The number of countable generic models for finite forcing

by

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Abstract. We give a characterization of forcing-complete theories and of types which are realized in generic models. Then, we use these results to prove a sufficient condition for the existence of $2^{\aleph_0}$ denumerable generic models.

We prove the following theorem:

THEOREM. *Suppose that T^f is a complete forcing-companion of a denumerable language, and suppose that T^f has no prime-model. Then T^f has $2^{\aleph_0}$ denumerable generic models.*

0. Notations. L is a denumerable first-order language with equality and T^f is the forcing-companion of a theory T in (see [1]). We denote by B_n ($n \in \omega$) the partial boolean algebras for T^f. For simplicity, we identify a formula with its equivalence class modulo T^f. In the sequel, T^f is assumed to be complete without a finite model. We denote by $B = \bigcup_{n\in\omega} B_n$ the boolean algebra for T^f. M is said to be a *prime-model* for T^f if M is elementarily embeddable in any other model of T^f. B_n is said to be *atomic* if for every $\varphi \neq 0$, there exists a ψ atom of B_n such that $\psi \leqslant \varphi$. A *type* is a proper filter of B. We denote by $\bigwedge_{i\in I} a_i$ the greatest lower bound (if it exists) of a family. The concept of the least upper bound is defined similarly. For notions and results concerning finite forcing in model theory, the reader is referred to [1], [7]. Other notions will be introduced when necessary.

1. Properties.

LEMMA 1. *Suppose T is a theory of L. M completes T if and only if for every formula $\varphi(x_0, \ldots, x_{n-1})$ of L and any n-tuple $\langle a_0, \ldots, a_{n-1}\rangle$ of $|M|$ (domain of M), if $M \models \varphi(a_0, \ldots, a_{n-1})$, there exists an existential formula $\psi(x_0, \ldots, x_{n-1})$ such that:*

(a) $\psi \leqslant \varphi$ (*for* T),
b) $M \models \psi(a_0, \ldots, a_{n-1})$.

Proof. M completes T if and only if $T \cup D(M)$ ($D(M)$: diagram of M) is a complete set of sentences for the language $L(M)$. A simple application of the theorem of compacity gives the result.

In [3], A. Macintyre uses this characterization to prove that T-generic structures are axiomatizable by a sentence of $L_{\omega_1\omega}$. In the sequel, we denote by $\Sigma_\varphi = \{\neg\psi / \psi$ an existential formula such that $\psi \leqslant \varphi\} \cup \{\varphi\}$.

Lemma 2. *M completes T if and only if M omits Σ_φ for every formula φ of L.*

Proof. If $M \models \varphi(a_0, \ldots, a_{n-1})$, then there exists an existential formula ψ such that $\psi \leqslant \varphi$ and $M \models \psi(a_0, \ldots, a_{n-1})$. So, $M \not\models \neg\psi(a_0, \ldots, a_{n-1})$ and Σ_φ is omitted in M. Conversely, if $M \models \varphi(a_0, \ldots, a_{n-1})$, since Σ_φ is omitted in M, there exists a $\neg\psi \in \Sigma_\varphi$ such that $M \not\models \neg\psi(a_0, \ldots, a_{n-1})$. In consequence, $M \models \psi(a_0, \ldots, a_{n-1})$ with ψ existential $\leqslant \varphi$; so M completes T (Lemma 1).

Proposition 1. *A complete theory T of L is equal to its forcing-companion T^f if and only if, for every formula φ, $\varphi = \bigvee \{\psi/\psi$ is an existential formula such that $\psi \leqslant \varphi\}$.*

Proof. Suppose T is equal to T^f; so, by [1] T has a model which completes T. From Lemma 2, Σ_φ is omitted in M for every formula φ. Since T is complete, this is equivalent to saying that $\bigwedge \Sigma_\varphi = 0$. But $\bigwedge \Sigma_\varphi = 0$ is equivalent to the fact that: $\bigwedge \{\neg\psi/\psi$ is an existential formula such that $\psi \leqslant \varphi\} = \neg\varphi$, which is also equivalent to $\bigvee \{\psi/\psi$ existential $\leqslant \varphi\} = \varphi$. Conversely, suppose that $\varphi = \bigvee \{\psi/\psi$ existential $\leqslant \varphi\}$; then $\bigwedge \Sigma_\varphi = 0$. This property being true for every formula φ of L and L being denumerable, the use of the omitting types theorem proves that there exists a denumerable model M of T which omits every Σ_φ. Lemma 2 proves that M completes T; so, by [1], $T = T^f$.

