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On internal composants of indecomposable plane continua

by

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Abstract. In [3] the author introduced the concept of internal composant and proved that the union of internal composants of an indecomposable continuum X is a second category subset of X. In the present paper we obtain some new results on internal composants. The main theorem states that the union of internal composants is a G_δ-subset of X. If X denotes the simplest indecomposable continuum defined by Knaster, then all composants of X, except one, are internal.

1. Introduction. Throughout this paper all sets are assumed to be subsets of the sphere S^2. Let X be a continuum lying in this sphere. The union of all proper subcontinua of X containing a fixed point $x \in X$ is denoted by $C(x)$ and is called a *composant* of X. If X is an indecomposable continuum, then the collection of all composants of X constitute a partition of X into $\mathfrak{c}$ connected dense and pairwise disjoint sets, with $\mathfrak{c}$ denoting the cardinal of the continuum. At first sight there is no difference between two distinct composants of X. However, as we shall see in the sequel, one can distinguish several important classes of composants.

The process of distinguishing composants in an indecomposable plane continuum was initiated by S. Mazurkiewicz in 1929, when he showed that the union of accessible composants of an indecomposable plane continuum X is a first category subsets of X (answering a question of Kuratowski). Let us recall that a composant C of X is said to be *accessible* provided there exist a point $p \in C$ and a non-degenerate continuum L such that $L \cap X = \{p\}$. Otherwise it is *inaccessible*. The above theorem found some applications in plane topology (see for example [2] and [3]). In the same year K. Kuratowski [7] defined a class of composants larger than that of accessible ones. Namely a composant C of X is called a *K-composant* [5] provided that there exist a continuum $D \subset C$ and a continuum L such that $L \cap X = D$ and $L \setminus X \neq \emptyset$. K. Kuratowski proved an analogue of the Mazurkiewicz theorem for the class of K-composants. Precisely, the union of K-composants is a first category subset of X. Up to now, these results have been the best known. In a conversation with the author, A. Lelek raised the question whether or not there exists in every indecomposable plane continuum X a composant C such that any arc intersecting both C and the complement of X has at least two distinct points in common with C. In 1964 H. Cook [1] proved a theorem which asserts that if X is an indecomposable continuum (not necessarily lying in S^2) and if F is a closed subset which intersects all composants of X, then there exists a non-empty closed subset A of F such that $C \cap A$ is dense in A, for every composant C of X. These facts were the starting point of the notion of an *internal composant* introduced by the author in [3]. Let us recall that a composant C of X is called *internal* if every continuum L intersecting both C and the complement of X intersects all composants of X. Otherwise it is *external*. Hence to solve Lelek's problem we had only to prove that every indecomposable continuum contains an internal composant. This was done in [3]. Thus, the answer to Lelek's problem is positive. In [3] we have proved more: The union of external composants of an indecomposable continuum is a subset of the first category in this continuum. Since every accessible composant or K-composant is external, this result generalizes simultaneously the classical results of Mazurkiewicz and Kuratowski. An easy and unexpected consequence of this theorem is this: every composant of X is a cut of S^2.

Let X be an indecomposable continuum. By $\boldsymbol{E}$ we denote the collection of all external composants of X and by $\boldsymbol{I}$ the collection of internal composants. It is not known whether or not is the union of accessible composants (or K-composants) of X an F_σ subset of X. One of the most important properties of external composants which we shall establish in this paper states that the union of external composants, i.e. $\boldsymbol{E}^*$, is an F_σ subset of X. At the end of this paper we prove that the simplest indecomposable plane continuum defined by Knaster contains exactly one external composant.

The concept of internality has found interesting applications in investigations of the boundaries of plane continua [4]. Relations of this notion to other notions were studied in [5].

2. Auxiliary properties of inaccessible composants. The letter X denotes an indecomposable non-degenerate continuum lying in S^2.

2.1. *Every composant of X is a first category subset of X* (see [6], p. 212).

The following theorem states a fundamental property of inaccessible composants.

