowski-Wojdyłowski theorem ([1], p. 78) any metrizable space $X$ may be considered as a closed subset of an $\mathbb{R}^\infty$-space $P$. Two metrizable spaces $X$ and $Y$ are said to be of the same shape in the
sense of Fox (notation: ShX ≡ ShY) if there exist two mutations $f$: $U(X, P) → V(Y, Q)$ and $g$: $V(Y, Q) → U(X, P)$ such that

$$fg ≅ u \text{ and } gf ≅ v$$

where $u$ and $v$ are mutations consisting of all inclusions in systems $U(X, P)$ and $V(Y, Q)$, respectively. If the mutations $f$ and $g$ satisfy the first condition of (1.10), then we say that the shape (in the sense of Fox) of $X$ dominates the shape of $Y$ (notation: ShX > ShY).

Remark. In [9] B. R. Fox introduced the notion of the shape in an arbitrary category and specialized this notion to metrizable spaces. Some definitions given above differ only formally from Fox's original definitions.

2. Extensions and restrictions of mutations. Let $X$ and $Y$ be closed subsets of ANR(3)-spaces $P$ and $Q$, respectively. Consider a map $f$: $X → X$. Then there exists a $U ∈ U(X, P)$ and a map $f$: $U → Q$ such that $f|_U = f(a)$ for every $a ∈ X$. The map $f$ determines uniquely a mutation $f$: $U(X, P) → V(Y, Q)$ consisting of all maps $g$: $f^{-1}(U) → U$ defined by $g(a) = f(a)$, where $U ⊂ U$, $U ∈ U(X, P)$ and $V ⊂ V(Y, Q)$. The mutation $f$ is called an extension of the map $f$ ([9], p. 34).

Let $X$ be a closed subset of a metrizable space $X'$ considered as a closed subset of an ANR(3)-space $P$ and let $Y$ be a closed subset of an ANR(3)-space $Q$. We say that a mutation $f$: $U(X', P) → V(Y, Q)$ is an extension of a mutation $f$: $U(X, P) → V(Y, Q)$ (and then $f$ is called a restriction of $f'$) if for every $f$ there exists an $f'$ such that $f'|_U = f(a)$ for every $a ∈ X$ and range $f' = range f$ (by range we denote the range of $f$).

Let us prove that

(2.1) If a map $f$: $X' → X$ is an extension of a map $f$: $X → Y$ and mutations $f$: $U(X, P) → V(Y, Q)$ and $f'$: $U'(X', P) → V(Y, Q)$ are extensions of the maps $f$ and $f'$, respectively, then $f'$ is an extension of $f$.

Proof. The mutation $f'$ consists of all maps $g$: $f'^{-1}(U) → U$ defined by $g(a) = f'(a)$, where $f$: $U → Q$, $U ∈ U(X, P)$, $f(a) = f(a)$ for every $a ∈ X$, $U ⊂ U$, $U ∈ U(X, P)$, $V ⊂ V(Y, Q)$. The mutation $f'$ consists of all maps $g$: $f'^{-1}(U) → U$ defined by $g(a) = f'(a)$, where $f$: $U → Q$, $U ∈ U(X, P)$, $f(a) = f(a)$ for every $a ∈ X$. The definition of $f'$ shows that $f' ≅ f$ (see (1.10)) and range $f' = range f$.

Thus, $f'$ is an extension of $f$.

(2.2) If $f$: $U(X, P) → U(X', P)$ is an extension of the inclusion $j$: $X → X'$, then for an arbitrary mutation $f'$: $U'(X', P) → V(Y, Q)$ the mutation $f'$: $U(X, P) → V(Y, Q)$ is a restriction of $f'$.

Proof. Take an arbitrary $f' = f'j$. By the definition $f' = f'j$, where $f'j = j$, that is, $f'j|_j = j$, where $j$: $U → Q$ is a map such that $j(x) = j(x)$ for every $x ∈ X$, $U ∈ U(X, P)$, $U ⊂ U$, $U ∈ U(X, P)$, $V ⊂ V(Y, Q)$, and for every $x ∈ X$ we obtain $f(x) = f'j(x) = f'(j)(x) = f'(j)(x)$ and obviously range $f' = range f'j$. Thus, $f'$ is a restriction of $f'$.

(2.3) THEOREM. If mutations $f$: $U(X, P) → V(Y, Q)$ and $f'$: $U'(X', P) → V(Y, Q)$ are both restrictions of a mutation $f$: $U'(X', P) → V(Y, Q)$, then $f' = f$.

First we prove the following

(2.4) LEMMA. If $X$ is a closed subset of a metric space $X'$, $Z$ is an ANR(3)-space and maps $j$: $X → Z$ are both extensions of a map $f$: $X → Z$, then there exists a neighbourhood $U$ of $X$ in $X'$ such that $j|_U → j|_U$.

Proof. Let $P$: $X × (0, 1)$ be a map defined by the formula

$$F(x, t) = \begin{cases} f(x) & \text{for } x ∈ X \text{ and } 0 ≤ t ≤ 1, \\ f'(x) & \text{for } x ∈ X' \text{ and } t = 0, \\ f'(x) & \text{for } x ∈ X' \text{ and } t = 1. \end{cases}$$

Obviously, $F$ is well defined and continuous. Since $Z$ is an ANR(3)-space and the set $X × (0, 1) → X × (0, 1)$ is closed in $X × (0, 1)$, there exists a neighbourhood $W$ of $X × (0, 1)$ in $X × (0, 1)$ such that $X × (0, 1)$ is a homotopy joining the maps $j|_U$ and $j|_U$.

