Form a direct system taking \( \mathcal{K}/S \) as the set of indices with the semilattice partial order \( \preceq \); take classes \([a]\) as components and for \( x \in [a], [a] \preceq [b], [a] \neq [b] \) put \( h_{[a]}^{a} = b, h_{[a]}^{b} = x \). Obviously \( L \) is a well-defined direct system with the in.h. property. If \( x \in [a], y \in [b] \), then \( x \cdot y \in [a \cdot b] \). Using (10) we have:

\[
x \cdot y = x^2 \cdot y^2 = a^2 \cdot b^2 = (a \cdot b)^2 = (a \cdot b)^2 = h_{[a]}^{a}(a \cdot b) \cdot h_{[b]}^{b}(y) 
\]

Thus \( \mathcal{K} \) is the sum of direct system \( L \). However, no identity of the form \( f(x, y) = x \) holds in \( K_{\mathcal{K}} \), because any binary term is equivalent to \( x \cdot y \) and \( x \cdot y \) is commutative. Hence if we had \( x \cdot y = x \), then we would have \( x \cdot y = y \) and \( x = y \), which is a contradiction because \( K_{\mathcal{K}} \) is not trivial (e.g. an algebra \((a, b); a \cdot y \) where \( a \cdot y = b \) belongs to \( \mathcal{K} \)).

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Non-planar embeddings of planar sets in \( \mathbb{R}^3 \)

by

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Abstract. Let \( X \) be a compact, 1-dimensional set in the plane, \( \mathbb{R}^2 \). Now let \( X \) be a subset of \( \mathbb{R}^3 \) homeomorphic to \( X \). It is shown that for each tame arc \( A \) in \( \mathbb{R}^3 \) with \( X \cap \mathbb{R}^2 \) and for each \( \varepsilon > 0 \), there is a homeomorphism \( h \) of \( \mathbb{R}^3 \) onto \( \mathbb{R}^3 \) that moves each point less than \( \varepsilon \), that is the identity off an \( \varepsilon \)-neighborhood of \( X \cap A \), and is such that \( X \cap h(A) = \emptyset \). An analog is also proven in the case in which \( X \) is 2-dimensional.

1. Introduction. A standard technique in studying geometric properties of the embedding of a compactum \( X \) in Euclidean \( n \)-space, \( \mathbb{R}^n \), requires that (for certain values of \( k \)) it should be possible to move \( k \)-simplices off \( X \) by a small homeomorphism of \( \mathbb{R}^n \). (See, e.g., [7] and [13].) If \( n \) is large, the conditions under which this can be done are fairly well understood. In the "simplest" nontrivial case, however, when \( n = 3, k = 1 \), and \( X \) is 1-dimensional, our intuition does not always serve us well. No obvious dimensional or algebraic obstructions suggest themselves, yet in general it cannot be done. References [5] and [11] give embeddings of Menger's universal 1-dimensional curve in \( \mathbb{R}^3 \) that cannot be freed from certain 1-simplices by any homeomorphism of \( \mathbb{R}^3 \) onto \( \mathbb{R}^3 \) that moves only points close to the universal curve. In [6], the ambient homeomorphism exists, but it cannot be close to the identity. (Caution: the term "tangled" is used with different meanings in [11] and [6].) It is easy to see from the constructions of these 1-dimensional sets that they are not locally embeddable in \( \mathbb{R}^3 \).

We show here (Theorem 3) that if a closed set \( X \) in \( \mathbb{R}^3 \) is a countable union of at-most-1-dimensional compact sets, each of which embeds in \( \mathbb{R}^3 \), then tame arcs can be freed from \( X \) by a homeomorphism of \( \mathbb{R}^3 \) that is close to the identity. On the other hand, if \( X \subset \mathbb{R}^3 \) is compact and \( X \) embeds in \( \mathbb{R}^3 \) (but perhaps \( \dim X = 2 \)), then we can still do the best possible considering the circumstances: we can push a tame arc off with a homeomorphism of \( \mathbb{R}^3 \), moving only points close to \( X \) (Lemma 4; cf. also Theorem 4). All of our theorems are stated for \( \mathbb{R}^3 \), but the proofs work for arbitrary boundary less 3-manifolds. In particular, Theorem 4 gives a new (and simpler) proof of the result [10] that a 2-cell topologically
embedded in any orientable 3-manifold-without-boundary has arbitrarily close neighborhoods that are cubes-with-handles.

