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Lusin density and Ceder's differentiable restrictions of arbitrary real functions

by

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Abstract. J. G. Ceder recently proved a theorem from which it follows that if A is an uncountable subset of the reals R, then for every $f\colon A\to R$, there exists a bilaterally dense in itself set $B\subset A$ such that $f|B$ is differentiable (infinite derivatives are allowed). Uncountability of A is necessary, and B cannot be made to have cardinality c (the cardinality of R). The main purpose of this paper is to characterize those sets $A\subset R$ for which it is true that for every $f\colon A\to R$, there exists a bilaterally c-dense in itself set $W\subset A$ and a dense in W set B such that $f|W$ is differentiable on B. A new notion of density results, and this notion is compared to known types of categoric density in metric spaces.

1. Introduction. A set B *is bilaterally dense* (c*-dense*) *in itself* if every closed interval containing an element of B contains points (c-many points) of B. A real function f *is differentiable at* x if and only if f is continuous at x, x is a limit point of the domain D_f of f, and it is true that there is an extended number m (possibly $+\infty$ or $-\infty$) such that if $\{x_n\}$ is a sequence of elements of $D_f-(x)$ converging to x, then $\{(f(x)-f(x_n))/(x-x_n)\}$ converges to m.

In [4] Ceder gives the following:

THEOREM C. *If A is an uncountable number set, then for every $f\colon A\to R$, there exists a countable set $C\subset A$ such that for each $x\in A-C$ there exists a bilaterally dense in itself set $B\subset A-C$ containing x such that $f|B$ is differentiable and monotonic.*

B cannot be made to have cardinality c. Ceder's argument for the monotonicity part of Theorem C has a mistake in it, but a correction is given in [7], and a short alternative correction is given in this paper. It is easy to show that *if $A\subset R$ is countable, there exists $f\colon A\to R$ which has no continuous restriction to any dense in itself subset of A.*

The primary purpose here is to prove the following two theorems:

THEOREM 1. *If A is an L_2 set, then for every $f\colon A\to R$, there exists an L_1 set $C\subset A$ such that for each $x\in A-C$ there exists a bilaterally c-dense in itself set $W\subset A-C$ and a dense in W set B containing x such that $f|W$ is differentiable on B.*

"L_1" and "L_2" are notions of "scarcity" and "abundance", respectively, analogous to countability and uncountability, respectively, and defined as follows. A *set* M *is Lusin in a set* N if $M \subset N$ and there is no subset of M of cardinality c which is nowhere dense in N (N. Lusin showed [9] that if the Continuum Hypothesis is true, there is a Lusin in R set of cardinality c). An L_1 *set* is the union of countably many sets, each Lusin in itself, and an L_2 *set* is one which is not an L_1 set. Properties L_1 and L_2 are termed "measures of Lusin density".

The phrase "and $f|W$ is monotonic" cannot be added to the end of the statement of Theorem 1. This author has found it necessary to leave open the question as to whether the phrase "and $f|B$ is monotonic" can be added to the end of the statement of Theorem 1.

THEOREM 2. *If* A *is an* L_1 *set, then there exists a function* $f\colon A \to R$ *such that if* $W \subset A$ *is bilaterally* c*-dense in itself and* $B \subset W$ *is dense in* W, *then* $f|W$ *is discontinuous at some element of* B.

The relationship between Theorem 1 and Ceder's theorem is analogous to the relationship between the theorems the author presented in [3] and a theorem of Bradford and Goffman [2] concerning Blumberg's theorem [1].

THEOREM BG [2]. *If* A *is a metric space, the following are equivalent*:

1) *for every* $f\colon A \to R$, *there exists* D, *dense in* A, *such that* $f|D$ *is continuous*,

2) A *is* $G_{\rm II}$ [6] (*i.e. every set open in* A *is 2nd category in* A).

