Mutational retracts and extensions of mutations

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Abstract. Let X be a closed subset of a metrizable space X' considered as a closed subset of an ANR$(\mathfrak{M})$-space P. A mutation [9] $\boldsymbol{r}$: $\boldsymbol{U}'(X', P) \to \boldsymbol{U}(X, P)$ is called a *mutational retraction* if $r(x) = x$ for every $r \in \boldsymbol{r}$ and $x \in X$. If there exists a mutational retraction $\boldsymbol{r}$: $\boldsymbol{U}'(X', P) \to \boldsymbol{U}(X, P)$ then the set X is called a *mutational retract* of the space X'.

Every fundamental retract [3] of a space X' is a mutational retract of X', but not conversely. Every compact mutational retract of a space X' is a fundamental retract of X'.

In the natural manner we define a *mutational neighbourhood retract* of a space X', *mutational absolute retract* (MAR) and *mutational absolute neighbourhood retract* (MANR).

Every FAR-space (FANR-space) [3] is a MAR-space (MANR-space), but not conversely. Every compact MAR-space (MANR-space) is a FAR-space (FANR-space).

In order to extend some standard notions of the homotopy theory onto arbitrary compacta K. Borsuk introduced in [2] the notion of the *fundamental sequence* from a compactum X to a compactum Y. Replacing maps by fundamental sequences one can obtain generalizations or modifications of many standard notions. In this manner K. Borsuk introduced in [3] the notions of the fundamental retract, the fundamental absolute retract and the fundamental absolute neighbourhood retract, which are generalizations of the notions of the retract, the absolute retract, and the absolute neighbourhood retract, respectively. Analogously in [4] K. Borsuk introduced the notion of *shape*, which is a modification of the homotopy type. In [5] and [6] these notions were extended to arbitrary metrizable spaces. Independently, in [9] R. H. Fox extended the notion of shape to arbitrary metrizable spaces, introducing the notion of the *mutation*, as a modification of the notion of the fundamental sequence.

In this paper, replacing fundamental sequences by mutations, we introduce the notions of the mutational retract, the mutational absolute retract and the mutational absolute neighbourhood retract, as generalizations of the notions of the fundamental retract, the fundamental absolute retract, and the fundamental absolute neighbourhood retract, respectively.

§ 1. Basic notions. In this section we recall some definitions from [5], [6], and [9].

Consider closed subsets X and Y of metrizable spaces P and Q, respectively. A *fundamental sequence* $\underline{f} = \{f_k, X, Y\}_{P,Q}$ from X to Y in P, Q is defined to be an ordered triple consisting of X, Y and a sequence of maps $f_k\colon P \to Q$, $k = 1, 2, \ldots$, satisfying the following two conditions:

(1.1) For every neighbourhood V of Y (in Q) there exists a neighbourhood U of X (in P) such that $f_k|U \simeq f_{k+1}|U$ in V for almost all k.

(1.2) For every compactum $A \subset X$ there exists a compactum $B \subset Y$ such that for every neighbourhood V of B (in Q) there exists a neighbourhood U of A (in P) such that $f_k|U \simeq f_{k+1}|U$ in V for almost all k.

Two fundamental sequences $\underline{f} = \{f_k, X, Y\}_{P,Q}$ and $\underline{g} = \{g_k, X, Y\}_{P,Q}$ are called *homotopic* (notation: $\underline{f} \simeq \underline{g}$) if the following two conditions are satisfied:

(1.3) For every neighbourhood V of Y (in Q) there exists a neighbourhood U of X (in P) such that $f_k|U \simeq g_k|U$ in V for almost all k.

(1.4) For every compactum $A \subset X$ there exists a compactum $B \subset Y$ such that for every neighbourhood V of B (in Q) there exists a neighbourhood U of A (in P) such that $f_k|U \simeq g_k|U$ in V for almost all k.

A *composition* of fundamental sequences $\underline{f} = \{f_k, X, Y\}_{P,Q}$ and $\underline{g} = \{g_k, Y, Z\}_{Q,R}$ is defined to be the fundamental sequence $\underline{gf} = \{g_k f_k, X, Z\}_{P,R}$. The fundamental sequence $\underline{i}_X = \{i_k, X, X\}_{P,P}$, where $i_k\colon P \to P$, $k = 1, 2, \ldots$, is the identity map, is called the *fundamental identity sequence.*

By the Kuratowski-Wojdysławski theorem ([1], p. 78) any metrizable space X may be considered as a closed subset of an $\mathrm{AR}(\mathfrak{M})$-space P. Two metrizable spaces X and Y are said to be of the same *shape* in the sense of Borsuk (notation: $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$) if there exist two fundamental sequences $\underline{f} = \{f_k, X, Y\}_{P,Q}$ and $\underline{g} = \{g_k, Y, X\}_{Q,P}$ where P and Q are $\mathrm{AR}(\mathfrak{M})$-spaces containing X and Y, respectively, as closed subsets, such that

(1.5) $$\underline{fg} \simeq \underline{i}_Y \quad \text{and} \quad \underline{gf} \simeq \underline{i}_X .$$

If the fundamental sequences $\underline{f}$ and $\underline{g}$ satisfy the first condition of (1.5), then the shape of X (in the sense of Borsuk) is said to *dominate* the shape of Y (notation: $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$).

If X is a closed subset of a metrizable space X' and X' is a closed subset of an $\mathrm{AR}(\mathfrak{M})$-space P, then a fundamental sequence

$\underline{r} = \{r_k, X', X\}_{P,P}$ such that $r_k(x) = x$ for every $x \in X$ and $k = 1, 2, \ldots$, is called a *fundamental retraction* of X' to X. If there exists a fundamental retraction of X' to X, then X is said to be a *fundamental retract* of X'. A closed subset X of a metrizable space X' is said to be a *fundamental neighbourhood retract of* X' if there exists a closed neighbourhood W of X in X' such that X is a fundamental retract of W.

A metrizable space is called a *fundamental absolute retract* (notation: $X \in \mathrm{FAR}$) if for every metrizable space X', containing X as a closed subset, the set X is a fundamental retract of X'. A metrizable space X is said to be a *fundamental absolute neighbourhood retract* (notation: $X \in \mathrm{FANR}$) if every metrizable space X' containing X as a closed subset, the set X is a fundamental neighbourhood retract of X'.

Let X be a closed subset of an $\mathrm{ANR}(\mathfrak{M})$-space P. The family $\mathbf{U}(X, P)$ of all open neighbourhoods of X in P is called a *complete neighbourhood system* of X in P.

Consider two arbitrary complete neighbourhood systems $\mathbf{U}(X, P)$ and $\mathbf{V}(Y, Q)$. A *mutation* $\mathbf{f}\colon \mathbf{U}(X, P) \to \mathbf{V}(Y, Q)$ from $\mathbf{U}(X, P)$ to $\mathbf{V}(Y, Q)$ is defined as a collection of maps $f\colon U \to V$, where $U \in \mathbf{U}(X, P)$, $V \in \mathbf{V}(Y, Q)$, such that

(1.6) If $f \in \mathbf{f}$, $f\colon U \to V$, $U' \subset U$, $U' \in \mathbf{U}(X, P)$, $V \subset V' \in \mathbf{V}(Y, Q)$ and $f'\colon U' \to V'$ is defined by $f'(x) = f(x)$ for $x \in U'$, then $f' \in \mathbf{f}$;

(1.7) Every neighbourhood $V \in \mathbf{V}(Y, Q)$ is the range of a map $f \in \mathbf{f}$;

(1.8) If $f_1, f_2 \in \mathbf{f}$ and $f_1, f_2\colon U \to V$, then there exists a $U' \in \mathbf{U}(X, P)$ such that $U' \subset U$ and $f_1|U' \simeq f_2|U'$.

If $\mathbf{U}(X, P)$ is a complete neighbourhood system, then the collection $\mathbf{u}$ of all inclusions $u\colon U' \to U$, where U', $U \in \mathbf{U}(X, P)$ and $U' \subset U$, is a mutation from $\mathbf{U}(X, P)$ to itself.

Consider two mutations $\mathbf{f}\colon \mathbf{U}(X, P) \to \mathbf{V}(Y, Q)$ and $\mathbf{g}\colon \mathbf{V}(Y, Q) \to \mathbf{W}(Z, R)$. The *composition* $\mathbf{gf}\colon \mathbf{U}(X, P) \to \mathbf{W}(Z, R)$ of the mutations $\mathbf{f}$ and $\mathbf{g}$ is the mutation constituting the collection of all compositions gf such that $f \in \mathbf{f}$, $g \in \mathbf{g}$ and gf is defined.