Let us recall some definitions and conventions. A 3-manifold is always connected. A closed manifold is compact and without boundary. A surface is a closed 2-manifold. An n-cell is a space homeomorphic to the standard n-simplex $\Delta^n$. An n-sphere is a space homeomorphic to $\partial \Delta^{n+1}$, where $\partial$ denotes "boundary". Homeomorphism is symbolized by $\approx_n$, and $\partial = \ast$. Let $X \subset T$ and $x \in X$. We say that $Y$ is locally p-connected at $x$ if for each open set $U$ of $X$ containing $x$, there is an open subset $V$ such that $x \in V \subset U$ and each mapping of $\partial \Delta^{n+1}$ into $V \cap Y$ is homotopic to a constant in $U \cap Y$. (Cf. [8].)

2. Sierpiński curves in $E^3$. A continuum is a compact, connected Hausdorff space. If $P$ and $Q$ are disjoint closed sets in a space $X$, then a closed $F \subset X - (P \cup Q)$ disrupts $P$ from $Q$ in $X$ if some continuum in $X$ intersects each of $P$ and $Q$ but each such continuum intersects $F$. A point set has a property irreducibly if it has the property, but none of its proper closed subnets has the property.

**Lemma 1.** Let $X$ be a compact set in $E^3$ ($n \geq 1$). Let $U$ be a component of $E^3 - X$, and $J$ a continuum in $U - U$. Let $p \in J$ and let $Q \subset X$ be a closed set that intersects $J$ but does not contain $p$. Suppose that $\{X_i : i \in T\}$ is an uncountable, disjoint collection of compact sets in $X - (p \cup Q)$ such that each $X_i$ irreducibly disrupts $p$ from $Q$ in $X$ but such that no continuum in $X_i$ separates two points of $Q$ in $E^3$. Then, there are only countably many $i \in T$ for which $X_i$ contains a continuum that separates $E^3$.

**Proof.** Suppose that for uncountably many $i \in T$, $X_i$ contains a continuum $C_i$ that separates $E^3$. Then we replace our original indexing set $T$ by the collection consisting of these troublesome values of $i$, and seek a contradiction. We continue to denote our new indexing set by $T$.

Suppose that $s \in T$ ($s \neq i$) and that $U_s$, $U_t$ are components of $E^3 - C_i$, $E^3 - C_t$, respectively, such that

$$X \cap U_s = \emptyset = X \cap U_t.$$

It follows that $U_s \cap U_t = \emptyset$. Hence, for only countably many $x \in T$ can it happen that some component of $E^3 - C_i$ fails to intersect in $X$. In particular, for all but countably many $i \in T$, $C_i$ separates in $E^3$ some pair of points of $X$. Since $X$ is separable, we can in fact assume without loss of generality that for each $i \in T$, $C_i$ separates in $E^3$ the point $p \in X$ from some fixed point $r \in X$.

Let $s, t \in T$, $s \neq t$. Now one of $C_s$ and $C_t$, say $C_s$, separates the other, $C_t$, from $p \in E^3$. In fact, $C_s$ separates $C_t$ from each point of $E^3 - C_t$ in $E^3$.

Write $E^3 - C_s = E \cup F$, where $E, F$ are disjoint, nonempty open sets and $G \subset E$, $J - G \subset F$. Since $X_i$ disrupts $p$ from $Q$ in $X$ and $J \cap Q = \emptyset$, it follows that $X_i \cap J = \emptyset$. Thus, $X_i$ has a separation

$$X_i = (X_i \cap E) \cup (X_i \cap F).$$

Since $X_i \cap E \neq X_i$, there is a continuum $Z \subset X - X_i \cap E$ such that $p \in Z$ and $Z \cap \emptyset = \emptyset$. Then $C_s \cup (Z \cap F)$ is a continuum in $X$ that contains $p$, intersects $Q$, and misses $X_i$. This contradiction completes the proof.

Let $D$ be a 2-cell, and $X$ a compact set in $D$. Let $D_1, D_2, \ldots, \ldots$ be any sequence of disjoint 2-cells in $D$. Let $D = D_1 \cup D_2 \cup \ldots$. Each point of $D$ that belongs to no accessible simple closed curve is said to be inaccessible. Clearly, $Z$ is a continuum. It can be shown that $Z$ is locally connected and that $Z - X$ has dimension one. In the case that $Z$ contains no open set in $D$, we obtain the classical Sierpiński curves (see, e.g., [2], [3], and [14]).