Corollary. *A complete theory T is equal to T^f if and only if, for every $\varphi \neq 0$, there exists an existential formula ψ such that: $\psi \neq 0$ and $\psi \leqslant \varphi$.*

Proof. The condition is necessary by Proposition 1. The sufficiency is similar to the proof that, in an atomic boolean algebra, every element a is the least upper bound of the atoms less or equal to a [6].

Remarks. 1. In particular, if T is atomic and if atoms are existential formulas, the corollary proves that $T = T^f$. This is the case, for instance, of complete arithmetic $\mathrm{Th}(N)$.

2. The complete theories which are forcing-companions have a characterization which is similar to those of atomic theories [8].

Let p be an ultrafilter of B_n. p is said to be *existential* if for every $\varphi \in p$ there exists an existential $\psi \in B_n$ such that $\psi \in p$ and $\psi \leqslant \varphi$. We write $\vec{x} = (x_0, \ldots, x_{n-1})$ in the sequel. p is said to be *sur-existential* if, for every formula $\varphi(\vec{x}, y_0, \ldots, y_m)$ of B such that $\exists y_0 \ldots y_m \varphi \in p$, there exists an existential formula $\psi(\vec{x}, y_0, \ldots, y_m)$ such that $\psi \leqslant \varphi$ and $\exists y_0 \ldots y_m \psi \in p$.

Lemma 3. *p is sur-existential if and only if p is the type of an n-uple $\langle a_0, \ldots, a_{n-1} \rangle$ of elements of a denumerable generic structure.*

Proof. From Lemma 1 it is clear that if p is the type of $\langle a_0, \ldots, a_{n-1} \rangle$ of a generic structure, p is sur-existential. Conversely, suppose that p is sur-existential and consider the language $L' = L(c_0, \ldots, c_{n-1})$ obtained by adding to L n new individuals $c_0, \ldots, c_{n-1}$. Let $p(\vec{c})$ be the set of sentences of L' obtained from p by replacing $x_0, \ldots, x_{n-1}$ by $c_0, \ldots, c_{n-1}$ respectively. $p(\vec{c})$ is complete in L'. Let $\delta(\vec{c}, y_0, \ldots, y_m)$ be a formula of L' whose free variables are among $y_0, \ldots, y_m$ (δ does not necessarily use every constant $c_0, \ldots, c_{n-1}$), and suppose that δ is consistant with $p(\vec{c})$, i.e. $\exists y_0 \ldots y_m \delta \in p(\vec{c})$. The formula $\delta(\vec{x}, y_0, \ldots, y_m)$ of L obtained by replacing $\vec{c}$ by $\vec{x}$ is consistent with T^f; so, by the corollary, there exists an existential formula $\psi(\vec{x}, y_0, \ldots, y_m)$ such that: $\psi \leqslant \delta$ and $\exists y_0 \ldots y_m \psi \in p$. The formula $\psi(\vec{c}, y_0, \ldots, y_m)$ of L' is existential and satisfies $p(\vec{c}) \vdash \psi(\vec{c}, y_0, \ldots, y_m) \to \delta(\vec{c}, y_0, \ldots, y_m)$. Since the formula $\psi(\vec{c}, y_0, \ldots, y_m)$ is consistent with $p(\vec{c})$, it follows from the corollary that $p(\vec{c}) = p(\vec{c})^f$. So, let M' be a denumerable $p(\vec{c})$-generic structure and let M be the L-reduct of M'; M' exists, see [1]. Let $N' \supset M'$ be such that $N' \models p(\vec{c})$; then $N' \succ M'$. If we consider the L-reduct M of M', we have: $M \models T^f$ and $M \models p(\vec{a}_c)$, where $\vec{a}_c$ is the assignment of $\vec{c}$ in M. Suppose that $N \models T^f$ and $N \supset M$. N realizes the existential formulas of $p(\vec{x})$ in $\vec{a}_c$. Since $p(\vec{x})$ is existential, we have $N \models p(\vec{a}_c)$. So, the structure $N' = (N, \vec{a}_c)$ is an extention of M' and is also a model of $p(\vec{c})$. Since M' completes $p(\vec{c})$, we have $M' \prec N'$. Consequently, the L-reducts M and N of M' and N' satisfy $M \prec N$. The structure M is T^f-generic and it realizes $p(\vec{x})$.