Proof of Theorem (2.3). Take two arbitrary maps $f$ and $g$, belonging to $f$ and $g$, respectively, with a common domain and a common range, i.e., $f_0$, $g_0$: $U → V$, where $U ∈ U(X, P)$, $V ⊂ V(Y, Q)$. Since $f'$ is an extension of $f$, there exists an $f'_0$: $U → V$, where $U ∈ U(X, P)$ and $f'_0(a) = f'(a)$ for every $a ∈ X$. Since $f'$ is also an extension of $g$, there exists an $f'_0$: $U → V$, where $U ∈ U(X, P)$ and $f'_0(a) = g(a)$ for every $a ∈ X$. By the definition of a mutation there exists a $U'_0$: $U → V$ such that $U'_0 ∈ U_0 ⊂ U'$ and $f_0|_U → f_0|_U$ (see (1.10)). By the first theorem of Hanner ([11], p. 96) $V_0$ is an ANR(3)-space. Hence by Lemma (2.4) there exists a $U_0$: $U → V$, such that $U_0 ∈ U_0 ⊂ U_0$ and $f_0|_U → f_0|_U$ and $f_0|_U → f_0|_U$. Therefore we have obtained $f_0|_U → f_0|_U$ and $f_0|_U → f_0|_U$. Thus, $f → g$ (see (1.9)) and the proof is completed.
It is obvious that the collection \( r' \) satisfies the condition (1.7).

Now, take two arbitrary maps \( V'_1, r'_1 \in r' \) with a common domain and a common range, i.e., \( r'_1 : V' \to V' \), where \( V' \subseteq V'(X', P) \subseteq V(X', Y, Q) \).

By the definition of the collection \( r' \) there exist \( U'_1, U_1 \in U(X, P) \), \( U'_1 \subseteq U'_2 \subseteq U(X', Y) \), and \( r_1, r'_1 \subseteq r' \) such that \( U_1 \subseteq U'_2 \subseteq U'_1 \), \( r_1 : U_1 \to U'_2 \), and

\[
\begin{align*}
    r'_2(y) &= h_0 r_2 h'_2(y) \\
    r_2(y) &= h_0 r_2 h'_2(y)
\end{align*}
\]

for every \( y \in V' \), where \( h'_2 : V' \to V'_2 \) and \( h'_2 : V' \to U'_1 \) are maps such that

\[
    h'_2(y) = h_0 r_2 h'_2(y) = h_0 r_2 h'_2(y) = h_0 r_2 h'_2(y) = \quad \text{for every } y \in V'.
\]

Let us prove the following

\[ (3.1) \text{ Theorem. Suppose } P \text{ and } Q \text{ are ANR}(3R)\text{-spaces, } X' \text{ and } Y' \text{ are closed subsets of } P \text{ and } Q \text{, and } X \text{ and } Y \text{ are closed subsets of } X' \text{ and } Y', \text{ respectively, and let } X' \to Y' \text{ be a homeomorphism such that } h(X) = Y. \text{ If there exists a mutual retraction } r : U'(X', P) \to U(X, P), \text{ then there exists a mutual retraction } r' : V'(Y', Q) \to V(Y, Q).
\]

Proof. Since \( Q \) is an ANR(3R)-space, then there exists a map \( h_0 : U_0 \to Q \) such that \( h_0(x) = h(x) \) for every \( x \in X \), where \( U_0 \subseteq U(X, P) \).

Take an arbitrary \( V \subseteq V(X, Q) \). Then \( h_0'(V) \subseteq h_0(U) \subseteq U(X, P) \).

Take an arbitrary \( U \subseteq U(X, P) \) such that \( U \subseteq h_0'(V) \).

Since \( r \) is a retraction then (see (1.7)) there exists an \( r \subseteq r' \) such that \( r : U \to U \), where \( U \subseteq U(X, P) \).

By the first theorem of Hanner (11) \( U \subseteq U ' \) is an ANR(3R)-space.

Therefore there exists a map \( h' : V' \to U' \) such that \( h'(y) = h^{-1}(y) \) for every \( y \in V' \), where \( V' \subseteq V(Y, Q) \).

Let us define the map \( r' : V' \to V' \) by the formula

\[
r'(y) = h_0 r h'(y) \quad \text{for every } y \in V'.
\]

Let \( r' \) be the collection of all maps \( r' \) defined in this way. Let us prove that \( r' : V'(Y, Q) \to V(Y, Q) \) is a retraction.

Take an arbitrary map \( r' \subseteq r' \), \( r'_1 : V'_1 \to V_1 \). Take arbitrary neighbourhoods \( V'_2 \subseteq V'(Y, Q) \) and \( V_2 \subseteq V(Y, Q) \) such that \( V'_2 \subseteq V'_1' \subseteq V_2 \subseteq V_1 \).

Consider the map \( r'_2 : V'_2 \to V_2 \) defined by the formula \( r'_2(y) = r'_2(y) \) for every \( y \in V'_2 \). We want to show that \( r'_2 \subseteq r' \) (see (1.6)). Since \( r'_2 \subseteq r' \), there exist \( U'_2 \subseteq U(X, P) \), \( U_2 \subseteq U(X', P) \), and \( r_2 \subseteq r' \) such that \( U_2 \subseteq h_0'(V'_2) \subseteq U'_2 \), \( r_2 : U_2 \to U'_2 \), and

\[
r'_2(y) = h_0 r'_2 h'_2(y) \quad \text{for every } y \in V'_2,
\]

where \( h'_2 : V'_2 \to U'_2 \) is a map such that \( h'_2(y) = h^{-1}(y) \) for every \( y \in V' \).

Let us define the map \( h'_2 : V'_2 \to U'_2 \) by the formula \( h'_2(y) = h'_2(y) \) for every \( y \in V'_2 \).

Then for every \( y \in V'_2 \) we have

\[
h'_2(y) = h_0 r'_2 h'_2(y) = r'_2(y) = r'_2(y).
\]

Therefore \( r'_2 \subseteq r' \). Thus the collection \( r' \) satisfies the condition (1.6).
sequence. Let us denote by \( f \) the collection of all maps \( f: U \to V \), where \( U \subseteq U(X, P), V \subseteq V(Y, Q) \), that are such that

\[
(3.5) \quad f \triangleq f_0 \subseteq U \quad \text{in} \quad V \quad \text{for almost all} \quad k
\]

(compare (4.5) of [9], p. 57). It follows by (1.1) that \( f \) is a mutation from \( U(X, P) \) to \( V(Y, Q) \) (cf. [9], p. 57). It will be called the mutation associated with the fundamental sequence \( f \).