2.2. [3]. *Let K and K' be disjoint open discs intersecting X and let $q \notin \overline{K} \cup X$. If C is an inaccessible composant of X, then there exist two continua $A \subset C$ and $B \subset \overline{K}$ and a point $p \in K' \cap X$ such that $A \cup B$ separates S^2 between p and q and $B \setminus A \subset K$. In particular, A separates $S^2 \setminus K$ between p and q.*

2.3. *Let K be an open disc intersecting X and let $\boldsymbol{R}$ be a collection of composants of X. Let L be a continuum disjoint with $\overline{K}$ and intersecting every member of $\boldsymbol{R}$. For each $C \in \boldsymbol{R}$ let $p_C \in L \cap C$ and denote by D_C the component of $X \setminus K$ which contains p_C. We assume that*

$$Q = \bigcup_{C \in \boldsymbol{R}} D_C$$

is not nowhere dense in X. Then, if $\hat{C}$ is an inaccessible composant of X, there exists a continuum $A \subset \hat{C}$ which separates $S^2 \setminus K$ between two points of L.

Proof. It is easy to see that $Q \setminus \hat{C}$ is not nowhere dense in X; hence there exists an open disc K' disjoint with K and such that

(1) $$\emptyset \neq K' \cap X \subset \overline{Q \setminus \hat{C}}\,.$$

Since $D_C \subset C$ and every composant is a boundary subset of X, $\boldsymbol{R}$ contains at least two composants. It follows that $L \not\subset X$ because X is irreducible between any two points from distinct composants. Let $q \in L \setminus X$. Then $q \notin \overline{K} \cup X$, hence by (1) and 2.2 there exist a continuum $A \subset \hat{C}$ and a point $p \in K' \cap X$ such that

(2) A separates $S^2 \setminus K$ between p and q.

Let M be the component of $S^2 \setminus (K \cup A)$ which contains p. Since $S^2 \setminus K$ is locally connected, M is open relative to this space. Furthermore, $M \cap K' \cap X$ is an open non-empty subset of X because this set contains p and $K \cap K' = \emptyset$. By (1) we have $M \cap K' \cap X \subset \overline{Q \setminus \hat{C}}$. This implies that $M \cap (Q \setminus \hat{C}) \neq \emptyset$; hence there exists a composant C' belonging to $\boldsymbol{R}$ which satisfies the conditions

(3) $$D_{C'} \cap M \neq \emptyset$$

and

(4) $$\hat{C} \cap C' = \emptyset\,.$$

Then $D_{C'} \subset S^2 \setminus K$ and by (4), $A \cap D_{C'} = \emptyset$. By (3) and (2) we infer that $D_{C'} \subset M$. Applying (2) we conclude that A separates $S^2 \setminus K$ between $D_{C'}$ and q. Thus A separates $S^2 \setminus K$ between two distinct points of L because $q \in L$ and $p_{C'} \in L \cap D_{C'}$. This finishes the proof.

2.4. *Let $\hat{C}$ be an inaccessible composant of X and let $\boldsymbol{R}$ be a collection of composants of X such that $\boldsymbol{R}^*$ is a second category subset of X, i.e. is not of the first category. If L is a continuum which intersects all composants belonging to $\boldsymbol{R}$ and does not contain X, then there exist an open neighborhood U of L (an open disc) and a continuum $A \subset \hat{C}$ which separates U between two distinct points of L.*

Proof. By hypothesis there exists a point $r \in X \setminus L$. Let $K_1, K_2, \ldots$ be a sequence of open discs with diameters converging to zero such that

(1) for each integer n, $r \in K_n$ and $L \cap \bar{K}_n = \emptyset$.

For each composant $C \in \boldsymbol{R}$ and for each index n let $p_C \in L \cap C$ and let $D_{C,n}$ denote the component of $X \setminus K_n$ which contains p_C.

(2) $$\boldsymbol{R}^* \setminus C(r) \subset \bigcup_n \bigcup_{C \in \boldsymbol{R}} D_{C,n}.$$

Let $x \in \boldsymbol{R}^* \setminus C(r)$ and let C be the member of $\boldsymbol{R}$ which contains x. Let $D \subset C$ be a continuum joining x and p_C. Since $C \cap C(r) = \emptyset$, we have $r \notin D$. There exists an integer n such that K_n is disjoint with D. Hence $x \in D \subset D_{C,n}$, which proves (2).