Two mutations $\mathbf{f}, \mathbf{g}\colon \mathbf{U}(X, P) \to \mathbf{V}(Y, Q)$ are *homotopic* (notation: $\mathbf{f} \simeq \mathbf{g}$) if

(1.9) For every $f \in \mathbf{f}$ and $g \in \mathbf{g}$ such that $f, g\colon U \to V$ there exists a $U' \in \mathbf{U}(X, P)$ such that $U' \subset U$ and $f|U' \simeq g|U'$.

By the Kuratowski-Wojdysławski theorem ([1], p. 78) any metrizable space X may be considered as a closed subset of an $\mathrm{ANR}(\mathfrak{M})$-space P. Two metrizable spaces X and Y are said to be of the same *shape* in the sense of Fox (notation: $\mathrm{Sh}\,X = \mathrm{Sh}\,Y$) if there exist two mutations $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ and $\boldsymbol{g}\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{U}(X, P)$ such that

$$\boldsymbol{f}\boldsymbol{g} \simeq \boldsymbol{v} \quad \text{and} \quad \boldsymbol{g}\boldsymbol{f} \simeq \boldsymbol{u} \tag{1.10}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are mutations consisting of all inclusions in systems $\boldsymbol{U}(X, P)$ and $\boldsymbol{V}(Y, Q)$, respectively. If the mutations $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfy the first condition of (1.10), then we say that the shape (in the sense of Fox) of X *dominates* the shape of Y (notation: $\mathrm{Sh}\,X \geqslant \mathrm{Sh}\,Y$).

Remark. In [9] R. H. Fox introduced the notion of the shape in an arbitrary category and specialized this notion to metrizable spaces. Some definitions given above differ only formally from Fox's original definitions.

§ 2. Extensions and restrictions of mutations. Let X and Y be closed subsets of $\mathrm{ANR}(\mathfrak{M})$-spaces P and Q, respectively. Consider a map $f\colon X \to Y$. Then there exists a $U_0 \in \boldsymbol{U}(X, P)$ and a map $\hat{f}\colon U_0 \to Q$ such that $\hat{f}(x) = f(x)$ for every $x \in X$. The map $\hat{f}$ determines uniquely a mutation $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ consisting of all maps $g\colon \hat{f}^{-1}(V) \cap U \to V$ defined by $g(x) = \hat{f}(x)$, where $U \subset U_0$, $U \in \boldsymbol{U}(X, P)$ and $V \in \boldsymbol{V}(Y, Q)$. The mutation $\boldsymbol{f}$ is called an *extension* of the map f ([9], p. 54).

Let X be a closed subset of a metrizable space X' considered as a closed subset of an $\mathrm{ANR}(\mathfrak{M})$-space P and let Y be a closed subset of an $\mathrm{ANR}(\mathfrak{M})$-space Q. We say that a mutation $\boldsymbol{f}'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$ is an *extension* of a mutation $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ (and then $\boldsymbol{f}$ is called a *restriction* of $\boldsymbol{f}'$) if for every $f \in \boldsymbol{f}$ there exists an $f' \in \boldsymbol{f}'$ such that $f'(x) = f(x)$ for every $x \in X$ and $\mathrm{range}\, f' = \mathrm{range}\, f$ (by $\mathrm{range}\, f$ we denote the range of f).

Let us prove that

(2.1) *If a map $f'\colon X' \to Y$ is an extension of a map $f\colon X \to Y$ and mutations $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ and $\boldsymbol{f}'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$ are extensions of the maps f and f', respectively, then $\boldsymbol{f}'$ is an extension of $\boldsymbol{f}$.*

Proof. The mutation $\boldsymbol{f}$ consists of all maps $g\colon \hat{f}^{-1}(V) \cap U \to V$ defined by $g(x) = \hat{f}(x)$, where $\hat{f}\colon U_0 \to Q$, $U_0 \in \boldsymbol{U}(X, P)$, $\hat{f}(x) = f(x)$ for every $x \in X$, $U \subset U_0$, $U \in \boldsymbol{U}(X, P)$, $V \in \boldsymbol{V}(Y, Q)$. The mutation $\boldsymbol{f}'$ consists of all maps $g'\colon \hat{f}'^{-1}(V) \cap U' \to V$ defined by $g'(x) = \hat{f}'(x)$, where $\hat{f}'\colon U_0' \to Q$, $U_0' \in \boldsymbol{U}'(X', P)$, $\hat{f}'(x) = f'(x)$ for every $x \in X'$ $U' \subset U_0'$, $U' \in \boldsymbol{U}'(X', P)$, $V \in \boldsymbol{V}(Y, Q)$.

Take an arbitrary $g \in \boldsymbol{f}$, $g\colon \hat{f}^{-1}(V) \cap U \to V$. From the definition of $\boldsymbol{f}'$ there exists a $g' \in \boldsymbol{f}'$ such that $g'\colon \hat{f}'^{-1}(V) \cap U' \to V$. For every $x \in X$ we have $g'(x) = \hat{f}'(x) = f'(x) = f(x) = \hat{f}(x) = g(x)$ and $\mathrm{range}\, g' = V = \mathrm{range}\, g$. Thus, $\boldsymbol{f}'$ is an extension of $\boldsymbol{f}$.

(2.2) *If $\boldsymbol{j}\colon \boldsymbol{U}(X, P) \to \boldsymbol{U}'(X', P)$ is an extension of the inclusion $j\colon X \to X'$, then for an arbitrary mutation $\boldsymbol{f}'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$ the mutation $\boldsymbol{f}'\boldsymbol{j}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ is a restriction of $\boldsymbol{f}'$.*

Proof. Take an arbitrary $f \in \boldsymbol{f}'\boldsymbol{j}$. By the definition $f = f'j'$, where $f' \in \boldsymbol{f}'$ and $j' \in \boldsymbol{j}$, i.e. $j'\colon \hat{j}^{-1}(V) \cap U \to V$, where $\hat{j}\colon U_0 \to Q$ is a map such that $\hat{j}(x) = j(x)$ for every $x \in X$, $U_0 \in \boldsymbol{U}(X, P)$, $U \subset U_0$, $U \in \boldsymbol{U}(X, P)$, $V \in \boldsymbol{V}(Y, Q)$. For every $x \in X$ we obtain $f(x) = f'j'(x) = f'\hat{j}(x) = f'j(x) = f'(x)$ and obviously $\mathrm{range}\, f = \mathrm{range}\, f'$. Thus, $\boldsymbol{f}'\boldsymbol{j}$ is a restriction of $\boldsymbol{f}'$.

(2.3) THEOREM. *If mutations $\boldsymbol{f}, \boldsymbol{g}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ are both restrictions of a mutation $\boldsymbol{f}'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$, then $\boldsymbol{f} \simeq \boldsymbol{g}$.*

First we prove the following

(2.4) LEMMA. *If X is a closed subset of a metric space X', Z is an $\mathrm{ANR}(\mathfrak{M})$-space and maps $\hat{f}, \tilde{f}\colon X' \to Z$ are both extensions of a map $f\colon X \to Z$, then there exists a neighbourhood U of X in X' such that $\hat{f}|U \simeq \tilde{f}|U$.*

Proof. Let $F\colon X \times \langle 0, 1\rangle \cup X' \times \{0, 1\} \to Z$ be a map defined by the formula

$$F(x, t) = \begin{cases} f(x) & \text{for} \quad x \in X \quad \text{and} \quad 0 \leqslant t \leqslant 1, \\ \hat{f}(x) & \text{for} \quad x \in X' \quad \text{and} \quad t = 0, \\ \tilde{f}(x) & \text{for} \quad x \in X' \quad \text{and} \quad t = 1. \end{cases}$$

Obviously, F is well defined and continuous. Since Z is an $\mathrm{ANR}(\mathfrak{M})$-space and the set $X \times \langle 0, 1\rangle \cup X' \times \{0, 1\}$ is closed in $X' \times \langle 0, 1\rangle$, there exists a neighbourhood W of $X \times \langle 0, 1\rangle \cup X' \times \{0, 1\}$ in $X' \times \langle 0, 1\rangle$ such that there exists an extension $\widetilde{F}\colon W \to Z$ of F. By the compactness of the unit interval there exists a neighbourhood U of X in X' such that $U \times \langle 0, 1\rangle \subset W$. The restriction $\widetilde{F}|U \times \langle 0, 1\rangle$ is a homotopy joining the maps $\hat{f}|U$ and $\tilde{f}|U$.