A compact set $X$ in the interior of an n-manifold $M^n$ is definable-by-cells in $M^n$ if there is a sequence of sets $N_1, N_2, \ldots$ such that each $N_i$ is a finite, disjoint union of $n$-cells, $N_1 \subset \text{Int} N_1$, and $X = \bigcup_{i=1}^{\infty} N_i$. If $M^n$ has a piecewise-linear structure and $n \neq 4$, then each component of $X_i$ can be taken to be a polyhedron.

**Theorem 1.** Let $X$ be a Sierpiński curve embedded in $E^3$. Let $a$ be a positive number. Then, there is a compact set $O \subset X$ such that if $O$ is a definable-by-cells in $E^3$, each component of $O$ has diameter less than $a$, and each component of $X - O$ has diameter less than $a$.

**Proof.** From examining a standard model of the Sierpiński curve, we find that $X$ contains a finite, disjoint collection of arcs $A_1, \ldots, A_n$, such that: (i) each $A_k$ has diameter less than $\epsilon/9$; (ii) each component of $X - \bigcup A_k$ has diameter less than $\epsilon/9$, and (iii) each $A_k$ has its endpoints in distinct accessible simple closed curves of $X$ and otherwise $A_k$ lies in the inaccessible part of $X$ (and hence $X - A_k$ is connected).

Let $i$ and $j$ be fixed for the moment ($1 \leq i < j \leq n$). Then, for each sufficiently small $a > 0$, the $\delta$-neighborhood of $A_i$ in $E^3$ contains uncountably many disjoint, polyhedral surfaces $(S_t : t \in T)$, each of which separates $A_i$ from $A_j$ in $E^3$. (Specifically, in order to do what follows, we require that $0 < \delta < \epsilon/9$ and that the $\delta$-neighborhood of $A_i$ in $E^3$ should miss not only each $A_k$, $k \neq i$, but also should miss some arc in $X - A_i$ that joins a point of $A_i$ to a point in some accessible simple closed curve of $X$ that meets $A_i$. Note that the $\delta$-neighborhood of $A_i$ in $E^3$ has diameter less than $\epsilon/3$.) There is a compact set $X_i \subset S_i \cap X$ such that $X_i$ irreducibly
separates $A_i$ from $A_j$ in $X$. By Lemma 1, there exists $i < j$ such that each component of $X_i$ is a continuum and its homeomorphic image in $E^3$ fails to separate $E^3$. Since $X_i$ lies in the polyhedral surface $S_i$, it follows that $X_i$ is definable-by-cells in $E^3$. Hence, for each fixed $i < j$ and each sufficiently small $\delta > 0$, the $\delta$-neighborhood of $A_i$ in $E^3$ contains a compact set $X_{ij} \subset X - \bigcup A_i$ such that $X_{ij}$ separates $A_i$ from $A_j$ in $X$, diameter $X_{ij}$ is less than $\delta/3$, and $X_{ij}$ is definable-by-cells in $E^3$.

Repeated application of the result of the previous paragraph yields disjoint compact sets

$$X_{1,2}, X_{1,3}, \ldots, X_{1,k}, X_{2,k}, X_{4,1}, \ldots, X_{4,n,1},$$

whose union

$$C \subset X - \bigcup A_i$$

has the following properties: $C$ is definable-by-cells in $E^3$; each component of $C$ has diameter less than $\delta/2$, and no component of $X - C$ contains more than one $A_i$. Suppose now that $K$ is a continuum in $X - C$. If $K$ meets no $A_i$, then $K$ has diameter less than $\delta/9$, as desired. Suppose then that $K$ meets exactly one $A_i$, say $A_i$. Then $K$ is contained in the union of $A_i$ with those components of $X - \bigcup A_k$ whose closures meet $A_i$. Hence, $K$ has diameter less than $\delta/3$. Further, by the above properties of $C$, $K$ meets no more than one $A_i$. Thus, each component of $X - C$ has diameter less than $\delta/3$, and the proof is complete.

**Theorem 2.** Let $X$ be a Sierpiński curve embedded in $E^3$. Let $A$ be a tame arc in $E^3$ with $X \cap \partial A = \emptyset$. Let $\alpha$ be a positive number. Then there is a homeomorphism $h$ of $E^3$ onto $E^3$ such that $h$ moves each point less than $\alpha$, $h$ reduces to the identity outside the $\varepsilon$-neighborhood of $U$ of $A \cap X$ in $E^3$, and $X \cap h(A) = \emptyset$. (That is, $X$ has the "strong arc-pushing property".)