Remarks. The notion of a sur-existential ultrafilter p is adequate for the types of generic models. The notion of an existential ultrafilter is, a priori, weaker; but we have no example of an existential ultrafilter which is not sur-existential. We could try to get such an example for $\mathrm{Th}(N)$ since there exists only one generic structure which is the standard model N. The following result (oral communication by A. Macintyre) proves that it is impossible.

There exists no non-principal existential ultrafilter for $\mathrm{Th}(N)$.

The reason is that there exists a formula $\theta(x)$ consistent with $\mathrm{Th}(N)$ such that, for every existential formula $\psi_1(x)$ and $\psi_2(x)$ satisfying $N \models \exists_\infty x \psi_1(x)$ and $N \models \exists_\infty x \psi_2(x)$ ($\exists_\infty$ is the quantifier "there is an infinity of ..."), we have $\mathrm{Th}(N) \neq \psi_1(x) \to \theta(x)$ and $\mathrm{Th}(N) \not\models \psi_2(x) \to \neg\theta(x)$. For every ultrafilter p, $\theta \in p$ or $\neg\theta \in p$. Suppose p is non-principal. In the first case, if $\psi(x)$ is an existential of p we have $N \models \exists_\infty x\psi(x)$, and so $\mathrm{Th}(N) \not\models \psi(x) \to \theta(x)$. The second case is identical. Note that $\theta(x)$ defines an immune and co-immune set.

Now, we remember that T^f has no prime-model; so there is an $n \geqslant 1$ such that B_n is a non-atomic boolean algebra. We denote by S_n the Stone space of B_n and by S the Stone space of B. σ_n is the function of Stone defined by

$$\sigma_n\colon\ \delta(\vec{x}) \mapsto \{p/p \in S_n \text{ and } \delta \in p\}\,.$$

Suppose $\theta(\vec{x}, y_0, \dots, y_m)$ is a formula of B. By Proposition 1 we know that $\theta = \bigvee \{\psi/\psi$ is an existential formula of B such that $\psi \leqslant \theta\}$. In fact, we can restrict the family of elements of the second member to the set of ψ whose free variables are among $x_0, \dots, x_{n-1}, y_0, \dots, y_m$:

$$\theta(\vec{x}, y_0, \dots, y_m) = \bigvee \{\psi(\vec{x}, y_0, \dots, y_m)/\dots\}\,.$$