THEOREM B. [3]. *If* A *is a separable metric space then the following are equivalent*:

1) *for every* $f\colon A \to R$, *there exists a* c*-dense in* A *set* W *and a dense in* W *set* D *such that* $f|W$ *is continuous on* D,

2) A *is* c*-typically dense in itself* (*i.e. no open in* A *set is the union of a 1st category in* A *set and a Lusin in* A *set*).

The separability of A in Theorem B is used only in showing 1) $\to$ 2).

In section 3, L_1 sets, L_2 sets, and L_2-density are defined for a general metric space setting, and these notions are investigated and compared to c-typical density and other forms of categoric density previously defined. Also, as a postscript, some possible variations of Theorem B are discussed.

2. Differentiable restrictions of real functions. A number x is a *limit* (c*-limit*) (L_2*-limit*) *point of a number set* A if for every open interval I containing x, $I \cap A$ is infinite (of cardinality c) (an L_2 set). The limits are termed *bilateral* if "open" can be replaced by "closed". Similarly, a point P of R^2 is a *bilateral limit* (c*-limit*) (L_2*-limit*) *point of a function* f if for every square S (includes interior) with a vertical side having center P, $\pi(S \cap f)$ is infinite (of cardinality c) (an L_2 set), where π denotes X-projection. A set A *is* (*bilaterally*) L_2*-dense in itself* if each element of A is a (bilateral) L_2-limit point of A.

It should be remarked that "cardinality c" could be replaced by "cardinality $\aleph_1$" in the definitions and in Theorem 1, but a difficulty would arise in the proof of the resulting Theorem 2 under this change.

The fact (see König's Theorem [6]) that the union of countably many sets having cardinality less than c has cardinality less than c is used repeatedly without so stating.

L_1 sets and L_2 sets have certain relationships to each other analogous to those which hold for countable sets and uncountable sets. Some of these are expressed in the following lemmas.

LEMMA 0. *Any set of cardinality less than* c *is Lusin in itself. Any subset of a Lusin in itself* (*respectively* L_1) *set is a Lusin in itself* (*respectively* L_1) *set. The union of countably many* L_1 *sets is an* L_1 *set.*

Proof. Follows directly from definitions.

LEMMA 1. *If* x *is an element of a bilaterally* L_2*-dense in itself set* M, *then there exists a bilaterally* c*-dense in itself nowhere dense in* M *subset* N *of* M *containing* x.

Proof. Suppose x is an element of a bilaterally L_2-dense in itself set M. For each positive integer n, $A_n = [x, x+1/n] \cap M$ is an L_2 set and therefore has a nowhere dense in A_n subset B_n of cardinality c, and B_n will have a bilaterally c-dense in itself subset R_n. Similarly, there will exist a bilaterally c-dense in itself subset L_n of $[x-1/n, x] \cap M$. Then $N = L_1 \cup L_2 \cup \ldots \cup (x) \cup R_1 \cup R_2 \cup \ldots$ is the desired set.

LEMMA 2. *If* A *is an* L_2 *set, then* $A = B \cup C$, *where* B *is an* L_1 *set and* C *is bilaterally* L_2*-dense in itself.*

Proof. Let B be the set of all points of A which are not bilateral L_2-limit points of A. For each $x \in A$, let I_x be the longest interval containing x such that $I_x \cap A$ is an L_1 set. $\{I_x|\ x \in B\}$ is a collection of a mutually exclusive intervals and therefore countable. Thus, $\{I_x \cap A|\ x \in A\}^* = B$ is an L_1 set (H^* denotes the union of the sets in a collection H of sets). Then each element of $C = A - B$ is a bilateral L_2-limit point of A, and since B is an L_1 set, it follows that C must be bilaterally L_2-dense in itself.