(3.6) Theorem. Suppose \( r = (r_\alpha : X, X, X)_\alpha \) is a fundamental retraction and \( r: U(X, P) \to U(X, P) \) is the mutation associated with \( r \). Then there exists a mutual retraction \( r': U'(X', P) \to U(X, P) \) homotopic to \( r \).

Proof. By hypothesis we have \( r_\alpha (a) = a \) for every \( a \in X \) and \( k = 1, 2, \ldots \). Let us denote by \( r' \) the collection of all maps \( r \in r \) such that \( r_\alpha (a) = a \) for every \( a \in X \). Let us prove that \( r' \) is a mutation.

Take an arbitrary map \( r_\alpha \in r' \), \( U_\alpha \subseteq U_\beta \), \( U_\alpha \subseteq U(X, P) \), and \( U \subseteq U(X, P) \).

Consider the map \( r_\alpha \in U'(X', P) \) and \( U \subseteq U(X, P) \) such that \( U_\alpha \subseteq U_\beta \), \( U_\beta \subseteq U \) and \( U \subseteq U(X, P) \).

Then \( U_\beta \subseteq U_\alpha \) is defined by the formula \( r_\alpha (a) = a \) (every \( a \in X \)). Since \( r_\alpha \in U(X, P) \), we have \( r_\alpha \in r' \), where \( r' \) is a mutation, we have by (1.6), \( r_\alpha \in r \). Moreover, for every \( a \in X \) we have \( r_\alpha (a) = a \). Therefore \( r \in r' \). Thus, the collection \( r' \) satisfies the condition (1.6).

Now, take an arbitrary neighbourhood \( U \subseteq U(X, P) \). Since \( r \) is a fundamental sequence, by (3.1) there exist a neighbourhood \( U' \subseteq U(X, P) \) and a natural number \( k \) such that \( r_\alpha (U') \subseteq U \). Let \( r' = U' \subseteq U \) be the map defined by the formula \( r_\alpha (a) = a \) (every \( a \in X \)). It is obvious that \( r \subseteq r' \). Thus, the collection \( r' \) satisfies the condition (1.7).

Now, take two arbitrary maps \( r_{\alpha_1}, r_{\alpha_2} \in r' \) with a common domain and a common range, i.e. \( r_{\alpha_1}, r_{\alpha_2}: U_1 \subseteq U_1 \), where \( U_1 \subseteq U'(X, P), U_1 \subseteq U(X, P) \).

Since \( r_\alpha \in r' \) and \( r \) is a mutation, by (1.8) there exists a neighbourhood \( U_1 \subseteq U(X, P) \) such that \( U_1 \subseteq U_1 \) and \( r_\alpha (U_1) = r_\alpha (U_1) \). Therefore the collection \( r' \) satisfies the condition (1.8). Thus, \( r' \) is a retraction.

Since for every \( r \in r' \) and for every \( a \in X \) we have \( r_\alpha (a) = a \), \( r \) is a mutation retraction.

It is obvious that \( r' \) is a restriction of \( r \). Hence by Theorem (2.3) we obtain \( r' = r \) and the proof is concluded.

By Theorem (3.6) we obtain the following

(3.7) Corollary. If \( X \) is a fundamental retraction of \( X' \), then \( X \) is a mutual retraction of \( X' \).

Remark. The converse of (3.7) is not true. A corresponding example will be given in § 6.

Since every retraction of \( X' \) is a fundamental retraction of \( X' \) (61), p. 84), by (3.7) we get the following

(3.8) Corollary. If \( X \) is a retraction of \( X' \), then \( X \) is a mutual retraction of \( X' \).

It is obvious that

\[
3.9 \quad \text{If} \quad r: U(X, P) \to U(X, P) \quad \text{and} \quad r': U'(X', P) \to U'(X', P) \quad \text{are mutual retractions, then the composition} \quad r': U'(X', P) \to U(X, P) \quad \text{is a mutual retraction.}
\]

Hence we obtain the following

(3.10) Corollary. If \( X \) is a mutual retraction of \( X' \), then \( X \) is a mutual retraction of \( X' \).

We say that a mutation \( f: U(X, P) \to U(Y, Q) \) is a \( h \)-mutation if there exists a mutation \( g: V(Y, Q) \to U(X, P) \) such that \( f = g \). The definitions of the \( h \)-fundamental sequence, (3), p. 57). By the definitions the existence of an \( h \)-mutation \( f: U(X, P) \to U(Y, Q) \) is equivalent to the inequality \( ShX \gg ShY \).

It follows by (3.5) of [10] that

(3.11) \( h \)-mutation associated an \( h \)-fundamental sequence is an \( h \)-mutation.

Let us prove that

(3.12) Every mutual retraction is an \( h \)-mutation.

Proof. Consider an arbitrary mutual retraction \( r: U'(X', P) \to U(X, P) \).

Let \( f = U(X, P) \to U(Y, Q) \) be the extension of the inclusion \( j: X \to X' \) and let \( u: U(X, P) \to U(Y, Q) \) be a mutation consisting of all inclusions of the system \( U(X, P) \).

We shall prove that \( f = u \).

By (2.3) if is a restriction of \( r \). Let us prove that \( u \) is also a restriction of \( r \).

Take an arbitrary map \( u \subseteq u \). Then \( u: U_1 \to U_1 \subseteq U(X, P) \).

Since \( r \subseteq r' \) and \( r \) is a mutation, by (1.7) there exist \( U_{\alpha_1} \subseteq U_{\alpha_2} \subseteq U(X, P) \) and \( u_\alpha (a) = a \) (every \( a \in X \)).

Therefore \( r \subseteq r' \). Hence, by (2.3), we obtain \( r' = r \). Thus, \( r \) is an \( h \)-mutation.