**Proof.** We can assume that $A$ is a straight line interval in the $z$-axis (which we picture as being vertical) in $E^3$; with $\partial A \subset E^3 - U$. Further, from the fact that $X$ is nowhere-dense in $E^3$, we see that for some homeomorphism $\sigma$ of $E^3$ onto $E^3$ that moves each point less than $\varepsilon/2$ and reduces to the identity on $E^3 - U$, $\sigma(X) \cap A$ is $0$-dimensional. If we can then find a homeomorphism $g$ of $E^3$ onto $E^3$ that moves each point less than $\varepsilon/2^2$ reduces to the identity on $E^3 - U$, and satisfies $g(X) \cap g(A) = \emptyset$, then we can take $h = \sigma^2 \circ g$. Thus, it clearly suffices to take up our original problem with the added hypothesis that $A \cap X$ is $0$-dimensional. This we do, with no change in notation.

Let $C_1, \ldots, C_n$ be a disjoint collection of solid cylinders in $U$ whose interiors cover $A \cap X$, such that each solid cylinder has the $z$-axis as an axis, each has diameter less than $\varepsilon/2$, and each intersects $A \cap X$ but has its ends in $E^3 - X$. Let $E_i$ denote the union of the two ends of $C_i$. Let

$$E_i \cap \sigma(A) = \{a_i, b_i\}.$$

There is a $\delta$ (with $0 < \delta < \varepsilon/2$) such that $K$ is a continuum of diameter less than $3\delta$ in some $E_i \cap E_j - E_i$, then $K$ fails to separate $a_i$ from $b_i$ in $E_i$. We also require that $\delta$ be less than the distance from $X$ to

$$(A - \bigcup \text{Int} C_i) \cup \bigcup E_i.$$

By Theorem 1, there is a disjoint collection $B_1, \ldots, B_n$ of polyhedral 3-cells of diameter less than $\delta$ in $E^3$, such that each $B_i$ intersects $X$, and each component of $X - \bigcup \text{Int} B_i$ has diameter less than $\delta$. Some homeomorphism $p$ of $E^3$ onto $E^3$ causes each $B_i \cap p(\partial C_i)$ to be empty, and is the identity outside a neighborhood of

$$\bigcup \{B_i, E_i \cap \partial C_i = \emptyset\}$$

that has components of diameter less than $\delta$ and misses

$$\bigcup \{E_i : E_i \cap \partial C_i = \emptyset\}.$$

(This neighborhood lies in $U$.) Note that each component of each $X \cap X \cap p(\partial C_i)$ has diameter less than $\delta$. Further, each component of each $p^{-1}(X) \cap \partial C_i$ misses $E_i$ and has diameter less than $3\delta$.

By the result of the previous paragraph, there is an arc

$$A_i \subset (E_i \cap p^{-1}(X))$$

from $a_i$ to $b_i$, for each $i$. Let $q$ be a homeomorphism of $E^3$ onto $E^3$ which moves each point less than $\varepsilon/2$, is the identity on $E^3 - U$ and on $A_i - \bigcup \text{Int} C_i$, and which results in

$$q(A \cap C_i) = A_i,$$

for each $i$.

Thus, $p^{-1}(X) \cap q(A) = \emptyset$. Finally, the desired homeomorphism is $h = p \circ q$.

3. Extensions and generalizations. We state the following result without proof. It involves straightforward changes (mainly the introduction of locally finite covers) in the usual proof (see [9]) that a compact, $n$-dimensional metric space embeds in $E^{n+1}$. The chief point to note is that the mapping $f$ given in the Lemma can be replaced by an approximation $g$ that extends $f|U$ and satisfies $g|U \cap q(X - U) = \emptyset$ (see Theorem 2 of [8]). The inability to accomplish this step is the only sticking point in the attempt to extend the result to higher dimensions. In fact, if $X$ is the disjoint union of a 2-cell $D$ and a Cantor set $C$, then $X$ is 2-dimensional, and for each $n > 3$ there is a mapping $f : X \to E^n$ such that $f|U$ is an embedding, $f|D$ is an embedding, and $f|\partial D$ represents a non-trivial element
of \pi_1(E^n - f(C)). Clearly, there is no embedding of X that closely approximates f and extends f(C). See [1] and [4] for the existence of the wild Cantor set f(C). (I am indebted to Michael P. Starbird for pointing out this example.)

**Lemma 2.** Let f be a mapping of a compact metric space X into R^n such that for some closed C \subset X, f(C) is an embedding. Suppose that X - C is one-dimensional, dim C \leq 2, and that R^n - f(C) is locally 0-connected at each point of f(C \cap X - C). Then, for each \epsilon > 0, there is an embedding F: X \rightarrow R^n such that F extends f(\epsilon) and d(f(x), F(x)) < \epsilon for each x \in X.