Proof of Theorem (2.3). Take two arbitrary maps f_0 and g_0 belonging to $\boldsymbol{f}$ and $\boldsymbol{g}$, respectively, with a common domain and a common range, i.e. $f_0, g_0\colon U_0 \to V_0$, where $U_0 \in \boldsymbol{U}(X, P)$, $V_0 \in \boldsymbol{V}(Y, Q)$. Since $\boldsymbol{f}'$ is an extension of $\boldsymbol{f}$, there exists an $f_0' \in \boldsymbol{f}'$ such that $f_0'\colon U_0' \to V_0$, where $U_0' \in \boldsymbol{U}'(X', P)$ and $f_0'(x) = f_0(x)$ for every $x \in X$. Since $\boldsymbol{f}'$ is also an extension of $\boldsymbol{g}$, there exists an $f_1' \in \boldsymbol{f}'$ such that $f_1'\colon U_1' \to V_0$, where $U_1' \in \boldsymbol{U}'(X', P)$ and $f_1'(x) = g_0(x)$ for every $x \in X$. By the definition of a mutation there exists a $U_2' \in \boldsymbol{U}'(X', P)$ such that $U_2' \subset U_0' \cap U_1'$ and $f_0'|U_2' \simeq f_1'|U_2'$ (see (1.8)). By the first theorem of Hanner ([1], p. 96) V_0 is an $\mathrm{ANR}(\mathfrak{M})$-space. Hence by Lemma (2.4) there exists a $U_1 \in \boldsymbol{U}(X, P)$ such that $U_1 \subset U_0 \cap U_2'$ and $f_0'|U_1 \simeq f_0|U_1$ and $f_1'|U_1 \simeq g_0|U_1$. Therefore we have obtained $f_0|U_1 \simeq f_0'|U_1 \simeq f_1'|U_1 \simeq g_0|U_1$. Thus $\boldsymbol{f} \simeq \boldsymbol{g}$ (see (1.9)) and the proof is concluded.

It is obvious that

(2.5) *If* f': $U'(X', P) \to V(Y, Q)$ *is an extension of a mutation* f: $U(X, P) \to V(Y, Q)$ *and* f'': $U''(X'', P) \to V(Y, Q)$ *is an extension of* f', *then* f'' *is an extension of* f.

§ 3. Mutational retractions and mutational retracts. Let X be a closed subset of a metrizable space X' considered as a closed sebset of an ANR($\mathfrak{M}$)-space P. We say that a mutation r: $U'(X', P) \to U(X, P)$ is a *mutational retraction* if $r(x) = x$ for every $r \in r$ and for every $x \in X$ (compare the definition of the *fundamental retraction*, [3], p. 58, [6] p. 82).

Let us prove the following

(3.1) THEOREM. *Suppose P and Q are* ANR($\mathfrak{M}$)*-spaces, X' and Y' are closed subsets of P and Q, X and Y are closed subsets of X' and Y', respectively, and h: $X' \to Y'$ is a homeomorphism such that $h(X) = Y$. If there exists a mutational retraction* r: $U'(X', P) \to U(X, P)$, *then there exists a mutational retraction* r': $V'(Y', Q) \to V(Y, Q)$.

Proof. Since Q is an ANR($\mathfrak{M}$)-space, then there exists a map h_0: $U_0 \to Q$ such that $h_0(x) = h(x)$ for every $x \in X$, where $U_0 \in U(X, P)$. Take an arbitrary $V \in V(Y, Q)$. Then $h_0^{-1}(V) \cap U_0 \in U(X, P)$. Take an arbitrary $U \in U(X, P)$ such that $U \subset h_0^{-1}(V) \cap U_0$. Since r is a mutation then (see (1.7)) there exists an $r \in r$ such that r: $U' \to U$, where $U' \in U'(X', P)$. By the first theorem of Hanner ([1], p. 96) U' is an ANR($\mathfrak{M}$)-space. Therefore there exists a map h': $V' \to U'$ such that $h'(y) = h^{-1}(y)$ for every $y \in Y'$, where $V' \in V'(Y', Q)$. Let us define the map r': $V' \to V$ by the formula

$$r'(y) = h_0 r h'(y) \quad \text{for every } y \in V'.$$

Let r' be the collection of all maps r' defined in this way. Let us prove that r': $V'(Y', Q) \to V(Y, Q)$ is a mutation.

Take an arbitrary map $r_1' \in r'$, r_1': $V_1' \to V_1$. Take arbitrary neighbourhoods $V_2' \in V'(Y', Q)$ and $V_2 \in V(Y, Q)$ such that $V_2' \subset V_1'$ and $V_1 \subset V_2$. Consider the map r_2': $V_2' \to V_2$ defined by the formula $r_2'(y) = r_1'(y)$ for every $y \in V_2'$. We want to show that $r_2' \in r'$ (see (1.6)). Since $r_1' \in r'$, there exist $U_1 \in U(X, P)$, $U_1' \in U'(X', P)$ and $r_1 \in r$ such that $U_1 \subset h_0^{-1}(V_1) \cap U_0$, r_1: $U_1' \to U_1$ and

$$r_1'(y) = h_0 r_1 h_1'(y) \quad \text{for every } y \in V_1',$$

where h_1': $V_1' \to U_1'$ is a map such that $h_1'(y) = h^{-1}(y)$ for every $y \in Y'$. Let us define the map h_2': $V_2' \to U_1'$ by the formula $h_2'(y) = h_1'(y)$ for every $y \in V_2'$. Then for every $y \in V_2'$ we have

$$h_0 r_1 h_2'(y) = h_0 r_1 h_1'(y) = r_1'(y) = r_2'(y).$$

Therefore $r_2' \in r'$. Thus the collection r' satisfies the condition (1.6).

It is obvious that the collection r' satisfies the condition (1.7).

Now, take two arbitrary maps $r_3', r_4' \in r'$ with a common domain and a common range, i.e. r_3', r_4': $V' \to V$, where $V' \in V'(Y', Q)$, $V \in V(Y, Q)$. By the definition of the collection r' there exist $U_3, U_4 \in U(X, P)$, $U_3', U_4' \in U'(X', P)$ and $r_3, r_4 \in r$ such that $U_3 \cup U_4 \subset h_0^{-1}(V) \cap U_0$, r_3: $U_3' \to U_3$, r_4: $U_4' \to U_4$ and

$$r_3'(y) = h_0 r_3 h_3'(y), \quad r_4'(y) = h_0 r_4 h_4'(y) \quad \text{for every } y \in V',$$

where h_3': $V' \to U_3'$ and h_4': $V' \to U_4'$ are maps such that

$$(3.2) \qquad h_3'(y) = h_4'(y) = h^{-1}(y) \quad \text{for every } y \in Y'.$$

Since r is a mutation, there exists a $U_5' \in U'(X', P)$ such that (see (1.8)) $U_5' \subset U_3' \cap U_4'$ and

$$(3.3) \qquad r_3|U_5' \simeq r_4|U_5' \quad \text{in} \quad U_3 \cup U_4.$$

Take $V_3' \in V'(Y', Q)$ such that

$$V_3' \subset V' \cap h_3'^{-1}(U_5') \cap h_4'^{-1}(U_5').$$

Moreover, by (3.2) and Lemma (2.4) we can require V_3' to be such that

$$(3.4) \qquad h_3'|V_3' \simeq h_4'|V_3' \quad \text{in } U_5'.$$

It follows by (3.3) and (3.4) that

$$r_3 h_3'|V_3' \simeq r_4 h_4'|V_3' \quad \text{in } U_3 \cup U_4.$$

Hence $h_0 r_3 h_3'|V_3' \simeq h_0 r_4 h_4'|V_3'$ in V. Therefore $r_3'|V_4' \simeq r_4'|V_3'$. Thus, the collection r' satisfies the condition (1.8). Therefore r' is a mutation.

Now, take an arbitrary map $r' \in r'$ and an arbitrary $y \in Y$. Then we have

$$r'(y) = h_0 r h'(y) = h_0 r h^{-1}(y) = h_0 h^{-1}(y) = h h^{-1}(y) = y.$$

Thus, r' is a mutational retraction and the proof is finished.

We say that a closed subset X of a metrizable space X' is a *mutational retract* of X' if there exists a mutational retraction r: $U'(X', P) \to U(X, P)$, where P is an ANR($\mathfrak{M}$)-space containing X' as a closed subset. By Theorem (3.1) the choice of an ANR($\mathfrak{M}$)-space P and the manner of imbedding of X' into P, as a closed subset, is immaterial. Moreover, it follows by Theorem (3.1) that the notion of mutational retract is topological invariant.