LEMMA 3. *If* A *is an* L_2 *set and* f *is a function from* A *into* R. *Then* f *contains a bilateral* L_2*-limit point of itself.*

Proof. A modification of the proof of Lemma 4 of [5] will be used. Suppose the lemma is false, and for each $x \in A$, let S_x be a square with a vertical side of length l_x having center $\langle x, f(x)\rangle$ such that $\pi(f \cap S_x)$ is an L_1 set. Assume that $A' = \{x|\ \langle x, f(x)\rangle$ belongs to the left side of $S_x\}$

is an L_2 set. For some positive integer n, $B = \{x \in A' \mid l_x > 1/n\}$ is an L_2 set. Therefore, B contains a bilateral L_2-limit point of itself, so let x denote such a point. Then there exists a vertical square S with right side of length $1/2n$ and right side having center $\langle x, j/2n\rangle$ for some integer j, such that $\pi(S \cap f|B) = D$ is an L_2 set. Let $y \in D$ be a bilateral L_2-limit point of D. Then $\pi(f \cap S_y)$ is an L_2 set, and this is a contradiction.

LEMMA 4. *If A is an L_2 set and f is a function from A into R, then there exists an L_1 subset B of A such that each point of $f|(A-B)$ is a bilateral L_2-limit point of $f|(A-B)$.*

Proof. Let B be the set of all points of A which are not bilateral L_2-limit points of f. It follows from Lemma 3 that B cannot be an L_2 set. Thus, each point of $f|(A-B)$ is a bilateral L_2-limit point of f, and since B is an L_1 set, it follows that each point of $f|(A-B)$ is a bilateral L_2-limit point of $f|(A-B)$.

LEMMA 5 (Ceder [4, Th. 3]). *If D is an uncountable set and f is a function from D into R, then there exists a countable subset E of D such that for each $x \in D-E$,*

$$D_L(f|(D-E), x) \cap D_R(f|(D-E), x) \neq \emptyset .$$

$D_L(f, x) = \{m \in [-\infty, \infty] \mid m = \lim_n (f(x)-f(x_n))/(x-x_n)$ for some increasing sequence $\{x_n\}$ of elements of D_f converging to $x\}$. $D_R(f, x)$ is defined similarly for decreasing sequences.

LEMMA 6. *If A is an L_2 set and f is a function from A into R, then there exists an L_1 subset C of A such that for each $x \in A-C$, there exists a bilaterally L_2-dense in itself subset M of $A-C$ containing x such that $f|M$ is differentiable at x.*

Proof. Let $B \subset A$ be the set described in Lemma 4. Then let $D = A-B$ and let E be the set described in Lemma 5. Now, let $C = B \cup E$. C is an L_1 set, and each point of $f|(A-C)$ is a bilateral L_2-limit point of $f|(A-C)$. Let $x \in A-C$. $D_L(f|(A-C), x) \cap D_R(f|(A-C), x) \neq \emptyset$, so let $\{x_n\}$ and $\{y_n\}$ be increasing and decreasing sequences, respectively, from $A-C$ converging to x such that

$$\lim_n (f(x_n)-f(x))/(x_n-x) = \lim_n (f(y_n)-f(x))/(y_n-x) .$$

Now, for each integer $n \geqslant 2$, let U_n be an L_2-dense in itself subset of the open in $A-C$ set $((x_{n-1}+x_n)/2, (x_n+x_{n+1})/2) \cap (A-C)$ such that for each $t \in U_n$, $|(f(t)-f(x))/(t-x) - (f(x_n)-f(x))/(x_n-x)| < 1/n$, and let V_n be a similar set associated with y_n. Then, $M = (U_2 \cup U_3 \cup \ldots) \cup (x) \cup (V_2 \cup V_3 \cup \ldots)$ is the desired set.

Proof of Theorem 1. Let A be an L_2 set and f be a function from A into R. Let C be the set of Lemma 6, and let $x \in A-C$. Let G_1 be the set whose only element is $A-C$, and proceed inductively as follows: Given G_n, then for each $S \in G_n$, (1) let M_S be a bilaterally L_2-dense in itself subset of S such that $f|M_S$ is differentiable at some element x_S of M_S (make sure $x_S = x$ on the first step), (2) let N_S be a nowhere dense in M_S subset of M_S containing x_S which is bilaterally c-dense in itself, (3) let K_S be the collection to which k belongs if and only if k is a non-empty intersection $M_S \cap I$ for some component I of $R-\mathrm{Cl}(N_S)$, and (4) let $G_{n+1} = \{K_S \mid S \in G_n\}^*$.