So, in the sequel we suppose this. We note $\theta(\vec{x}, \vec{y}) = \bigvee \{\psi(\vec{x}, \vec{y})/\dots\}$ It is also easy to prove that $\exists\vec{y}\,\theta(\vec{x}, \vec{y}) = \bigvee \{\exists\vec{y}\psi(\vec{x}, \vec{y})/\psi(\vec{x}, \vec{y})$ is an existential formula $\leqslant \theta(\vec{x}, \vec{y})\}$. With each formula $\delta(\vec{x})$ of B_n we associate the set $E_\delta = \{\psi(\vec{x})/\psi(\vec{x}) \in B_n$ and $\psi(\vec{x}) \leqslant \delta(\vec{x})\}$, and withe each formula $\theta(\vec{x}, \vec{y})$ of B we associate the set $E'_\theta = \{\exists\vec{y}\psi(\vec{x}, \vec{y})/\psi(\vec{x}, \vec{y})$ which is an existential formula of $B \leqslant \theta(\vec{x}, \vec{y})\}$. We have $\bigvee E_\delta = \delta(\vec{x})$ for each $\delta \in B_n$, and $\bigvee E'_\theta = \exists\vec{y}\,\theta(\vec{x}, \vec{y})$ for each $\theta \in B$. We get a denumerable set of least upper bounds and we want to preserve them. We recall that an ultrafilter U of a boolean algebra A is said to *preserve* the least upper bound $a = \bigvee_{i \in I} a_i$ if $a \notin U$; otherwise there exists an $i \in I$ such that $a_i \in U$. In the sequel we use the following fondamental result (see, for instance, [5]):

LEMMA 4. *Let $a = \bigvee_{i \in I} a_i$. The set $\{p/p \in S(A)$ and p preserve $a\}$ is a dense open set in the Stone space $S(A)$ of A.*

COROLLARY. *The set of ultrafilters which preserve a denumerable family of least upper bounds of A is a dense subset of $S(A)$.*

Let φ be an incompletable formula of B_n; we have the following properties:

LEMMA 5. 1) *The space $\sigma_n(\varphi)$ has no isolated point. In fact, $\sigma_n(\varphi)$ is homeomorphic to the Cantor space 2^ω.*

2) *The set V of $p \in S_n$ which preserve both the families $\bigvee E_\delta$, for every $\delta \in B_n$, and the families $\bigvee E'_\theta$, for every $\theta \in B$, is a denumerable intersection of dense open sets of $\sigma_n(\varphi)$.*

3) *V is not meager in $\sigma_n(\varphi)$.*

4) *V is uncountable.*

5) $\mathrm{Card}(V) = 2^{\aleph_0}$.

6) *Each element p of V is sur-existential.*

Proof. 1) and 2) result from Lemma 4 and its corollary. 3) is a property of Baire spaces (see [2]). 4) follows from 3) and 1) for if V were denumerable, it would be a countable union of nowhere dense sets since the space $\sigma_n(\varphi)$ has no isolated point, and so V would be meager. 5) V is in fact a borelien set since it is a countable intersection of dense open sets. Since V is uncountable, $\mathrm{Card}(V) = 2^{\aleph_0}$. 6) We can easily verify that each p is sur-existential; this follows from the definition of V and from the definition of sur-existential ultrafilters.

In order to prove the theorem, we remark that each countable generic structure realizes only a denumerable set of sur-existentials. Since, by Lemma 3, each sur-existential is realized in a denumerable generic model, there are $2^{\aleph_0}$ such models.

2. Applications. By using recursivity and specific properties of groups and division rings, A. Macintyre has proved that there exist $2^{\aleph_0}$ denumerable generic groups and $2^{\aleph_0}$ denumerable generic division rings (see [3] and [4]). Here, using the above theorem and an omitting types theorem for generic structures (see [3]), we also get this result.

THEOREM (A. Macintyre). 1) *There exist $2^{\aleph_0}$ denumerable generic groups.*

2) *There exist $2^{\aleph_0}$ denumerable generic division rings.*

Proof. It is implicit in [3] (oral communication of A. Macintyre) that there exists no prime-generic model for T^f the forcing-companion of the theory of groups (M is a prime-generic model for T^f if M is elementarily embeddable in all other generics). Suppose T^f has a prime-model M_0; it is easy to see that M_0 is generic. In particular, M_0 is prime-generic: a contradiction. The theorem we have proved shows that T^f has $2^{\aleph_0}$ denumerable generic models. These models are the generic models for the theory of groups. The same thing is true for division rings.

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Reçu par la Rédaction le 29. 1. 1973