By (3.12) we obtain the following

(3.13) Corollary. If \( X \) is a mutual retraction of \( X' \), then \( ShX \gg ShX' \).

By propositions (4.6) and (4.7) of [9], p. 57) for compacts the relation \( ShX \gg ShX' \) is equivalent to the relation \( ShX \gg ShX' \).

Hence by (3.13) we obtain the following

(3.14) Corollary. If \( X \) is a mutual retraction of a compact \( X' \), then \( ShX \gg ShX' \).

Remark. For non-compact metrizable spaces (3.14) is not true. A corresponding example will be given in § 6.

Let us prove that
(3.15) If $X'$ is a closed subset of a metrizable space $X''$, $X$ is a closed subset of $X'$ and $X$ is a mutational retract of $X''$, then $X$ is a mutational retract of $X'$.

Proof. By hypothesis there exists a mutational retraction $r: U''(X'', P) \to U(X, P)$. Let $f: U(X', P) \to U''(X'', P)$ be the extension of the inclusion $j: X' \to X''$. It is easy to see that $r f: U(X', P) \to U(X, P)$ is a mutational retraction.

Let us denote by $\bigoplus_{t \in T} X_t$ the sum of the family $(X_t)_{t \in T}$ of disjoint spaces (see [5], p. 70).

(3.16) Theorem. If $X_t$ is a mutational retract of $X_t'$ for every $t \in T$, then $\bigoplus_{t \in T} X_t$ is a mutational retract of $\bigoplus_{t \in T} X_t'$.

Proof. By hypothesis, for every $t \in T$ there exists a mutational retraction $r_t: U_t(X_t', P_t) \to U_t(X_t, P_t)$, where $P_t$ is an ANR(3R)-space containing $X_t'$ as a closed subset. Let $X = \bigoplus_{t \in T} X_t$, $X' = \bigoplus_{t \in T} X_t'$, $P = \bigoplus_{t \in T} P_t$.

Obviously $X$ is closed in $X'$ and $X'$ is closed in $P$. By Theorem 8.1 of [11] (p. 99) $P$ is an ANR(3R)-space. Consider the complete neighbourhood systems $U(X', P)$ and $U(X', P)$. Let $r$ be the collection of all combinations of maps $r_t \in r_t$, i.e. $r$ consists of all maps $r: U \to U'$ where $U \in U(X', P)$ and $U' \in U(X', P)$, such that for every $x \in U \cap P$ we have $r(x) = r_t(x)$, where $t \in r_t$. It is obvious that $r: U(X', P) \to U(X, P)$ is a mutational retraction. Thus, the proof is concluded.

(3.17) Theorem. If $X$ is a mutational retract of a metrizable space $X'$ contained as a closed subset in an ANR(3R)-space $P$, then for every mutation $f: U(X, P) \to V(Y, Q)$ there exists an extension $f': U(X', P) \to V(Y, Q)$ of $f$.

Proof. By hypothesis and Theorem (3.1) there exists a mutational retraction $r: U'(X', P) \to U(X, P)$. Let us put $f' = fr: U'(X', P) \to V(Y, Q)$. Let us show that $f'$ is an extension of $f$. Take an arbitrary map $f \in f$; $f: U \to V$. Since $r$ is a mutation, then by (1.8) there exists an $r \in r$ such that $fr: U \to U$. Let us put $f' = fr$. Then we have $f' \in f$, $f'(x) = fr(x) = f(x)$ for every $x \in X$ and range$f' = range f$. Thus, $f'$ is an extension of $f$ and the proof is finished.

Let $X$ be a closed subset of a metrizable space $X'$. We say that $X$ is a mutational neighbourhood retract of $X'$ if there exists a closed neighbourhood $W$ of $X$ in $X'$ such that $X$ is a mutational retract of $W$ (compare the definition of the fundamental neighbourhood retract, [3] p. 59, [6] p. 84). Obviously, this notion is a topological invariant.

By (3.7) we obtain the following

(3.18) Corollary. If $X$ is a fundamental neighbourhood retract of $X'$ then $X$ is a mutational neighbourhood retract of $X'$.

Remark. The converse of (3.18) is not true. A corresponding example will be given in § 6.

By Theorem (3.17) we get the following

(3.19) Corollary. If $X$ is a mutational neighbourhood retract of a metrizable space $X'$ contained as a closed subset in an ANR(3R)-space $P$, then there exists a closed neighbourhood $W$ of $X$ in $X'$ such that for every mutation $f: U(X, P) \to V(Y, Q)$ there exists an extension $f': U(W, P) \to V(Y, Q)$ of $f$.

By Theorem (3.16) we obtain the following

(3.20) Corollary. If $X_t$ is a mutational neighbourhood retract of $X_t$ for every $t \in T$, then $\bigoplus_{t \in T} X_t$ is a mutational neighbourhood retract of $\bigoplus_{t \in T} X_t'$.

Remark. For fundamental retracts and fundamental neighbourhood retracts theorems analogous to (3.16) and (3.20) are not true. A corresponding example will be given in § 6. If the set $T$ is finite, then analogous theorems for fundamental retracts are also true. The simple proofs are left to the reader (compare the proof of Theorem (4.3) of [10]).

§ 4. Mutational absolute retracts and mutational absolute neighbourhood retracts.

We say that a metrizable space $X$ is a mutational absolute retract (shortly: MAR) if, for every metrizable space $X'$ containing $X$ as a closed subset, the set $X$ is a mutational retract of $X'$. A metrizable space $X$ is said to be a mutational absolute neighbourhood retract (shortly: MANR) if for every metrizable space $X'$ containing $X$ as a closed subset, the set $X$ is a mutational neighbourhood retract of $X'$ (compare definitions of MAR and MANR, [3] p. 65, [6] p. 94). By Theorem (3.1) these notions are topological invariants.

It is obvious that

(4.1) Every MAR-space is a MANR-space.