Now, let $W = \{t \mid$ there is a positive integer n and a set S in G_n such that $t \in N_S\}$, and let $B = \{t \mid$ there is a positive integer n and a set S in G_n such that $t = x_S\}$. Each set N_S is bilaterally c-dense in itself, so W is also. To show that B is dense in W, suppose $t \in W$ and $d > 0$. There is a positive integer n and an element S of G_n such that $t \in N_S$. N_S is bilaterally c-dense in itself, so there is another element t' of N_S within $d > 0$ of t. There is a set T of K_S which lies between t and t'. $T \in G_{n+1}$, and x_T will be an element of B within $d > 0$ of t.

Now, let $t \in B$. There is a positive integer n and a set S in G_n such that $t = x_S$. $f|M_S$ is differentiable at x_S, and $M_S \cap W$ is open in W, so $f|W$ is differentiable at t.

Proof of Theorem 2. Suppose A is an L_1 set. Then there exists a sequence $M_1, M_2, \ldots$ of sets, each Lusin in itself, such that $A = M_1 \cup M_2 \cup \ldots$ Let $g\colon R \to R$ be the function of Zygmund and Sierpiński [8] which has no continuous restriction of cardinality c. Let $f\colon A \to R$ be defined by $f(x) = 2n\pi + \mathrm{Arctan}(g(x))$ if $x \in M_n \setminus \bigcup_{k<n} M_k$. Now, suppose $W \subset A$ is bilaterally c-dense in itself, D is dense in W, and $f|W$ is continuous on D. Let t be an element of D and n be the integer such that $t \in M_n$. It follows that there is a set V, open in W, which is a subset of M_n. $f|V$ is continuous on a dense subset of V, and therefore continuous on a dense G_δ subset U of V. Since M_n is Lusin in itself, so is V. It follows that U has cardinality c and $f|U$ is continuous, which is a contradiction.

The remainder of section 2 concerns the monotonicity of $f|B$ in Ceder's theorem.

The problem in Ceder's argument occurs in his Lemma 3, which states that *if B is bilaterally dense in itself and $f\colon B \to R$ is differentiable, then there exists $A \subset B$ such that A is bilaterally dense in itself and $f|A$ is differentiable and monotonic*. Consider the following example. Let B be the set to which x belongs if and only if there is a sequence $\{t_n\}$, each term of which is either -1, 0, or 1 and only finitely many terms of which are not 0 such that $x = \Sigma_n t_n 10^{-n}$, in which case, $f(x) = \Sigma_n (t_n 10^{-n})^2$. f has derivative zero at each point. However, each point of f is a local proper minimum for f, so f can have no monotonic restriction to a *bilaterally dense in itself subset of B*.

In proving the monotonicity part of Theorem C, care must be taken to avoid local maxima and minima so that the sequences Ceder describes on lines 7-9 of the argument for Lemma 3 of [4] will exist. Therefore, define $g'(x) = +0$ (respectively -0) if and only if $g'(x) = 0$, and there exist sequences $\{x_n\}$ and $\{y_n\}$, increasing and decreasing, respectively, with limit x such that $\{g(x_n)\}$ and $\{g(y_n)\}$ are increasing and decreasing (decreasing and increasing), respectively. Then g *is monotonically differentiable at* x if and only if g is differentiable at x and either $g'(x) \neq 0$ or else $g'(x) = \pm 0$.

Notice that if "L_1" and "L_2" are replaced by "countable" and "uncountable", respectively, in Lemmas 2-6, the resulting statements (to be referred to as Lemmas 2′-6′, respectively) are true.