By 2.1 the set $C(r)$ is of the first category; hence $\boldsymbol{R}^* \setminus C(r)$ is still a second category subset of X. Thus (2) implies the existence of an index n_0 with the property that the set $\bigcup_{C \in \boldsymbol{R}} D_{C,n_0}$ is not nowhere dense in X. This result together with both (1) and 2.3 imply that there exists a continuum $A \subset \hat{C}$ which separates $S^2 \setminus K_{n_0}$ between two distinct points of L. Setting $U = S^2 \setminus \bar{K}_{n_0}$, we obtain by (1) the desired open neighborhood of L. This completes the proof.

3. Properties of internal composants. In this section we list only some of the most important properties of internal composants of an indecomposable continuum X; their proofs are presented in the next section.

3.1. *The union of external composants of X, i.e. $\boldsymbol{E}^*$, is a first category subset of X.*

3.2. *The set $\boldsymbol{E}^*$ is an F_σ-set in X. Consequently, the union of internal composants of X, i.e. I^*, is a $\boldsymbol{G}_\delta$-set dense in X.*

3.3. *The collection of internal composants of X is of the power of the continuum.*

3.4. *If L is a continuum which intersects both an internal composant of X and the complement of X, then there exists a non-empty closed subset A of L such that $C \cap A$ is dense in A for every composant C of X.*

3.5. *If L is an arc intersecting both an internal composant of X and the complement of X, then there exists a Cantor set A in L such that for every composant C the set $C \cap A$ is dense in A.*

3.6. *Let C be an internal composant. If L is a continuum which intersects both C and the complement of X and does not contain X, then there exist an open neighbourhood U of L and a continuum $A \subset C$ which separates U between two distinct points of L.*

3.7. *Let C be an internal composant of X. If $L_1, L_2, \ldots$ is a sequence of continua converging topologically to a continuum L and if for each integer n the set L_n is disjoint with C, then L is also disjoint with C provided that $X \not\subset L \not\subset X$.*

3.8. *Let C be an internal composant. Let V be an open set (in S^2) intersecting X and let D be a compact set lying in the complement of X, i.e. $D \subset S^2 \setminus X$. Consider the set F consisting of all those points x of X for which there exists a continuum lying in $S^2 \setminus V$, containing x, intersecting D and disjoint with C. Then F is a closed subset of X.*

3.9. *Each internal composant is inaccessible.*

4. Proofs of the properties from section 3.

Proof of 3.1. See [3], Main Theorem.

Proof of 3.4. This follows from the Second Theorem in [3].

Proof of 3.5. Let A be as in 3.4. Then A is 0-dimensional for otherwise there would exist an arc M in A. Then $\mathrm{Int}_A M$ would be non-empty and M would be a subset of a composant of X, which is impossible. Since A is compact, to prove that A is a Cantor set it remains to show only that A is dense in itself. Let $x \in A$. We have to prove that $x \in \overline{A \setminus \{x\}}$. But $A \subset X$, hence $x \in C(x)$. If C is a composant of X distinct from $C(x)$, then we have $x \in A = \overline{C \cap A} \subset \overline{(X \setminus C(x)) \cap A} \subset \overline{A \setminus \{x\}}$, which completes the proof.

Proof of 3.9. This fact is an immediate consequence of the definition of internal composant.

Proof of 3.6. According to 3.9 the composant C is inaccessible. By the assumptions on C and L the set L intersects all composants of X. Hence 3.6 follows at once from 2.4, where we substitute for $\boldsymbol{R}$ the collection of all composants of X.

Proof of 3.7. Suppose L intersects C. Then L intersects both C and the complement of X and does not contain X. By 3.6 there exist an open neighbourhood U of L and a continuum $A \subset C$ which separates U between two distinct points of L, say p and q. Hence we can write

(1) $$U \setminus A = G \cup H, \quad p \in G \text{ and } q \in H,$$

where G and H are open and disjoint. Then $\boldsymbol{B}(U, G, H)$ (¹) is an open subset of 2^{S^2} containing L and $\mathrm{Lim} L_n = L$, hence for some index n we have $L_n \in \boldsymbol{B}(U, G, H)$. This means that $L_n \subset U$ and $L_n \cap G \neq \emptyset \neq L_n \cap H$. By (1) the set L_n intersects A and therefore C, contrary to our assumptions on L_n. This finishes the proof.