**Remark.** The hypotheses stated in the second sentence of Lemma 2 will automatically hold if (1) X is 1-dimensional, or (2) if X - C is 1-dimensional and X embeds in R^n. That (1) suffices, follows from the fact that no compact, 1-dimensional space can separate an open, connected set in R^n. That (2) suffices, follows from the next lemma.

**Lemma 3.** Let Z be a closed set in R^n, and let z \in Z be a limit point of \partial Z - 2. Suppose that h: Z \rightarrow R^n is an embedding with h(z) = x \in Z and h(z) = x \in Z. Then R^n - x is locally 0-connected at h(x) = x.

**Proof.** Let U \subset R^n be the interior of a closed 3-cell B, where x \in U. Let D \subset R^n be a (round) closed 2-cell such that z \in Int D, (R^n - Z) \cap \partial D \neq \emptyset, and D \cap Z \subset h^{-1}(U). (Note that each component of Z \cap \partial D is a point or an arc.) Let V \subset U be an open, connected set in R^n with x \in V and h^{-1}(V \cap Z) \subset Int D.

We will claim that each pair of points p, q \in V - X can be joined by a path in U - X. If this is not so, then K = X \cap B separates p from q in B. But K is the union of K = K \cap h^{-1}(Z - Int D) and K = K \cap h(\partial Z).

Let P be a path in V from p to q. Note that P \cap K = \emptyset, since K \subset V. Since K \subset U, there is also a path Q from p to q in U - X. (No compact set that embeds in R^n can separate U \supset \emptyset. Note that K \cap K = h(\partial Z \cap \partial D) is a compact set in U, and each component of K \cap K is a point or an arc. Hence, P \cap Q represents a cycle in U that bounds in U - K \cap K. By the Alexander Addition Theorem ([15], page 60), there is a path from p to q in U - K. The proof is complete.

**Theorem 3.** Let X be a closed set in R^n. Suppose that X can be expressed as the union of a countable number of compact sets of dimension at most one, each of which can be embedded in R^n. Then X has the strong arc-pushing property.

**Proof.** We consider only the case where X is compact, 1-dimensional, and embeds in R^n. Once this has been done, we will leave to the reader the easy argument that the strong arc-pushing property is preserved under countable unions. Let A be a tame arc in R^n with x \cap \partial A = \emptyset. Let \epsilon be a positive number, and U be the \epsilon-neighborhood of A \cap X in R^n.

For some compact, nowhere-dense set Z \subset R^n, there is a homeomorphism f of Y onto X. Let V be an open set in R^n such that X \subset V and \partial A \subset U. Then for some finite disjoint collection Z_1, \ldots, Z_m of Sierpiński curves in E^n, each point of Y belongs to the inaccessible part of some Z_i, and there is an extension F of f to a continuous mapping of Z = \bigcup_i Z_i into V. By Lemma 2, F can in fact be chosen to be an embedding of Z into V. Let \delta (0 < \delta < \epsilon) be a number so small that the \delta-neighborhood in R^n of A \cap F(Z) is contained in U. By Theorem 3, there is a homeomorphism \lambda of R^n onto R^n such that \lambda moves each point less than \delta, \lambda reduces to the identity outside the \delta-neighborhood in R^n of A \cap F(Z), and F(Z) \cap \lambda(A) = \emptyset. Then \lambda is the required homeomorphism.

**Corollary 3.1.** Suppose X satisfies the hypotheses of Theorem 3.

Let M be a 2-manifold embedded as a closed subset of R^n. Then, for each \epsilon > 0, there is a homeomorphism h of R^n onto R^n that moves each point less than \epsilon, is the identity off a preassigned neighborhood of M \times X, and is such that X \times h(M) is zero-dimensional.

**Proof.** It follows from [2] that M has a sequence of triangulations T_1, T_2, ..., whose meshes converge to zero, and each of whose one-skeleta is tame. The result of Theorem 3 is used to construct a suitable sequence of homeomorphisms h_1, h_2, ..., such that the limit of the h_i's is a homeomorphism h of R^n onto R^n with each h_i((\partial T_i)\cap Z) missing X. Details are left to the reader.