Consider two metrizable spaces X and Y contained in ANR($\mathfrak{M}$)-spaces P and Q, respectively, and let $\underline{f} = \{f_k, X, Y\}_{P,Q}$ be a fundamental sequence. Let us denote by $\boldsymbol{f}$ the collection of all maps $f\colon U\to V$, where $U\in \boldsymbol{U}(X,P)$, $V\in \boldsymbol{V}(Y,Q)$, that are such that

(3.5) $$f\simeq f_k|U \quad \text{in } V \text{ for almost all } k$$

(compare (4.5) of [9], p. 57). It follows by (1.1) that $\boldsymbol{f}$ is a mutation from $\boldsymbol{U}(X,P)$ to $\boldsymbol{V}(Y,Q)$ (cf. [9], p. 57). It will be called the mutation *associated* with the fundamental sequence $\underline{f}$.

(3.6) THEOREM. *Suppose $\underline{r}=\{r_k,X',X\}_{P,P}$ is a fundamental retraction and $\boldsymbol{r}\colon \boldsymbol{U}'(X',P)\to \boldsymbol{U}(X,P)$ is the mutation associated with $\underline{r}$. Then there exists a mutational retraction $\boldsymbol{r}'\colon \boldsymbol{U}'(X',P)\to \boldsymbol{U}(X,P)$ homotopic to $\boldsymbol{r}$.*

Proof. By hypothesis we have $r_k(x)=x$ for every $x\in X$ and $k=1,2,\ldots$ Let us denote by $\boldsymbol{r}'$ the collection of all maps $r\in\boldsymbol{r}$ such that $r(x)=x$ for every $x\in X$. Let us prove that $\boldsymbol{r}'$ is a mutation.

Take an arbitrary map $r_0'\in\boldsymbol{r}'$, $r_0'\colon U_0'\to U_0$, $U_0'\in\boldsymbol{U}'(X',P)$, $U_0\in\boldsymbol{U}(X,P)$. Take $U_1'\in\boldsymbol{U}'(X',P)$ and $U_1\in\boldsymbol{U}(X,P)$ such that $U_1'\subset U_0'$ and $U_0\subset U_1$ and consider the map $r_1'\colon U_1'\to U_1$ defined by the formula $r_1'(x)=r_0(x)$ for every $x\in U_1$. Since $r_0'\in\boldsymbol{r}'$, we have $r_0'\in\boldsymbol{r}$, and since $\boldsymbol{r}$ is a mutation, we have by (1.6), $r_1'\in\boldsymbol{r}$. Moreover, for every $x\in X$ we have $r_1'(x)=r_0'(x)=x$. Therefore $r_1'\in\boldsymbol{r}'$. Thus, the collection $\boldsymbol{r}'$ satisfies the condition (1.6).

Now, take an arbitrary neighbourhood $U\in\boldsymbol{U}(X,P)$. Since $\underline{r}$ is a fundamental sequence, by (1.1) there exist a neighbourhood $U'\in\boldsymbol{U}'(X',P)$ and a natural number k such that $r_k(U')\subset U$. Let $r\colon U'\to U$ be the map defined by the formula $r(x)=r_k(x)$ for every $x\in U'$. It is obvious that $r\in\boldsymbol{r}'$. Thus, the collection $\boldsymbol{r}'$ satisfies the condition (1.7).

Now, take two arbitrary maps $r_2',r_3'\in\boldsymbol{r}'$ with a common domain and a common range, i.e. $r_2',r_3'\colon U_2'\to U_2$, where $U_2'\in\boldsymbol{U}'(X',P)$, $U_2\in\boldsymbol{U}(X,P)$. Since $r_2',r_3'\in\boldsymbol{r}$ and $\boldsymbol{r}$ is a mutation, by (1.8) there exists a neighbourhood $U_3'\in\boldsymbol{U}'(X',P)$ such that $U_3'\subset U_2'$ and $r_2'|U_3'\simeq r_3'|U_3'$. Therefore the collection $\boldsymbol{r}'$ satisfies the condition (1.8). Thus, $\boldsymbol{r}'$ is a mutation.

Since for every $r\in\boldsymbol{r}'$ and for every $x\in X$ we have $r(x)=x$, $\boldsymbol{r}$ is a mutational retraction.

It is obvious that $\boldsymbol{r}'$ is a restriction of $\boldsymbol{r}$. Hence by Theorem (2.3) we obtain $\boldsymbol{r}'\simeq\boldsymbol{r}$ and the proof is concluded.

By Theorem (3.6) we obtain the following

(3.7) COROLLARY. *If X is a fundamental retract of X', then X is a mutational retract of X'.*

Remark. The converse of (3.7) is not true. A corresponding example will be given in § 6.

Since every retract of X' is a fundamental retract of X' ([6], p. 84), by (3.7) we get the following

(3.8) COROLLARY. *If X is a retract of X', then X is a mutational retract of X'.*

It is obvious that

(3.9) *If $\boldsymbol{r}\colon \boldsymbol{U}'(X',P)\to\boldsymbol{U}(X,P)$ and $\boldsymbol{r}'\colon \boldsymbol{U}''(X'',P)\to\boldsymbol{U}'(X',P)$ are mutational retractions, then the composition $\boldsymbol{r}\boldsymbol{r}'\colon \boldsymbol{U}''(X'',P)\to\boldsymbol{U}(X,P)$ is a mutational retraction.*

Hence we obtain the following

(3.10) COROLLARY. *If X is a mutational retract of X' and X' is a mutational retract of X'', then X is a mutational retract of X''.*

We say that a mutation $\boldsymbol{f}\colon \boldsymbol{U}(X,P)\to\boldsymbol{V}(Y,Q)$ is a *h-mutation* if there exists a mutation $\boldsymbol{g}\colon \boldsymbol{V}(Y,Q)\to\boldsymbol{U}(X,P)$ such that $\boldsymbol{f}\boldsymbol{g}\simeq\boldsymbol{v}$, where $\boldsymbol{v}\colon \boldsymbol{V}(Y,Q)\to\boldsymbol{V}(Y,Q)$ is a mutation consisting of all inclusions of the system $\boldsymbol{V}(Y,Q)$ (compare the definition of the *h-fundamental sequence*, [3] p. 59). By the definitions the existence of a h-mutation $\boldsymbol{f}\colon \boldsymbol{U}(X,P)\to\boldsymbol{V}(Y,Q)$ is equivalent to the inequality $\operatorname{Sh}X\geqslant\operatorname{Sh}Y$.

It follows by (3.5) of [10] that

(3.11) *A mutation associated an h-fundamental sequence is an h-mutation.*

Let us prove that

(3.12) *Every mutational retraction is an h-mutation.*

Proof. Consider an arbitrary mutational retraction $\boldsymbol{r}\colon \boldsymbol{U}'(X',P)\to\boldsymbol{U}(X,P)$. Let $\boldsymbol{j}\colon \boldsymbol{U}(X,P)\to\boldsymbol{U}'(X',P)$ be the extension of the inclusion $j\colon X\to X'$ and let $\boldsymbol{u}\colon \boldsymbol{U}(X,P)\to\boldsymbol{U}(X,P)$ be a mutation consisting of all inclusions of the system $\boldsymbol{U}(X,P)$. We shall prove that $\boldsymbol{r}\boldsymbol{j}\simeq\boldsymbol{u}$.

By (2.2) $\boldsymbol{r}\boldsymbol{j}$ is a restriction of $\boldsymbol{r}$. Let us prove that $\boldsymbol{u}$ is also a restriction of $\boldsymbol{r}$. Take an arbitrary map $u\in\boldsymbol{u}$. Then $u\colon U_1\to U_2$, $U_1,U_2\in\boldsymbol{U}(X,P)$, $U_1\subset U_2$ and $u(x)=x$ for every $x\in U_1$. Since $\boldsymbol{r}$ is a mutation, by (1.7) there exist $U'\in\boldsymbol{U}'(X',P)$ and $r\in\boldsymbol{r}$ such that $r\colon U'\to U_2$. Since $\boldsymbol{r}$ is a mutational retraction, we have $r(x)=x=u(x)$ for every $x\in X$. Therefore $\boldsymbol{u}$ is a restriction of $\boldsymbol{r}$. Hence, by Theorem (2.3), we obtain $\boldsymbol{r}\boldsymbol{j}\simeq\boldsymbol{u}$. Thus, $\boldsymbol{r}$ is an h-mutation.