(¹) Symbol 2^X denotes the set of all non-empty closed subsets of X with the exponential topology. Then $B(G_0, G_1, \ldots, G_n)$ denotes the set of all elements of 2^X which are contained in G_0 and intersect all G_j, for $1 \leqslant j \leqslant n$ (see [6], p. 45).

Proof of 3.8. Let $x_1, x_2, \dots$ be a sequence of points belonging to F and converging to x. We have to show that $x \in F$. For each integer n there exists a continuum L_n such that

(1) $$x_n \in L_n \subset S^2 \setminus V,$$

(2) $$L_n \cap D \neq \emptyset$$

and

(3) $$L_n \cap C = \emptyset.$$

Without loss of generality we may assume that $\{L_n\}$ is a convergent sequence with the limit continuum L (for otherwise we may choose a convergent subsequence of $\{L_n\}$). By (1), $x \in L \subset S^2 \setminus V$ and by (2) the set L intersects D. Hence $X \not\subset L \not\subset X$. Thus by (3) and 3.7 we infer that L is disjoint with C. But these facts together imply that x is a point of F, which completes the proof.

Proof of 3.2. We have to show that E^* is an F_σ-set. By 3.1 there exists an internal composant in X, say C_0. Let $S^2 \setminus X = D_1 \cup D_2 \cup \dots$ be the union of compact subsets of the sphere. There exists a sequence $K_1, K_2, \dots$ of open discs intersecting X with diameters converging to zero such that the sets $U_n = K_n \cap X \neq \emptyset$ constitute a base for X. For each pair of integers n, i let F_{ni} be the subset of X consisting of all those points $x \in X$ for which there exists a continuum lying in $S^2 \setminus K_n$ which contains x, intersects D_i and is disjoint with C_0. By 3.8, F_{ni} is closed in X and it is obvious that F_{ni} is a subset of E^*. So, to prove the whole theorem, we need only to show that E^* is contained in $\bigcup_{n,i} F_{ni}$, because this will imply that E^* is an F_σ-set. Let $x \in E^*$ and let C be the external composant which contains x. Hence there exists a continuum M which intersects both C and the complement of X and not all composants of X are intersected by it. It follows that M is disjoint will C_0. There exists an index i such that $M \cap D_i \neq \emptyset$. Let N be a subcontinuum of C containing x and intersecting M. Put $L = M \cup N$. The set $X \setminus L$ is non-empty because it contains C_0. Hence there exists an index n such that L is a subset of $S^2 \setminus K_n$. Moreover, L contains x and intersects D_i. Since L is disjoint with C_0, these results imply that x belongs to F_{ni}, which finishes the proof.

Proof of 3.3. In [1] Cook proved that if M is a G_δ-set, in an indecomposable continuum Y, which contains a composant of Y, then M contains $\mathfrak{c}$ composants of Y. Hence by 3.1 and 3.2 the set I contains at least $\mathfrak{c}$ composants, and therefore card $I = \mathfrak{c}$ because every indecomposable non-degenerate continuum contains exactly $\mathfrak{c}$ distinct composants.

5. An example. In this final section we shall show that all composants of the simplest indecomposable plane continuum X_0, defined by Knaster (see [6], p. 204), except the accessible one, are internal. Let us recall the construction of X_0. This continuum consists of

(i) all semi-circles with ordinates $\geqslant 0$ and centre $(1/2, 0)$ which pass through every point of the Cantor set $\mathfrak{C}$.

(ii) all semi-circles with ordinates $\leqslant 0$, which have for $n \geqslant 1$ the centre at $(5/2 \cdot 3^n, 0)$ and pass through each point of the set $\mathfrak{C}$ lying in the interval $2/3^n \leqslant x \leqslant 1/3^{n-1}$.