LEMMA 7′. *If $A \subset R$ is uncountable and $f\colon A \to R$ has no uncountable horizontal restriction, then there exists a countable set $F \subset A$ such that if $x \in A - F$, there exists a bilaterally uncountably-dense in itself subset M of $A - F$ such that $f|M$ is monotonically differentiable at x.*

Proof. Suppose f satisfies the hypothesis, but there exists no bilaterally uncountably dense in itself subset M of A such that $f|M$ is monotonically differentiable at some element x of M. Let C be the set of Lemma 6′. It follows that for each $x \in A - C$, there exist positive numbers c_x and d_x such that either $U_x = \{t|\ x < t < x + d_x$ and $0 < (f(x) - f(t))/(x - t) < c_x\}$ or $V_x = \{t|\ x < t < x + d_x$ and $-c_x < (f(x) - f(t))/(x - t) < 0\}$ is countable. Assume that the set $B = \{x \in A - C|\ U_x$ is countable$\}$ is uncountable, and let n be a positive integer such that $D = \{x \in B|\ c_x > 1/n$ and $d_x > 1/n\}$ is uncountable. Now, let M be a bilaterally uncountably dense in itself subset of D such that $f|M$ is differentiable at some element y of M (Lemma 6′). $(f|M)'(y) = 0$ and $(f|M)'(y) \neq -0$ and U_y is countable, so there must exist an element x of M such that $y - 1/2n < x < y$ and $0 < (f(x) - f(y))/(x - y) < 1/2n$. It follows that U_x is uncountable, and this is a contradiction.

Proof of Theorem C. First it will be proved that there exists *some* bilaterally dense in itself subset B of A such that $f|B$ is differentiable and monotonic. If f has an uncountable horizontal restriction, the proof is trivial, so assume otherwise. Let $G_1 = (A)$ and make the following changes in the inductive process of the proof of Theorem 1. Replace "L_1" and "L_2" by "countable" and "uncountable", respectively. On part (1), make sure $f|M_S$ is *monotonically* differentiable at x_S. On part (2) just let N_S be a nowhere dense in M_S subset of M_S containing x_S as a bilateral limit point. Ignore W and define B as before. To show that B is bilaterally dense in itself, let $t \in B$ and $d > 0$. There is a positive integer n and a set S in G_n such that $t = x_S$. x_S is a bilateral limit point of N_S, so there are elements a and b of N_S to the left and right, respectively, of x_S and within $d > 0$ of x_S. Then there exist sets T and U in G_{n+1} such that T lies between a and x_S and U lies between x_S and b. Then x_T and x_U will be elements of B to the left and right, respectively, of t within $d > 0$ of t.

Thus, B is a bilaterally dense in itself subset of A and $f|B$ is differentiable (proof as before), which is the point to which Ceder arrives on line 13 of p. 357 of [4]. However, $f|B$ is actually *monotonically* differentiable now, and with this additional hypothesis, Ceder's Lemma 3 and the rest of his argument for Theorem C is valid with only minor changes.

3. Lusin density in a metric setting. It is clear from Lemma 2 that the conclusion of Theorem 1 holds for a subset A of R if and only if A contains an L_2-dense in itself subset. To give a better idea of which sets do have this property, L_2-density will be compared with other types of categoric density. Because of possible future applications of this concept in more general situations, the comparison is given in a general metric setting.