By (3.12) we obtain the following

(3.13) COROLLARY. *If X is a mutational retract of X', then $\operatorname{Sh}X\leqslant\operatorname{Sh}X'$.*

By propositions (4.6) and (4.7) of [9] (p. 57) for compacta the relation $\operatorname{Sh}X\leqslant\operatorname{Sh}X'$ is equivalent to the relation $\operatorname{Sh}(X)\leqslant\operatorname{Sh}(X')$. Hence by (3.13) we obtain the following

(3.14) COROLLARY. *If a compactum X is a mutational retract of a compactum X', then $\operatorname{Sh}(X)\leqslant\operatorname{Sh}(X')$.*

Remark. For non-compact metrizable spaces (3.14) is not true. A corresponding example will be given in § 6.

Let us prove that

(3.15) *If X' is a closed subset of a metrizable space X'', X is a closed subset of X' and X is a mutational retract of X'', then X is a mutational retract of X'.*

Proof. By hypothesis there exists a mutational retraction $\boldsymbol{r}: \boldsymbol{U}''(X'', P) \to \boldsymbol{U}(X, P)$. Let $\boldsymbol{j}: \boldsymbol{U}'(X', P) \to \boldsymbol{U}''(X'', P)$ be the extension of the inclusion $j: X' \to X''$. It is easy to see that $\boldsymbol{rj}: \boldsymbol{U}'(X', P) \to \boldsymbol{U}(X, P)$ is a mutational retraction.

Let us denote by $\bigoplus_{t \in T} X_t$ the *sum* of the family $\{X_t\}_{t \in T}$ of disjoint spaces (see [8], p. 70).

(3.16) Theorem. *If X_t is a mutational retract of X'_t for every $t \in T$, then $\bigoplus_{t \in T} X_t$ is a mutational retract of $\bigoplus_{t \in T} X'_t$.*

Proof. By hypothesis, for every $t \in T$ there exists a mutational retraction $\boldsymbol{r}_t: \boldsymbol{U}'_t(X'_t, P_t) \to \boldsymbol{U}_t(X_t, P_t)$, where P_t is an $\mathrm{ANR}(\mathfrak{M})$-space containing X'_t as a closed subset. Let $X = \bigoplus_{t \in T} X_t$, $X' = \bigoplus_{t \in T} X'_t$, $P = \bigoplus_{t \in T} P_t$. Obviously X is closed in X' and X' is closed in P. By Theorem 8.1 of [11] (p. 98) P is an $\mathrm{ANR}(\mathfrak{M})$-space. Consider the complete neighbourhood systems $\boldsymbol{U}(X, P)$ and $\boldsymbol{U}'(X', P)$. Let $\boldsymbol{r}$ be the collection of all combinations of maps $r_t \in \boldsymbol{r}_t$, i.e. $\boldsymbol{r}$ consists of all maps $r: U' \to U$, where $U' \in \boldsymbol{U}'(X', P)$ and $U \in \boldsymbol{U}(X, P)$, such that for every $x \in U' \cap P_t$ we have $r(x) = r_t(x)$, where $r_t \in \boldsymbol{r}_t$. It is obvious that $\boldsymbol{r}: \boldsymbol{U}'(X', P) \to \boldsymbol{U}(X, P)$ is a mutational retraction. Thus, the proof is concluded.

(3.17) Theorem. *If X is a mutational retract of a metrizable space X' contained as a closed subset in an $\mathrm{ANR}(\mathfrak{M})$-space P, then for every mutation $\boldsymbol{f}: \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ there exists an extension $\boldsymbol{f}': \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$ of $\boldsymbol{f}$.*

Proof. By hypothesis and Theorem (3.1) there exists a mutational retraction $\boldsymbol{r}: \boldsymbol{U}'(X', P) \to \boldsymbol{U}(X, P)$. Let us put $\boldsymbol{f}' = \boldsymbol{fr}: \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$. Let us show that $\boldsymbol{f}'$ is an extension of $\boldsymbol{f}$. Take an arbitrary map $f \in \boldsymbol{f}$, $f: U \to V$. Since $\boldsymbol{r}$ is a mutation, then by (1.8) there exists an $r \in \boldsymbol{r}$ such that $r: U' \to U$. Let us put $f' = fr$. Then we have $f' \in \boldsymbol{f}'$, $f'(x) = fr(x) = f(x)$ for every $x \in X$ and $\operatorname{range} f' = \operatorname{range} f$. Thus, $\boldsymbol{f}'$ is an extension of $\boldsymbol{f}$ and the proof is finished.

Let X be a closed subset of a metrizable space X'. We say that X is a *mutational neighbourhood retract* of X' if there exists a closed neighbourhood W of X in X' such that X is a mutational retract of W (compare the definition of the *fundamental neighbourhood retract*, [3] p. 59, [6] p. 84). Obviously, this notion is a topological invariant.

By (3.7) we obtain the following

(3.18) Corollary. *If X is a fundamental neighbourhood retract of X' then X is a mutational neighbourhood retract of X'.*

Remark. The converse of (3.18) is not true. A corresponding example will be given in § 6.

By Theorem (3.17) we get the following

(3.19) Corollary. *If X is a mutational neighbourhood retract of a metrizable space X' contained as a closed subset in an $\mathrm{ANR}(\mathfrak{M})$-space P, then there exists a closed neighbourhood W of X in X' such that for every mutation $\boldsymbol{f}: \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ there exists an extension $\boldsymbol{f}': \boldsymbol{U}'(W, P) \to \boldsymbol{V}(Y, Q)$ of $\boldsymbol{f}$.*

By Theorem (3.16) we obtain the following

(3.20) Corollary. *If X_t is a mutational neighbourhood retract of X'_t for every $t \in T$, then $\bigoplus_{t \in T} X_t$ is a mutational neighbourhood retract of $\bigoplus_{t \in T} X'_t$.*

Remark. For fundamental retracts and fundamental neighbourhood retracts theorems analogous to (3.16) and (3.20) are not true. A corresponding example will be given in § 6. If the set T is finite, then analogous theorems for fundamental retracts are also true. The simple proofs are left to the reader (compare the proof of Theorem (4.3) of [10]).

§ 4. Mutational absolute retracts and mutational absolute neighbourhood retracts. We say that a metrizable space X is a *mutational absolute retract* (shortly: MAR) if, for every metrizable space X' containing X as a closed subset, the set X is a mutational retract of X'. A metrizable space X is said to be a *mutational absolute neighbourhood retract* (shortly: MANR) if for every metrizable space X' containing X as a closed subset, the set X is a mutational neighbourhood retract of X' (compare definitions of FAR and FANR, [3] p. 65, [6] p. 94). By Theorem (3.1) these notions are topological invariants.

It is obvious that

(4.1) *Every* MAR*-space is a* MANR*-space.*

It follows by (3.7) that

(4.2) *Every* FAR*-space is a* MAR*-space.*

Hence by (22.3) of [6]

(4.3) *Every* $\mathrm{AR}(\mathfrak{M})$*-space is* MAR*-space.*

It follows by (3.18) that

(4.4) *Every* FANR*-space is a* MANR*-space.*

Hence by (22.3) of [6]

(4.5) *Every* $\mathrm{ANR}(\mathfrak{M})$*-space is a* MANR*-space.*

Remark. The converses of (4.1)-(4.5) are not true. Corresponding examples will be given in § 6.

(4.6) Theorem. *Every mutational retract of a* MAR*-space is a* MAR*-space.*

First we prove the lemma concerning the metrizability of the matching $X \overset{h}{\cup} Y$ of spaces X and Y (see [1], p. 116).

(4.7) LEMMA. *If X and Y are disjoint metrizable spaces, X_0 is a closed subset of X, $h: X_0 \to Y$ is a homeomorphical imbedding such that $h(X_0)$ is a closed subset of Y, then $X \overset{h}{\cup} Y$ is a metrizable space.*

In [12] (p. 95) J. Nagata has proved the following

(4.8) THEOREM (Nagata). *If a topological space R can be represented in the form $R = \bigcup_{\alpha \in A} S_\alpha$, where $\{S_\alpha | \alpha \in A\}$ is locally finite and S_α are closed metrizable subspaces, then R is metrizable.*

Proof of Lemma (4.7). Let $\varphi: X \oplus Y \to X \overset{h}{\cup} Y$ be a natural mapping (see [8], p. 83). It follows by the hypotheses that the set $\varphi(X)$ is homeomorphic to X, $\varphi(Y)$ is homeomorphic to Y and the sets $\varphi(X)$ and $\varphi(Y)$ are closed in $X \overset{h}{\cup} Y$. Hence by Theorem (4.8) we obtain Lemma (4.7).