It follows by (3.7) that

(4.2) Every MAR-space is a MAR-space.

Hence by (22.3) of [6]

(4.3) Every MAR-space is a MAR-space.

It follows by (3.18) that

(4.4) Every MANR-space is a MAR-space.

Hence by (22.3) of [6]

(4.5) Every MANR-space is a MANR-space.

Remark. The converses of (4.1)-(4.5) are not true. Corresponding examples will be given in § 6.

(4.6) Theorem. Every mutational retract of a MAR-space is a MAR-space.
First we prove the lemma concerning the metrizability of the matching \( X \triangleleft Y \) of spaces \( X \) and \( Y \) (see [1], p. 116).

(4.7) Lemma. If \( X \) and \( Y \) are disjoint metrizable spaces, \( X_0 \) is a closed subset of \( X \), \( h: X_0 \to Y \) is a homeomorphic imbedding such that \( h(X_0) \) is a closed subset of \( Y \), then \( X \triangleleft Y \) is a metrizable space.

In [12] (p. 95) J. Nagata has proved the following

(4.8) Theorem (Nagata). If a topological space \( R \) can be represented in the form \( R = \bigsqcup_{i \in A} S_i \), where \( S_i \) \( i \in A \) is locally finite and \( S_0 \) is closed metrizable subspaces, then \( R \) is metrizable.

Proof of Lemma (4.7). Let \( \varphi: X \oplus Y \to X \triangleleft Y \) be a natural mapping (see [8], p. 63). It follows by the hypotheses that the set \( \varphi(X) \) is homeomorphic to \( X \), \( \varphi(Y) \) is homeomorphic to \( Y \) and the sets \( \varphi(X) \) and \( \varphi(Y) \) are closed in \( X \triangleleft Y \). Hence by Theorem (4.8) we obtain Lemma (4.7).

Proof of Theorem (4.6). Take an arbitrary MAR-space \( X \) and let \( X_0 \) be a mutual retraction of \( X \). Take an arbitrary metrizable space \( X' \) containing \( X \) as a closed subset. We can assume that \( X \triangleleft X' \) is closed in \( X \). By Lemma (4.7) we can define a metric in the set \( X \triangleleft X' \) such that the spaces \( X \) and \( X' \) are closed subspaces of \( X \triangleleft X' \). By the Kuratowski-Wojdyslawski theorem ([1], p. 78) the space \( X \triangleleft X' \) may be considered as a closed subset of an ANR(\( \mathbb{R} \))-space \( P \). By the hypotheses \( X_0 \) is a mutual retraction of \( X \) and \( X_0 \) is a MAR-space, it is a mutual retraction of \( P \). Hence by (3.10) \( X_0 \) is a mutual retraction of \( P \). Therefore there exists a mutual retraction \( r: U(P, P) \to U(X_0, P) \). Let \( j: U(X', P) \to U(P, P) \) be the extension of the inclusion \( j: X' \to P \). It is evident that \( i: U(X', P) \to U(X_0, P) \) is a mutual retraction. Therefore \( X_0 \) is a mutual retraction of every metrizable space \( X' \) containing \( X \) as a closed subset. Thus \( X_0 \) is a MAR-space and the proof is concluded.

(4.9) Theorem. MAR-spaces are the same as mutual retractions of ANR(\( \mathbb{R} \))-spaces.

Proof. Take an arbitrary MAR-space \( X \). By the Kuratowski-Wojdyslawski theorem ([1], p. 78) there exists a MAR-space \( P \) containing \( X \) as a closed subset. Since \( X \) is a MAR-space, \( X \) is a mutual retraction of \( P \).

Now, suppose that \( X \) is a mutual retraction of an ANR(\( \mathbb{R} \))-space \( P \). By (4.3) \( P \) is a MAR-space and hence, by Theorem (4.8), \( X \) is a MAR-space. Thus the proof is finished.

By (3.13) and (4.9) we obtain the following

(4.10) Corollary. The shape \( Sh \) of a MAR-space \( X \) is trivial.
Remark. For \textit{FANR} spaces the theorem analogous to (4.13) is not true. A corresponding example will be given in § 6. If the set \( T \) is finite, then the analogous theorem for \textit{FANR} spaces is also true. The simple proof is left to the reader (compare the remark concerning (3.9)).

(4.14) \textbf{Theorem.} A metrizable space \( X \) is a \textit{MAR} space if and only if for every metrizable space \( X' \) containing \( X \) as a closed subset and for every \textit{ANR}(3\[2\]) space \( P \) containing \( X' \) as a closed subset and for every mutation \( f: \text{U}(X, P) \to \text{V}(Y, Q) \) there exists an extension \( f': \text{U}(X', P) \to \text{V}(Y, Q) \) of \( f \).

\textbf{Proof.} Suppose \( X \) is a \textit{MAR} space. Then the required extension \( f' \) exists by Theorem (3.17).

Now, suppose that the second part of Theorem is satisfied. Consider the mutation \( u: \text{U}(X, P) \to \text{U}(X, P) \) consisting of all inclusions. By hypothesis there exists an extension \( u': \text{U}(X', P) \to \text{U}(X, P) \) of \( u \). Let us denote by \( r \) the collection of all maps \( r \in u' \) such that \( r(x) = x \) for every \( x \in X \). Let us show that \( r \) is a mutation.

Take an arbitrary map \( r_1 \in r, r_2 \in \text{U}_1 \). Take arbitrary neighbourhoods \( \text{U}_1 \in \text{U}(X', P) \) and \( \text{U}_2 \in \text{U}(X, P) \) such that \( \text{U}_1 \subset \text{U}_2 \) and \( \text{U}_2 \subset \text{U}_1 \). Consider the map \( r_1 \text{U}_1 \to \text{U}_2 \) defined by the formula \( r_1(x)(s) = r_1(x)(s) \) for every \( s \in \text{U}_1 \). Since \( r_1 \in u' \) and \( u' \) is a mutation, by (1.6) \( r_1 \in u' \). Moreover, for every \( x \in X \) we have \( r_1(x) = r_1(x) = x \). Therefore \( r_1 \in u' \). Thus, the collection \( r \) satisfies the condition (1.6).