The definition of a number set M being Lusin in a number set N could be generalized directly to sets in a metric space, but this leads to certain complications in non-separable spaces. The author encountered some of these complications in the preparation of an earlier paper [3], and the referee of that paper suggested defining the concept as follows: *M is Lusin in N* if and only if every nowhere dense in N subset L of M is a countable union $L = L_1 \cup L_2 \cup \ldots$, where each L_i is of local cardinality less than c (i.e. such that if $x \in L_i$ for some i, then there is a neighborhood U of x such that $U \cap L_i$ has cardinality less than c). The necessity for the change to "local cardinality" is fairly obvious due to non-separable spaces such as the following; let X be the unit square disc and δ be the metric defined by $\delta(\langle a, b\rangle, \langle c, d\rangle) = |b - d|$ if $a = c$ and 1 otherwise. Then the set M of points in X with rational ordinates is Lusin in X because M is itself of local cardinality less than c. However, M would not satisfy the direct generalization of the definition given in Section 1 because M contains nowhere dense in X sets (containing just one point for each abscissa) with cardinality c. In this space, a set M is Lusin in X if and only if each nowhere dense in X subset L of M is *itself* of local cardinality less than c. However, this simplification of the definition would prove unsatisfactory because of non-separable spaces such as the following; let X be the set of all number sequences and δ be the metric defined by $\delta(\{x_n\}, \{y_n\}) = 1/2^k$, where k is the least positive integer such that $x_k \neq y_k$. Then, the set M of all sequences in X which have even subscripted terms equal to zero and which terminate in zeros is itself nowhere dense in X but of local cardinality c. However, $M = M_1 \cup M_2 \cup \ldots$ where $M_i = \{\{x_n\} \in M|\ x_j = 0$ if $j \geqslant i\}$, and each M_i is locally finite.

First it will be pointed out that the new definition does reduce to the direct generalization of the definition given in Section 1 in separable spaces.

THEOREM 3. *If $M \subset N \subset X$, where X is separable, then M is Lusin in N if and only if there is no subset of M of cardinality c which is nowhere dense in N.*

Proof. Suppose M is Lusin in N and $L \subset M$ is nowhere dense in N. Then $L = L_1 \cup L_2 \cup \ldots$, where each L_i is of local cardinality less than c. Since each L_i is separable, it follows that each L_i is of cardinality less than c. Therefore $L = L_1 \cup L_2 \cup \ldots$ is of cardinality less than c. The proof in the opposite direction is obvious.

The notions of an L_1 *set* and an L_2 *set*, as well as L_2-*density* are defined as before but in terms of the generalized notion of a Lusin set. Now, the notion of a subset M of a metric spaces X being a T_1 *set* (called "first c-type" in [3]) shall mean that it is the union of a first category set and a Lusin in X set. A T_2 *set* (called "second c-type" in [3]) is one which is not a T_1 set, and M *is* T_2-*dense in* X (called "c-typically dense in X" in [3]) if and only if for every open set Q, $Q \cap M$ is a T_2 set.

THEOREM 4. *A metric space X is T_2-dense in itself if and only if no open set is the union of a first category set and an L_1 set.*

Proof. Suppose X is T_2-dense in itself. If X is not G_{II}, the proof is immediate, so assume X is G_{II}, but that there is an open set $O = K \cup N$, where K is first category and N is an L_1 set. Then $N = N_1 \cup N_2 \cup \ldots$, where each N_i is Lusin in itself. Some N_i must be second category, so for each N_i, let $N_i' = \{x \in N_i |\ x$ is an element of some open set in which N_i is dense$\}$. Then $K' = K \cup (N_1 - N_1') \cup (N_2 - N_2') \cup \ldots$ is still first category. Let $N' = N_1' \cup N_2' \cup \ldots$ Then $O = K' \cup N'$. Now let Q be a nowhere dense in X subset of N'. Then if i is a positive integer such that N_i' is not empty, N_i' is a dense subset of some open set U, so that $Q \cap N_i'$ is nowhere dense in N_i' and nowhere dense in N_i. Thus, $Q \cap N_i' = A_1^i \cup A_2^i \cup A_3^i \cup \ldots$, where each A_j^i is of local cardinality less than c. So $Q = A_1^1 \cup A_1^2 \cup A_2^1 \cup A_1^3 \cup A_2^2 \cup A_3^1 \cup \ldots$, and N' is Lusin in X. Therefore O is a T_1 set and this is a contradiction. The proof in the opposite direction is obvious.