Proof of Theorem (4.6). Take an arbitrary MAR-space X and let X_0 be a mutational retract of X. Take an arbitrary metrizable space X' containing X_0 as a closed subset. We can assume that $X \cap X' = X_0$. By Lemma (4.7) we can define a metric in the set $X \cup X'$ such that the spaces X and X' are closed subspaces of $X \cup X'$. By the Kuratowski-Wojdysławski theorem ([1], p. 78) the space $X \cup X'$ may be considered as a closed subset of an $\mathrm{ANR}(\mathfrak{M})$-space P. By the hypotheses X_0 is a mutational retract of X and X, as a MAR-space, is a mutational retract of P. Hence by (3.10) X_0 is a mutational retract of P. Therefore there exists a mutational retraction $\boldsymbol{r}: \boldsymbol{U}(P, P) \to \boldsymbol{U}_0(X_0, P)$. Let $\boldsymbol{j}: \boldsymbol{U}'(X', P) \to \boldsymbol{U}(P, P)$ be the extension of the inclusion $j: X' \to P$. It is evident that $\boldsymbol{rj}: \boldsymbol{U}'(X', P) \to \boldsymbol{U}_0(X_0, P)$ is a mutational retraction. Therefore X_0 is a mutational retract of every metrizable space X' containing X as a closed subset. Thus, X_0 is a MAR-space and the proof is concluded.

(4.9) THEOREM. MAR-*spaces are the same as mutational retracts of* $\mathrm{AR}(\mathfrak{M})$-*spaces.*

Proof. Take an arbitrary MAR-space X. By the Kuratowski-Wojdysławski theorem ([1], p. 78) there exists an $\mathrm{AR}(\mathfrak{M})$-space P containing X as a closed subset. Since X is a MAR-space, X is a mutational retract of P.

Now, suppose that X is a mutational retract of an $\mathrm{AR}(\mathfrak{M})$-space P. By (4.3) P is a MAR-space and hence, by Theorem (4.6), X is a MAR-space. Thus, the proof is finished.

By (3.13) and (4.9) we obtain the following

(4.10) COROLLARY. *The shape* $\mathrm{Sh}\,X$ *of a* MAR-*space* X *is trivial.*

(4.11) THEOREM. MANR-*spaces are the same as mutational retracts of* $\mathrm{ANR}(\mathfrak{M})$-*spaces.*

Proof. Take an arbitrary MANR-space X. By the Kuratowski-Wojdysławski theorem ([1], p. 78) there exists an $\mathrm{ANR}(\mathfrak{M})$-space P containing X as a closed subset. Since X is a MANR-space, there exists a closed neighbourhood W of X in P such that there exists a mutational retraction $\boldsymbol{r}: \boldsymbol{U}(W, P) \to \boldsymbol{U}(X, P)$. Let V be an open neighbourhood of X in P contained in W. By the first theorem of Hanner ([1], p. 96) V is an $\mathrm{ANR}(\mathfrak{M})$-space. Let $\boldsymbol{j}: \boldsymbol{U}(V, V) \to \boldsymbol{U}(W, P)$ be the extension of the inclusion $j: V \to W$, and let $\boldsymbol{i}: \boldsymbol{U}(X, P) \to \boldsymbol{U}(X, V)$ be the extension of the identity $i: X \to X$. It is evident that $\boldsymbol{irj}: \boldsymbol{U}(V, V) \to \boldsymbol{U}(X, V)$ is a mutational retraction. Thus, X is a mutational retract of the $\mathrm{ANR}(\mathfrak{M})$-space V.

Now, suppose that X is a mutational retract of an $\mathrm{ANR}(\mathfrak{M})$-space P. Take an arbitrary metrizable space X' containing X as a closed subset. We can assume that $X' \cap P = X$ and by Lemma (4.7) we can define a metric in the set $X' \cup P$ such that the spaces X' and P are closed subspaces of $X' \cup P$. By the Kuratowski-Wojdysławski theorem ([1], p. 78) the space $X' \cup P$ may be considered as a closed subset of an $\mathrm{ANR}(\mathfrak{M})$-space Q. Since P is an $\mathrm{ANR}(\mathfrak{M})$-space, by (4.5) it is a MANR-space. Therefore there exists a closed neighbourhood W of P in Q such that P is a mutational retract of W. Hence by (3.10) X is a mutational retract of W, and by (3.15) X is a mutational retract of $W \cap X'$. Obviously W is a closed neighbourhood of X in Q, therefore $W \cap X'$ is a closed neighbourhood of X in X'. Hence X is a mutational neighbourhood retract of X'. Thus, X is a MANR-space and the proof is concluded.

(4.12) THEOREM. *Every mutational retract of a* MANR-*space is a* MANR-*space.*

Proof. Let X_0 be a mutational retract of a MANR-space X. By Theorem (4.11) there exists an $\mathrm{ANR}(\mathfrak{M})$-space P such that X is a mutational retract of P. Hence by (3.10) X_0 is a mutational retract of P and, by Theorem (4.11), X_0 is a MANR-space.

(4.13) THEOREM. *If X_t are* MANR-*spaces for every $t \in T$, then $\bigoplus_{t \in T} X_t$ is a* MANR-*space.*

Proof. By Theorem (4.11) X_t is a mutational retract of an $\mathrm{ANR}(\mathfrak{M})$-space P_t for every $t \in T$. Obviously we can assume that the spaces P_t $(t \in T)$ are disjoint. Hence by Theorem (3.16) $\bigoplus_{t \in T} X_t$ is a mutational retract of $\bigoplus_{t \in T} P_t$. By Theorem (8.1) of [11] (p. 98) $\bigoplus_{t \in T} P_t$ is an $\mathrm{ANR}(\mathfrak{M})$-space. Hence, by Theorem (4.11) $\bigoplus_{t \in T} X_t$ is a MANR-space.

Remark. For FANR-spaces the theorem analogous to (4.13) is not true. A corresponding example will be given in § 6. If the set T is finite, then the analogous theorem for FANR-spaces is also true. The simple proof is left to the reader (compare the remark concerning (3.20)).

(4.14) THEOREM. *A metrizable space X is a MAR-space if and only if for every metrizable space X' containing X as a closed subset and for every ANR($\mathfrak{M}$)-space P containing X' as a closed subset and for every mutation $f\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ there exists an extension $f'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$ of f.*

Proof. Suppose X is a MAR-space. Then the required extension f' exists by Theorem (3.17).

Now, suppose that the second part of Theorem is satisfied. Consider the mutation $\boldsymbol{u}\colon \boldsymbol{U}(X, P) \to \boldsymbol{U}(X, P)$ consisting of all inclusions. By hypothesis there exists an extension $\boldsymbol{u}'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{U}(X, P)$ of $\boldsymbol{u}$. Le us denote by $\boldsymbol{r}$ the collection of all maps $r \in \boldsymbol{u}'$ such that $r(x) = x$ for every $x \in X$. Let us show that $\boldsymbol{r}$ is a mutation.

Take an arbitrary map $r_1 \in \boldsymbol{r}$, $r_1\colon U_1' \to U_1$. Take arbitrary neighbourhoods $U_2' \in \boldsymbol{U}'(X', P)$ and $U_2 \in \boldsymbol{U}(X, P)$ such that $U_2' \subset U_1'$ and $U_1 \subset U_2$. Consider the map $r_2\colon U_2' \to U_2$ defined by the formula $r_2(x) = r_1(x)$ for every $x \in U_2'$. Since $r_1 \in \boldsymbol{u}'$ and $\boldsymbol{u}'$ is a mutation, by (1.6) $r_2 \in \boldsymbol{u}'$. Moreover, for every $x \in X$ we have $r_2(x) = r_1(x) = x$. Therefore $r_2 \in \boldsymbol{r}$. Thus, the collection $\boldsymbol{r}$ satisfies the condition (1.6).

Now, take an arbitrary neighbourhood $U \in \boldsymbol{U}(X, P)$. Take $u \in \boldsymbol{u}$ with range $u = U$. By the definition of an extension there exists a $u' \in \boldsymbol{u}'$ such that range $u' =$ range $u = U$ and $u'(x) = u(x) = x$ for every $x \in X$. Therefore $u' \in \boldsymbol{r}$ and range $u' = U$. Thus, the collection r satisfies the condition (1.7).