Now, take an arbitrary neighbourhood \( U \in \text{U}(X, P) \). Take \( u \in u' \) with range \( u = U \). By the definition of an extension there exists a \( u \in u' \) such that range \( u' \equiv \text{range}u = U \) and \( u(x)(s) = u(x)(s) = x \) for every \( x \in X \). Therefore \( u \in u' \) and \( u' \in u' \). Thus, the collection \( r \) satisfies the condition (1.7).

Now, take two arbitrary maps \( r_1, r_2 \in \text{U}_1 \) with a common domain and a common range, \( r_1, r_2 : \text{U}_1 \to \text{U}_2 \). Since \( r_1, r_2 \in u' \) and \( u' \) is a mutation, by (1.8) there exists a \( U \in \text{U}(X', P) \) such that \( r_1, r_2 \in \text{U}(X', P) \). Therefore the collection \( r \) satisfies the condition (1.9). Thus, \( r \) is a mutation and, obviously, it is a metrizable retraction. Therefore \( X \) is a metrizable retract of \( X' \). Thus, \( X \) is a \textit{MAR} space, and the proof is finished.

(4.15) \textbf{Theorem.} A metrizable space \( X \) is a \textit{MAR} space if and only if for every metrizable space \( X' \) containing \( X \) as a closed subset there exists a closed neighbourhood \( W \) of \( X \) in \( X' \) such that for every \textit{ANR}(3\[2\]) space \( P \) containing \( X' \) as a closed subset and for every mutation \( f: \text{U}(X, P) \to \text{V}(Y, Q) \) there exists an extension \( f': \text{U}(W, P) \to \text{V}(Y, Q) \) of \( f \).

\textbf{Proof.} Suppose \( X \) is a \textit{MAR} space and take arbitrary spaces \( X' \) and \( P \) satisfying the hypotheses. Then the required neighbourhood \( W \) and an extension \( f' \) exist by (3.19).

Now, suppose that the second part of Theorem is satisfied. Consider the mutation \( u: \text{U}(X, P) \to \text{U}(X, P) \) consisting of all inclusions. By hypothesis there exists an extension \( u': \text{U}(W, P) \to \text{U}(X, P) \) of \( u \). Analogously as in the proof of Theorem (4.14) we define the collection \( r \) and prove that \( r: \text{U}(W, P) \to \text{U}(X, P) \) is a metrizable retraction. Thus, \( X \) is a \textit{MAR} space and the proof is concluded.

\section{§ 5. Compact mutational retracts.} Let us prove the following

(5.1) \textbf{Theorem.} If a compactum \( X \) is a mutational retract of a metrizable space \( X' \), then \( X \) is a fundamental retract of \( X' \).

\textbf{Proof.} By hypothesis there exists a metrizable retraction \( r: \text{U}(X', P) \to \text{U}(X, P) \). By the Kuratowski-Woźniakiewicz theorem and Theorem (3.1) we can assume that \( P \in \text{ANR}(\mathbb{R}) \). Let \( U_k \in \text{U}(X, P), k = 1, 2, \ldots \), be a sequence of neighbourhoods of \( X \) in \( P \) such that

\begin{equation}
U_{k+1} \subset U_k \quad \text{for} \quad k = 1, 2, \ldots \quad \text{and} \quad \bigcap_{k=1}^{\infty} U_k = X.
\end{equation}

Let \( r_k : U_{k+1} \to U_k, k = 1, 2, \ldots \), be a constituent of the mutation \( r \); obviously we can require that \( U_k \subset U_{k+1} \). Let us define maps \( r_k : U_k \to U_k \), \( k = 1, 2, \ldots \), setting \( r_k = r_k(U_k) \).

Now, we shall define by induction a sequence of neighbourhoods

\begin{equation}
W_i \in \text{U}(X', P) \quad \text{and a sequence of maps} \quad r_i : W_i \to U_i \quad \text{for} \quad i = 1, 2, \ldots, k \quad \text{and} \quad k = 1, 2, \ldots, \text{satisfying the following conditions:}
\end{equation}

\begin{align}
(5.3) & U_{k+1} \subset U_k \cap U_{k+1} \quad \text{for} \quad k = 2, 3, \ldots, \\
(5.4) & r_k = r_k(U_k) \quad \text{for} \quad k = 1, 2, \ldots, \\
(5.5) & r_2(1) = r_1(1) \quad \text{for} \quad x \in U_1 \quad \text{and} \quad i = j < k, \\
(5.6) & r_i = r_i(U_i) \quad \text{for} \quad i = k \quad \text{and} \quad k < j.
\end{align}

Let us put \( W_k = U_k \) and \( r_i = r_i(U_i) \). Consider the maps \( r_i : W_i \to U_i \) and \( r_k : U_k \to U_k \). Since \( r_k \) and \( r_i \) are restrictions of the constituents of \( r \) and \( U_i \), \( U_k \subset U_k \), there exists a neighbourhood \( U_i' \in \text{U}(X', P) \) such that \( U_i' \subset U_i \) and \( r_i(U_i') = r_k(U_k) \) in \( U_i \).

Let us put \( r_i = r_i(U_i) \). Since \( U_i \in \text{ANR}(\mathbb{R}) \) and \( r_k(U_k) \) is an extension of \( r_i(U_i) \), by Borsuk's homotopy extension theorem ([1], p. 94) the map \( r_i(U_i) \) has an extension \( r_i(U_i) \to U_i \) such that \( r_i(U_i) = r_i(U_i) \), \( U_i' \subset U_i \) and the neighbourhoods \( U_i', U_k \) and the maps \( r_i, r_k \) satisfy the required conditions (5.3)-(5.6).