Now, a standard version of the Baire Category Theorem states that every complete metric space is G_{II}. Hausdorff [6] defines various "density" or "compactness" properties which are intermediate to completeness and G_{II}. It is shown that for metric spaces,

$$\text{completeness} \rightarrow \text{absolute } G_\delta \rightarrow F_{\mathrm{II}} \rightarrow G_{\mathrm{II}},$$

but that none of these implications is reversible. It will now be shown (using the Continuum Hypothesis CH when necessary) that in perfect metric spaces, L_2-density and T_2-density compare nicely with property G_{II} and *property* F_{II}, which means that each perfect set is second category in itself.

THEOREM 5. *If X is a perfect (i.e. having no isolated points) metric space, then*

$$F_{\mathrm{II}} \xrightarrow{\mathrm{CH}} T_2\text{-density} \begin{array}{c} \nearrow\ L_2\text{-density}\ \searrow \\ \searrow\ \ G_{\mathrm{II}}\ \ \nearrow \end{array} \text{uncountable density},$$

but none of the implications is reversible.

Proof. Suppose X is F_{II} but not T_2-dense in itself. Then there exists an open set O such that $O = N_1 \cup N_2 \cup \ldots \cup L$, where each N_i is nowhere dense and L is Lusin in X. We can assume, without loss of generality, that each N_i is closed and $L = O - (N_1 \cup N_2 \cup \ldots)$. Since $F_{\mathrm{II}} \rightarrow G_{\mathrm{II}}$, L must be dense in some open set W. Let $G_1 = (W)$, P_W be a point of $W \cap L$, and proceed inductively as follows: Given G_n and P_J for each $J \in G_n$, then for each $J \in G_n$, (1) let Q_J be a point of $J \cap L$ distinct from P_J, (2) let U and V be mutually exclusive neighborhoods having centers P_J and Q_J, respectively, radii less than $1/n$, and closures lying in $J - N_n$, (3) rename $P_J = P_U$ and $Q_J = P_V$, and (4) let $G_{n+1} = \{g |$ for some set $J \in G_n$, g is either the set U or the set V described in (2)$\}$. Then, the set $N = L \cap \cap (G_1^* \cap G_2^* \cap \ldots)$ is a separable perfect subset of L. Since N is nowhere dense in X, it is the union of countable many sets each having cardinality less than c, and since N is separable, it follows that N is of cardinality less than c. *Assuming* CH, it follows that N is countable. So N is a perfect subset of X which is not second category in itself. This is a contradiction.

That T_2-density implies both L_2-density and G_{II} follows immediately from Theorem 4. Furthermore, if there is a countable open set O, it follows that since X is perfect, O is both first category and an L_1 set, so the last two implications hold.

To show that the implications are not reversible, *assume* CH and consider a Lusin set L which is uncountably dense in R but has no uncountable nowhere dense subset. Let $C_1, C_2, \ldots$ be a sequence of mutually exclusive Cantor sets such that $C_1 \cup C_2 \cup \ldots$ is dense in R, and for each i, let C_i' be a subset of C_i which is uncountably dense in C_i but which has no uncountable nowhere dense in C_i subset. Then, $X = C_1' \cup C_2' \cup \ldots$ would be uncountably dense in itself but neither L_2-dense in itself nor G_{II}. Furthermore, L is G_{II} but not L_2-dense in itself, $C = C_1 \cup C_2 \cup \ldots$ is L_2-dense in itself but not G_{II}, and $L \cup C$ is both G_{II} and L_2-dense in itself but not T_2-dense in itself. Finally, let E be the set of endpoints of C_1 and $Y = (R - C_1) \cup E$. Then, Y is T_2-dense in itself but not F_{II}.