Let $p = (0, 0)$ and, for each integer $n \geqslant 1$, let $p_n = (1/3^{n-1}, 0)$. By C_0 we denote the composant of X_0 which contains p. Hence C_0 is accessible and therefore it is an external composants of X_0. For each $n \geqslant 1$ let L_n be the arc in C_0 which joins p and p_n, and let M_n be the semi-circle with ordinates $\leqslant 0$ and centre at $(1/2 \cdot 3^{n-1}, 0)$. Clearly, each M_n contains p. Denote by D_n the closed disc bounded by the simple closed curve $L_n \cup M_n$. Hence D_n's constitute a decreasing sequence of discs such that

(1) $$X_0 = \bigcap_n D_n$$

(see Figure 1).

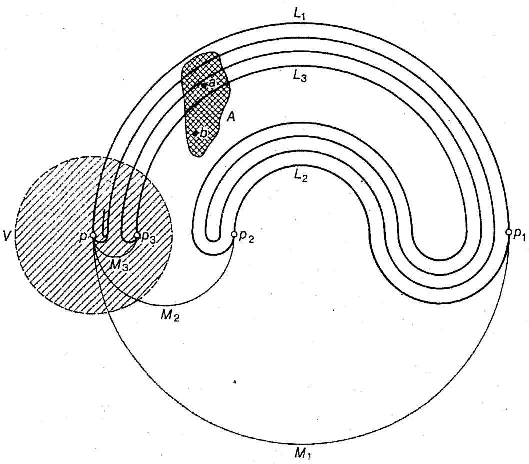

Fig. 1

We now are ready to prove the mentioned theorem.

THEOREM. *Every composant C_1 of X_0 distinct from C_0 is internal.*

Proof. Let A be a plane continuum which intersects both C_1 and the complement of X_0. Let $a \in A \cap C_1$. We have to show that A intersects all composants. Consider two following cases.

(I) For each open neighbourhood V of p which does not contain a the component of $A \setminus V$ which contains a is contained in X_0.

Since a and p lie in distinct composants, then X_0 is irreducible between them. Let $V_1, V_2, \ldots$ be open neighbourhoods of p with diameters converging to zero such that no V_n contains a, and let A_n be the component of $A \setminus V_n$ which contains a. Since A intersects V_n (otherwise A would be a subset of X_0), A_n intersects the boundary of V_n. It follows that $B = \overline{\bigcup_n A_n}$ is a subcontinuum of X_0 containing both a and p. Hence $B = X_0$. On the other hand B is a subset of A and therefore $X_0 \subset A$, which proves our theorem in the first case.

(II) There exists an open neighbourhood V of p which does not contain a and such that the component of $A \setminus V$ which contains a intersects the complement of X_0.

Without loss of generality we may assume that V is a circular open disc with centre at p and A is a continuum disjoint with V, i.e.

$$A \cap V = \emptyset . \tag{2}$$

Let $b \in A \setminus X_0$. Since diameters of M_n's converge to zero, then by (1), there exists a sufficiently large index n_0 such that

$$b \notin D_{n_0} \tag{3}$$

and

$$M_{n_0} \subset V . \tag{4}$$

Put $D = D_{n_0}$, $L = L_{n_0}$ and $M = M_{n_0}$. Since $L \subset C_0$, $a \in C_1$, and $M \cap X_0 \subset L$, then

$$a \in \operatorname{Int} D , \tag{5}$$

hence by (3) and (5) we obtain

(6) $L \cup M$ separates the plane between a and b.

Let C be an arbitrary composant of X_0. By the construction of X_0 and by (4) it is easy to show that there exists a homotopy

$$F\colon (M \cup L) \times I \to E^2 \setminus \{a, b\}$$

satisfying the conditions

$$F(x, 0) = x , \quad \text{for every } x \in M \cup L , \tag{7}$$

$$M_1' = F(M \times \{1\}) \subset V \tag{8}$$

and

$$L_1' = F(L \times \{1\}) \subset C . \tag{9}$$

By (6) and (7) we infer that $L_1' \cup M_1'$ separates the plane between a and b (see [6], p. 473). Since A is a connected set joining a and b, A intersects $L_1' \cup M_1'$. But, by (2) and (8), A is disjoint with M_1'. Hence A intersects L_1' and therefore by (9) the set A intersects C, which completes the proof of the theorem.

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