Now, suppose that we have defined neighbourhoods \( U_i \) and maps \( r_i : U_i \to U_i \) for \( k = 1, 2, \ldots, n \) and \( i = 1, 2, \ldots, k \) in a such manner that the conditions (5.3)-(5.6) are satisfied. We shall define a neighbourhood \( W_{n+1} \) and maps \( r_{n+1} : U_{n+1} \to U_i \) for \( i = 1, 2, \ldots, n+1 \) satisfying the required conditions.
Consider the maps $\tilde{r}_n: \mathcal{U}_n \to \mathcal{U}_n$ and $\tilde{r}_{k+1}: \mathcal{U}_{k+1} \to \mathcal{U}_n$. Since these maps are restrictions of the constiituents $r_n: \mathcal{U}_n \to \mathcal{U}_n$ and $r_{k+1}: \mathcal{U}_{k+1} \to \mathcal{U}_n$, for $r$ and $\mathcal{U}_{k+1} \subset \mathcal{U}_n$, there exists a neighbourhood $\mathcal{U}_n^{k+1} \subset \mathcal{U}(X', \mathcal{P})$ such that $\mathcal{U}_n^{k+1} \subset \mathcal{U}_n \cap \mathcal{U}_n^{k+1}$ and $\tilde{r}_n | \mathcal{U}_n^{k+1} = \tilde{r}_{k+1} | \mathcal{U}_n^{k+1}$ in $\mathcal{U}_n$.

Let us put $\mathbf{r}_{n,k+1} = \tilde{r}_n | \mathcal{U}_n^{k+1}$. Since $\mathbf{r}_{n,k+1} | \mathcal{U}_n^{k+1}$ is an extension of $\tilde{r}_n | \mathcal{U}_n^{k+1}$ by Borsuk's homotopy extension theorem the map $\mathbf{r}_{n,k+1}$ has an extension $\tilde{r}_{k+1}: \mathcal{U}_n \to \mathcal{U}_n$ such that $\mathbf{r}_{n,k+1} | \mathcal{U}_n^{k+1}$ is homotopic to $\tilde{r}_{k+1}$. Therefore, the neighbourhood $\mathcal{U}_n^{k+1}$ and the maps $\mathbf{r}_{n,k+1}$ and $\tilde{r}_n$ satisfy the required conditions (5.3)-(5.6).

Now, suppose that we have defined maps $\mathbf{r}_{n,k+1}, \mathbf{r}_{n,k+2}, \ldots, \mathbf{r}_{n,n}$ satisfying the conditions (5.4)-(5.6). We shall define $\mathbf{r}_{n,n+1}$. By (5.6) we have $\mathbf{r}_{n,n+1} | \mathcal{U}_n^{n+1} = \tilde{r}_n | \mathcal{U}_n^{n+1}$ and by (5.5) the map $\tilde{r}_n$ considered as a map into $\mathcal{U}_n^{n+1}$ has the extension $\tilde{r}_n^{n+1}: \mathcal{U}_n^{n+1} \to \mathcal{U}_n$. Hence by Borsuk's homotopy extension theorem the map $\mathbf{r}_{n,n+1}$ has an extension $\mathbf{r}_{n,n+1}^{n+1}: \mathcal{U}_n^{n+1} \to \mathcal{U}_n$ homotopic to $\tilde{r}_n^{n+1}$. Therefore, the map $\mathbf{r}_{n,n+1}^{n+1}$ satisfies the required conditions. Thus, the sequences of neighbourhoods $\mathcal{U}_n^{k+1}$ and maps $\mathbf{r}_n$ defined above satisfy the conditions (6.5)-(6.6).

Since $\mathcal{P} \in \mathcal{AR}(\mathcal{M})$, for every $k = 1, 2, \ldots$ there exists a map $\tilde{r}_2: \mathcal{P} \to \mathcal{P}$ such that $\tilde{r}_2(\mathcal{P}) = \mathcal{P}$ for every $x \in \mathcal{P}$.

Let us prove that $r = (\tilde{r}_2, X, \mathcal{P})$ is a fundamental retraction. Take an arbitrary neighbourhood $U \in \mathcal{U}(\mathcal{P}, \mathcal{E})$. By (5.3) and the compactness of $X$ there exists a natural number $n$ such that $U \subset U \cap U^{n+1}$ for $n > k \geq n$. Let us observe that $\tilde{r}_n | \mathcal{U}_n^{n+1} = \tilde{r}_n | \mathcal{U}_n$ in $\mathcal{U}_n$ for $k \geq n$.

Indeed, by (5.4) we obtain $\tilde{r}_n | \mathcal{U}_n^{n+1} = \tilde{r}_n | \mathcal{U}_n^{n+1} = \tilde{r}_n | \mathcal{U}_n^{n+1} = \tilde{r}_n | \mathcal{U}_n^{n+1}$ in $\mathcal{U}_n$.

Hence by (5.6) we obtain (5.7). Therefore, $r$ is a fundamental sequence.

Since $r$ is a mutual retraction, by (5.4), (5.5) and the definition of $\tilde{r}_2$ it follows at once that $\tilde{r}_2(x) = x$ for every $x \in X$. Thus, $r$ is a fundamental retraction and the proof is concluded.

Theorem 5.1 implies at once the following

(5.8) COROLLARY. Every compact MAR-space is a FAR-space and every compact MANR-space is a FANR-space.

§ 6. Examples. First we give an example of a space which is not a MAR-space.

(6.1) Solenoids of Van Dantzig are not MAR-spaces.

Proof. By (1.3) of [7] solenoids are not movable (for the definition of movability see [7], p. 137). Therefore by (3.12) of [7] they are not FANR-spaces. Hence by Corollary (5.8) and the compactness of solenoids we obtain (6.1).

Now, we give an example of a MAR-space which is not a FAR-space.