COROLLARY. *If the domain A of a real function f contains a perfect in R set, then the conclusion of Theorem 1 holds.*

Properties G_{II} and F_{II} have been shown to have great significance in some theorems. For example, it is shown in [6, p. 287] that *if A is G_{II}, then every $g \in \mathcal{B}(A)$ (the collection of pointwise limits of sequences of continuous real valued functions with domain A) is at most pointwise discontinuous (i.e. continuous on a dense subset of A)*. It follows immediately [6, p. 288] that *if A is F_{II}, then for every $f \in \mathcal{B}(A)$ and every closed set $M \subset A$, $f|M$ is at most pointwise discontinuous*. In a similar manner, Theorems BG, B, 1, 2, and C lead immediately to the following four theorems.

THEOREM BG+. *If A is a metric space, then the following are equivalent:*

1) *for every f: $A \to R$ and perfect $M \subset A$, there exists D, dense in M, such that $f|D$ is continuous,*

2) *A is F_{II}.*

THEOREM B+. *If A is a separable metric space, then the following are equivalent:*

1) *for every f: $A \to R$ and perfect $M \subset A$, there exists a c-dense in M set W and a dense in W set D such that $f|W$ is continuous on D,*

a) *every perfect subset of A is T_2-dense in itself.*

THEOREM 1+. *If $A \subset R$, the following are equivalent:*

1) *for every f: $A \to R$ and perfect in A subset M of A, there exists a bilaterally c-dense in itself subset W of M and a dense in W set B such that $f|W$ is differentiable on B,*

b) *every perfect in A subset of A is an L_2 set.*

THEOREM C+. *If $A \subset R$, the following are equivalent:*

1) *for every f: $A \to R$ and perfect in A subset M of A, there exists a bilaterally dense in itself subset B of M such that $f|B$ is differentiable and monotonic,*

c) *every perfect in A subset of A is uncountable.*

On the basis of Theorem 5, it is easily seen that condition b and property F_{II} are both intermediate in strength to conditions a and c. It is interesting that the following turns out to be true.

THEOREM 6. *In metric spaces A, property F_{II} and conditions* a, b, *and* c *are all equivalent.*

Proof. It is easily seen that condition c and F_{II} are equivalent [6, p. 336]. So it is necessary to show only that F_{II} implies condition a. If condition a fails, it follows that there is a perfect subset M of A such that M is not T_2-dense in itself. Then, as in the first part of the proof of Theorem 5, it can be shown that there is a countable set N which is perfect in M. N will also be perfect in A, but of first category in itself, so A is not F_{II}.

On more possible variation on Theorem B will now be considered. It follows from Theorem B that *if f: $R \to R$, there exists a c-dense in R set W such that $f|W$ is at most pointwise discontinuous*. Considering the relationship between pointwise discontinuity and Baire's first class $\mathcal{B}$, it is natural to ask, "*can W be chosen so that $f|W$ is also in $\mathcal{B}(W)$?*" To show the impossibility of this, consider the function f: $R \to R$ of Sierpiński and Zygmund constructed in the third paragraph on p. 423 of [8], except let Φ be the family of all real valued Borel functions defined on G_δ subsets of R. Suppose $W \subset R$ is c-dense in R and $f|W$ is in $\mathcal{B}(W)$. Let $f_1, f_2, \ldots$ be a sequence of continuous functions with domain W converging to f. For each n, let g_n be a continuous extension of f_n to a G_δ set M_n; let M be the G_δ set $M = M_1 \cap M_2 \cap \ldots$, and for each n, let $h_n = g_n|M$. Now, $N = \{x|\ h_1(x), h_2(x), \ldots \text{ converges}\}$ is $F_{\sigma\delta}$ relative to M [6, p. 307]. So if for each n, $k_n(x) = h_n(x)$ if $x \in N$ and $k_n(x) = 0$ if $x \in M-N$, then k_n is a Borel function on M, and the pointwise limit k of $k_1, k_2, \ldots$ is a Borel function on M and agrees with f on W. But f does not coincide with any function of the family Φ on any set of cardinality c. This is a contradiction.

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