Now, take two arbitrary maps $r_3, r_4 \in \boldsymbol{r}$ with a common domain and a common range, $r_3, r_4\colon U_0' \to U_0$. Since $r_3, r_4 \in \boldsymbol{u}'$ and $\boldsymbol{u}'$ is a mutation, by (1.8) there exists a $U_3' \in \boldsymbol{U}'(X', P)$ such that $r_3|U_3' \simeq r_4|U_3'$. Therefore the collection $\boldsymbol{r}$ satisfies the condition (1.8). Thus, $\boldsymbol{r}$ is a mutation and, obviously, it is a mutational retraction. Therefore X is a mutational retract of X'. Thus, X is a MAR-space, and the proof is finished.

(4.15) THEOREM. *A metrizable space X is a MANR-space if and only if for every metrizable space X' containing X as a closed subset there exists a closed neighbourhood W of X in X' such that for every ANR($\mathfrak{M}$)-space P containing X' as a closed subset and for every mutation $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ there exists an extension $\boldsymbol{f}'\colon \boldsymbol{U}'(W, P) \to \boldsymbol{V}(Y, Q)$ of f.*

Proof. Suppose X is a MANR-space and take arbitrary spaces X' and P satisfying the hypotheses. Then the required neighbourhood W and an extension f' exist by (3.19).

Now, suppose that the second part of Theorem is satisfied. Consider the mutation $\boldsymbol{u}\colon \boldsymbol{U}(X, P) \to \boldsymbol{U}(X, P)$ consisting of all inclusions. By hypothesis there exists an extension $\boldsymbol{u}'\colon \boldsymbol{U}'(W, P) \to \boldsymbol{U}(X, P)$ of $\boldsymbol{u}$. Analogously as in the proof of Theorem (4.14) we define the collection $\boldsymbol{r}$ and prove that $\boldsymbol{r}\colon \boldsymbol{U}'(W, P) \to \boldsymbol{U}(X, P)$ is a mutational retraction. Thus, X is a MANR-space and the proof is concluded.

§ 5. Compact mutational retracts. Let us prove the following

(5.1) THEOREM. *If a compactum X is a mutational retract of a metrizable space X', then X is a fundamental retract of X'.*

Proof. By hypothesis there exists a mutational retraction $\boldsymbol{r}\colon \boldsymbol{U}'(X', P) \to \boldsymbol{U}(X, P)$. By the Kuratowski-Wojdysławski theorem and Theorem (3.1) we can assume that $P \in \mathrm{AR}(\mathfrak{M})$. Let $U_k \in \boldsymbol{U}(X, P)$, $k = 1, 2, \ldots$, be a sequence of neighbourhoods of X in P such that

$$(5.2)\qquad U_{k+1} \subset U_k \quad \text{for} \quad k = 1, 2, \ldots \quad \text{and} \quad \bigcap_{k=1}^{\infty} U_k = X.$$

Let $r_k\colon U_{k-1}' \to U_k$, $k = 1, 2, \ldots$, be a constituent of the mutation $\boldsymbol{r}$; obviously we can require that $\overline{U}_k' \subset U_{k-1}'$. Let us define maps $\bar{r}_k\colon \overline{U}_k' \to U_k$, $k = 1, 2, \ldots$, setting $\bar{r}_k = r_k|\overline{U}_k'$.

Now, we shall define by induction a sequence of neighbourhoods $U_k'' \in \boldsymbol{U}'(X', P)$ and a sequence of maps $r_k^i\colon \overline{U}_i'' \to U_i$ for $i = 1, 2, \ldots, k$ and $k = 1, 2, \ldots$ satisfying the following conditions:

$$(5.3)\qquad \overline{U}_k'' \subset U_{k-1}'' \cap U_k' \quad \text{for} \quad k = 2, 3, \ldots,$$

$$(5.4)\qquad r_k^k = \bar{r}_k|U_k'' \quad \text{for} \quad k = 1, 2, \ldots,$$

$$(5.5)\qquad r_k^i(x) = r_k^j(x) \quad \text{for} \quad x \in \overline{U}_j'' \text{ and } i \leqslant j \leqslant k,$$

$$(5.6)\qquad r_k^i \simeq r_l^i\colon \overline{U}_i'' \to U_i \quad \text{for} \quad i \leqslant k \text{ and } i \leqslant l.$$

Let us put $U_1'' = U_1'$ and $r_1^1 = \bar{r}_1\colon \overline{U}_1'' \to U_1$. Consider the maps $\bar{r}_1\colon \overline{U}_1' \to U_1$ and $\bar{r}_2\colon \overline{U}_2' \to U_2$. Since $\bar{r}_1$ and $\bar{r}_2$ are restrictions of the constituents $r_1\colon U_0' \to U_1$ and $r_2\colon U_1' \to U_2$ of $\boldsymbol{r}$ and $U_2 \subset U_1$, there exists a neighbourhood $U_2'' \in \boldsymbol{U}'(X', P)$ such that $\overline{U}_2'' \subset U_2'$ and

$$\bar{r}_1|\overline{U}_2'' \simeq \bar{r}_2|\overline{U}_2'' \quad \text{in } U_1.$$

Let us put $r_2^2 = \bar{r}_2|\overline{U}_2''\colon \overline{U}_2'' \to U_2$. Since $U_1 \in \mathrm{ANR}(\mathfrak{M})$ and $\bar{r}_1|\overline{U}_2''$ has the extension $\bar{r}_1\colon \overline{U}_1'' \to U_1$, by Borsuk's homotopy extension theorem ([1], p. 94) the map $\bar{r}_2|\overline{U}_2''$ has an extension $r_2^1\colon \overline{U}_1'' \to U_1$ such that $r_2^1 \simeq \bar{r}_1 = r_1^1$. Therefore, the neighbourhoods U_1'', U_2'' and the maps r_1^1, r_2^1, r_2^2 satisfy the required conditions (5.3)-(5.6).

Now, suppose that we have defined neighbourhoods U_k'' and maps $r_k^i\colon \overline{U}_i'' \to U_i$ for $k = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, k$ in a such manner that the conditions (5.3)-(5.6) are satisfied. We shall define a neighbourhood U_{n+1}'' and maps r_{n+1}^i for $i = 1, 2, \ldots, n+1$ satisfying the required conditions.

Consider the maps $\bar{r}_n: \bar{U}'_n \to U_n$ and $\bar{r}_{n+1}: \bar{U}'_{n+1} \to U_{n+1}$. Since these maps are restrictions of the constituents $r_n: U'_{n-1} \to U_n$ and $r_{n+1}: U'_n \to U_{n+1}$ of $\boldsymbol{r}$ and $U_{n+1} \subset U_n$, there exists a neighbourhood $U''_{n+1} \in \boldsymbol{U}'(X', P)$ such that $\bar{U}''_{n+1} \subset U''_n \cap U'_{n+1}$ and

$$\bar{r}_n|\bar{U}''_{n+1} \simeq \bar{r}_{n+1}|\bar{U}''_{n+1} \quad \text{in} \quad U_n .$$

Let us put $r^{n+1}_{n+1} = \bar{r}_{n+1}|\bar{U}''_{n+1}: \bar{U}''_{n+1} \to U_{n+1}$. Since $r^n_n = \bar{r}_n|\bar{U}''_n$ is an extension of $\bar{r}_n|\bar{U}''_{n+1}$, by Borsuk's homotopy extension theorem the map $\bar{r}_{n+1}|\bar{U}''_{n+1}$ has an extension $r^n_{n+1}: \bar{U}''_n \to U_n$ such that $r^n_{n+1} \simeq r^n_n$. Therefore, the neighbourhood U''_{n+1} and the maps r^{n+1}_{n+1} and r^n_{n+1} satisfy the required conditions (5.3)-(5.6).

Now, suppose that we have defined maps $r^{n+1}_{n+1}, r^n_{n+1}, \ldots, r^i_{n+1}$ satisfying the conditions (5.4)-(5.6). We shall define r^{i-1}_{n+1}. By (5.6) we have $r^i_{n+1} \simeq r^i_n: \bar{U}''_i \to U_i$ and by (5.5) the map r^i_n considered as a map into U_{i-1} has the extension $r^{i-1}_n: \bar{U}_{i-1} \to U_{i-1}$. Hence by Borsuk's homotopy extension theorem the map r^i_{n+1} has an extension $r^{i-1}_{n+1}: \bar{U}''_{i-1} \to U_{i-1}$ homotopic to r^{i-1}_n. Therefore, the map r^{i-1}_{n+1} satisfies the required conditions. Thus, the sequences of neighbourhoods U''_k and maps r^i_k defined above satisfy the conditions (6.3)-(6.6).