Let us denote by $A_n$ (for $n = 1, 2, \ldots$) the closed half-line lying in the plane $\mathcal{P}$ with the end-point $(0, 0)$ and containing the point $(1, 1)$, and by $A_{n,n}$ the half-line with the end-point $(0, 0)$ and containing the point $(1, 0)$. Consider the subspace $X = \bigcup A_n$ of $\mathcal{P}$. Let us prove that

(6.2) The space $\mathcal{X}$ is a MAR-space and it is not a FAR-space.

Proof. Obviously, $X$ is a closed subset of $\mathcal{P}$. It is easy to see that for every neighbourhood $U$ of $X$ in $\mathcal{P}$ there exists a homeomorphical Imbedding $r: \mathcal{P} \to \mathcal{P}$ such that $r(x) = x$ for every $x \in X$. The collection $r$ of all maps $r$ obtained in this way is a mutual retraction $r: \mathcal{R}(\mathcal{P}, \mathcal{P}) \to \mathcal{U}(\mathcal{X}, \mathcal{P})$. Hence by Theorem (4.9) $X$ is a MAR-space.

Now suppose, on the contrary, that $X$ is a FAR-space. Then there exists a fundamental retraction $r = (r_2, \mathcal{P}, X, \mathcal{P})$. It follows by the definitions that

(6.3) $r_2(x) = x$ for every $x \in X$ and every $k = 1, 2, \ldots$

and

(6.4) for every neighbourhood $U$ of $X$ in $\mathcal{P}$ $r_2 | \mathcal{U}_n = r_2 | \mathcal{U}_n$ in $\mathcal{U}_n$ for almost all $k$.

Denote by $B_2$ (for $k = 1, 2, \ldots$) the closed square with vertices $(k, 0), (k+1, 0), (k+1, 1), (k, 1)$. Let us show that

(6.5) $B_2 \not\subset \mathcal{R}(\mathcal{P}) - X \not\subset \mathcal{P}$ for every $k = 1, 2, \ldots$.

Since $B_2$ is a locally connected continuum and $B_2$ is continuous, $r_2(B_2)$ is a locally connected continuum. It follows by (6.3) that $B_2 \subset \mathcal{P}$ and $r_2(B_2)$ contains some points belonging to $B_2$ which do not belong to $B_2 \subset \mathcal{P}$; otherwise $r_2(B_2)$ would not be locally connected.

Thus $r_2(B_2) \subset \mathcal{P} - \mathcal{P} = \mathcal{P}$ and hence we obtain (6.5).

Take an arbitrary point $x_0 \in \mathcal{P} - \bigcup A_n$ for $k = 1, 2, \ldots$. The set $\{x_0\}$ is closed in $\mathcal{P}$, because it has no accumulation points. Therefore, the set $\mathcal{U} = \mathcal{P} - \bigcup A_n$ is a neighbourhood of $X$ in $\mathcal{P}$. Since $x_0 \not\in \mathcal{R}(\mathcal{P})$, $r_2(x_0)$ is not contained in $\mathcal{U}$ for any $k = 1, 2, \ldots$, which contradicts (6.4). Thus, $X$ is not a FAR-space and the proof is concluded.
The same example shows that the converses of (3.7) and (4.2) are not true.

Now, we give an example of a MANR-space which is not a FANR-space. This example, due to S. Nowak, was described for other purposes in [10].

Consider the subset $Z$ of the plane $\mathbb{E}^2$ which is the closure of the set consisting of all points $(x, y) \in \mathbb{E}^2$ such that $y = \cos(\pi x)$ where $-1 < x < 1$ and $x \neq 0$ and of all points $(x, y) \in \mathbb{E}^2$ such that $x^2 + (y + 1)^2 = 1$ and $y < -1$. Denote by $\mathcal{N}$ the set of natural numbers with the discrete topology and put $X = Z \times \mathcal{N}$. Let us prove that

(6.6) The space $X$ is a MANR-space and it is not a FANR-space.

Proof. Let $X_n = Z \times \{n\}$ for $n = 1, 2, \ldots$. Obviously $X = \bigoplus_n X_n$.

By Theorem (3.1) of [2] $Z$ is a fundamental retract of an annulus. Hence by (3.7) $Z$ is a mutational retract of an annulus, and by Theorem (4.11), it is a MANR-space; therefore so is $X_n$. Thus, by Theorem (4.13), $X$ is a MANR-space.

Now suppose, on the contrary, that $X$ is a FANR-space. Obviously, we can assume that $X$ is a closed subset of $\mathbb{E}^2$. Then there exists a closed neighbourhood $U$ of $X$ in $\mathbb{E}^2$ such that $X$ is a fundamental retract of $U$. The neighbourhood $U$ contains a closed neighbourhood $V$ of $X$ which is the union of a countable family of mutually disjoint annula. Therefore $X$ is a fundamental retract of $V$. Hence, by Theorem (3.2) of [6], $\text{Sh}(X) \leq \text{Sh}(V)$.

Let $S$ be a circle and put $Y = S \times X$. The spaces $V$ and $Y$ are of the same homotopy type; therefore $\text{Sh}(Y) = \text{Sh}(X)$. Hence $\text{Sh}(X) \leq \text{Sh}(Y)$. In [10] S. Nowak has proved that the shapes $\text{Sh}(X)$ and $\text{Sh}(Y)$ are incomparable, which contradicts the inequality obtained. Thus, $X$ is not a FANR-space and the proof is finished.

The same example shows that for non-compact spaces (3.14) is not true if we put $X' = V$, where $V$ is defined in the last proof.

Also the same example shows that the converse of (3.18) is not true. Namely, the space $X$ may be considered as a closed subset of the plane $\mathbb{E}^2$. Since $X$ is a MANR-space, it is a mutational neighbourhood retract of $\mathbb{E}^2$. But, as we have shown, $X$ is not a fundamental neighbourhood retract of $\mathbb{E}^2$.

This example shows also that for fundamental retracts, fundamental neighbourhood retracts, and FANR-spaces theorems analogous to (3.16), (3.29) and (4.13), respectively, are not true.

It follows by the same example that the converse of (4.1) is not true, i.e.

(6.7) The space $X$ is not a MAR-space.