Since $P \in \mathrm{AR}(\mathfrak{M})$, for every $k = 1, 2, \ldots$ there exists a map $\tilde{r}_k: P \to P$ such that

$$\tilde{r}_k(x) = r^1_k(x) \quad \text{for every } x \in \bar{U}''_1.$$

Let us prove that $\underline{r} = \{\tilde{r}_k, X', X\}_{P,P}$ is a fundamental retraction. Take an arbitrary neighbourhood $U \in \boldsymbol{U}(X, P)$. By (5.2) and the compactness of X there exists a natural number n such that $U_k \subset U$ for $k \geqslant n$. Let us observe that

(5.7) $$\tilde{r}_k|U''_n \simeq \tilde{r}_{k+1}|U''_n \quad \text{in } U_n \quad \text{for} \quad k \geqslant n .$$

Indeed, by (5.4) we obtain

$$\tilde{r}_k|U''_n = r^1_k|U''_n = r^n_k: \ U''_n \to U_n,$$
$$\tilde{r}_{k+1}|U''_n = r^1_{k+1}|U''_n = r^n_{k+1}: \ U''_n \to U_n .$$

Hence by (5.6) we obtain (5.7). Therefore, $\underline{r}$ is a fundamental sequence.

Since $\boldsymbol{r}$ is a mutational retraction, by (5.4), (5.5) and the definition of $\tilde{r}_k$ it follows at once that $\tilde{r}_k(x) = x$ for every $x \in X$. Thus, $\underline{r}$ is a fundamental retraction and the proof is concluded.

Theorem (5.1) implies at once the following

(5.8) Corollary. *Every compact* MAR-*space is a* FAR-*space and every compact* MANR-*space is a* FANR-*space.*

§ 6. Examples. First we give an example of a space which is not a MANR-space.

(6.1) *Solenoids of Van Dantzig are not* MANR-*spaces.*

Proof. By (1.3) of [7] solenoids are not movable (for the definition of movability see [7], p. 137). Therefore by (3.12) of [7] they are not FANR-spaces. Hence by Corollary (5.8) and the compactness of solenoids we obtain (6.1).

Now, we give an example of a MAR-space which is not a FAR-space.

Let us denote by A_n (for $n = 1, 2, \ldots$) the closed half-line lying in the plane E^2 with the end-point $(0, 0)$ and containing the point $(n, 1)$, and by A_0 the half-line with the end-point $(0, 0)$ and containing the point $(1, 0)$. Consider the subspace $X = \bigcup_{n=0}^{\infty} A_n$ of E^2. Let us prove that

(6.2) *The space X is a* MAR-*space and it is not a* FAR-*space.*

Proof. Obviously X is a closed subset of E^2. It is easy to see that for every neighbourhood U of X in E^2 there exists a homeomorphical imbedding $r: E^2 \to U$ such that $r(x) = x$ for every $x \in X$. The collection $\boldsymbol{r}$ of all maps r obtained in this way is a mutational retraction $\boldsymbol{r}: \boldsymbol{V}(E^2, E^2) \to \boldsymbol{U}(X, E^2)$. Hence by Theorem (4.9) X is a MAR-space.

Now suppose, on the contrary, that X is a FAR-space. Then there exists a fundamental retraction $\underline{r} = \{r_k, E^2, X\}_{E^2,E^2}$. It follows by the definitions that

(6.3) $$r_k(x) = x \quad \text{for every } x \in X \text{ and every } k = 1, 2, \ldots$$

and

(6.4) for every neighbourhood U of X in E^2 $r_k \simeq r_{k+1}$ in U for almost all k.

Denote by B_k (for $k = 1, 2, \ldots$) the closed square with vertices $(k, 0)$, $(k+1, 0)$, $(k+1, 1)$, $(k, 1)$. Let us show that

(6.5) $$B_k \cap r_k(E^2) - X \neq \emptyset \quad \text{for every } k = 1, 2, \ldots$$

Since B_k is a locally connected continuum and r_k is continuous, $r_k(B_k)$ is a locally connected continuum. It follows by (6.3) that $B_k \cap X \subset r_k(B_k)$. Therefore $r_k(B_k)$ contains some points belonging to B_k which do not belong to $B_k \cap X$, otherwise $r_k(B_k)$ would not be locally connected. Thus $r_k(B_k) \cap B_k - X \neq \emptyset$ and hence we obtain (6.5).

Take an arbitrary point $x_k \in (B_k \cap r_k(E^2) - X)$ for $k = 1, 2, \ldots$ The set $\bigcup_{k=1}^{\infty} \{x_k\}$ is closed in E^2, because it has no accumulation points. Therefore the set $U = E^2 - \bigcup_{k=1}^{\infty} \{x_k\}$ is a neighbourhood of X in E^2. Since $x_k \in r_k(E^2)$, $r_k(E^2)$ is not contained in U for any $k = 1, 2, \ldots$, which contradicts (6.4). Thus, X is not a FAR-space and the proof is concluded.

The same example shows that the converses of (3.7) and (4.2) are not true.

Now, we give an example of a MANR-space which is not a FANR-space. This example, due to S. Nowak, was described for other purposes in [10].

Consider the subset Z of the plane E^2 which is the closure of the set consisting of all points $(x, y) \in E^2$ such that $y = \cos(\pi/x)$ where $-1 \leqslant x \leqslant 1$ and $x \neq 0$ and of all points $(x, y) \in E^2$ such that $x^2+(y+1)^2 = 1$ and $y \leqslant -1$. Denote by N the set of natural numbers with the discrete topology and put $X = Z \times N$. Let us prove that

(6.6) *The space X is a* MANR*-space and it is not a* FANR*-space.*

Proof. Let $X_n = Z \times \{n\}$ for $n = 1, 2, \ldots$ Obviously $X = \bigoplus_{n=1}^{\infty} X_n$. By Theorem (3.1) of [3] Z is a fundamental retract of an annulus. Hence by (3.7) Z is a mutational retract of an annulus, and by Theorem (4.11), it is a MANR-space; therefore so is X_n. Thus, by Theorem (4.13), X is a MANR-space.

Now suppose, on the contrary, that X is a FANR-space. Obviously, we can assume that X is a closed subset of E^2. Then there exists a closed neighbourhood U of X in E^2 such that X is a fundamental retract of U. The neighbourhood U contains a closed neighbourhood V of X which is the union of a countable family of mutually disjoint annula. Therefore X is a fundamental retract of V. Hence, by Theorem (18.2) of [6], $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(V)$.

Let S be a circle and put $Y = S \times N$. The spaces V and Y are of the same homotopy type; therefore $\mathrm{Sh}(V) = \mathrm{Sh}(Y)$. Hence $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$. In [10] S. Nowak has proved that the shapes $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ are incomparable, which contradicts the inequality obtained. Thus, X is not a FANR-space and the proof is finished.

The same example shows that for non-compact spaces (3.14) is not true if we put $X' = V$, where V is defined in the last proof.

Also the same example shows that the converse of (3.18) is not true. Namely, the space X may be considered as a closed subset of the plane E^2. Since X is a MANR-space, it is a mutational neighbourhood retract of E^2. But, as we have shown, X is not a fundamental neighbourhood retract of E^2.

This example shows also that for fundamental retracts, fundamental neighbourhood retracts, and FANR-spaces theorems analogous to (3.16), (3.20) and (4.13), respectively, are not true.

It follows by the same example that the converse of (4.1) is not true, i.e. that

(6.7) *The space X is not a* MAR*-space.*

Proof. Suppose, on the contrary, that X is a MAR-space. We can consider X as a closed subset of E^2. Then there exists a mutational retraction $\boldsymbol{r}\colon \boldsymbol{V}(E^2, E^2) \to \boldsymbol{U}(X, E^2)$. Take a neighbourhood $U \in \boldsymbol{U}(X, E^2)$ which is a union of a countable family of mutually disjoint annula U_n $(n = 1, 2, \ldots)$. By (1.7) there exists an $r \in \boldsymbol{r}$ such that $r\colon E^2 \to U$. Moreover, $r(x) = x$ for every $x \in X$. Therefore $r(E^2) \cap U_n \neq \emptyset$ for every $n = 1, 2, \ldots$ Hence $r(E^2)$ is not connected, which is not possible. Thus we have obtained a contradiction and the proof is finished.

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