**Lemma 4.** Let Z \subset R^n be compact, and let g: Z \rightarrow R^n be an embedding with g(Z) = X. Let A be a tame arc in R^n with x \cap \partial A = \emptyset. Then, for each \epsilon > 0, there is a homeomorphism h of R^n onto R^n such that h is the identity off the \epsilon-neighborhood U of Z, and X \times h(A) = \emptyset. (That is, X has the "arc-pushing property".)

**Proof.** We only sketch the proof, since many of the ideas involved have already been introduced. Consider first the case that Z is a generalized Sierpiński curve. Let V be the interior of Z relative to R^n, and let B = Z - V. Since g(B) is an at-most-1-dimensional, compact set that embeds in R^n, Theorem 3 allows us to assume without loss of generality that A \cap g(B) = \emptyset.

Let T be a nice tubular neighborhood of A that misses g(B), and whose ends miss X. We assume that T is so thin that it can be partitioned into a finite number of smaller tubes (3-cells) C_1, \ldots, C_n such that each C_i \cap C_{i+1} is a 2-cell in C_i \cap \partial C_{i+1}, C_i \cap C_j = \emptyset if |i - j| > 1, and each C_i
either lies in \( U \) or misses \( X \). We assume also that consecutive \( G_i \)'s do not both intersect \( X \). Let \( K \) be the union of all the \( G_i \)'s that miss \( X \). Let \( L \) and \( R \) be disjoint arcs in \( \partial T \), each joining the ends of \( T \). Then there are disjoint arcs \( P, Q \) in \( X - A \) such that \( X \cap L \cup P \cap Q \cap R \), each of \( P \) and \( Q \) meets the accessible part of \( X \) precisely in its endpoints, which are on the same accessible simple closed curve of \( X \), and otherwise each of \( P \) and \( Q \) misses the accessible part of \( X \).

Using Lemma 1 as we did in the proof of Theorem 1, some compact set
\[
G \subset X - (A \cup P \cup Q)
\]
is definable by cells in \( B^p \), and separates \( P \) from \( Q \) in \( X \). Hence, there is a finite, disjoint collection \( B_1, \ldots, B_k \) of polyhedral 3-cells in \( U - (A \cup K \cup P \cup Q \cup L \cup R) \) such that each \( B_i \) meets \( X \) and \( X \cap B_i \) separate \( P \) from \( Q \) in \( X \). Some homeomorphism \( p \) of \( B^p \) onto \( B^p \) is then identity on
\[
A \cap K \cap L \cap P \cap Q \cap R \cap U \quad (B^p - U)
\]
and causes each \( B_i \) in \( p(\partial T) \) to be empty. Thus, no continuum in \( X \cap K \cap L \cap P \cap Q \cap R \cap U \) forms \( A \) that misses \( X \) and joins the ends of \( \partial A \), in the union of \( A \cap K \) with the boundaries of those \( p(G_i) \) whose corresponding \( G_i \)'s meet \( X \). Finally, some homeomorphism of \( B^p \) onto \( B^p \) throws \( A \) onto \( A' \) as required.

For the general case, use Lemmas 2 and 3 to extend \( p \) to an embedding of the union of a finite, disjoint collection of generalized Sierpiński curves into \( U \). Then apply the results of the special case to each of these.

We state the next theorem in what we consider a convenient form.

More general hypotheses are possible. For example, one need only assume that \( X \) has an upper semicontinuous decomposition into a countable number of compact sets, each embeddable in \( B^p \).

**Theorem 4.** Let \( X \subset B^p \). Suppose that \( X \) has only countably many components, each embeddable in \( B^p \). Then \( X \) has the arc-pushing property.

**Proof.** There is a transfinite, properly nested “sequence” of compact sets
\[
X = X_0 \supset X_1 \supset \cdots \supset X_n \supset \cdots
\]
such that \( X_{\alpha+1} \) is obtained from \( X_{\alpha} \) by removing the components of \( X_{\alpha} \) that open in \( X_{\beta} \) and for each limit ordinal \( \alpha \),
\[
X_{\alpha} = \bigcap \{X_{\beta} : \beta < \alpha \}.
\]

It follows that for some countable ordinal \( \alpha \), \( X_{\alpha} \) has a finite (nonzero) number of components. We call \( \alpha \) the degree of \( X \), and prove the theorem by induction on \( \alpha \).

If \( \alpha = 1 \), the result is immediate from Lemma 4. Suppose then that the theorem is proven for all compact \( X \subset B^p \) that satisfy our hypotheses and have degree less than \( \alpha \). Let \( A \subset B^p \) satisfy \( \partial A = \emptyset \), and let \( \varepsilon \) be a given positive number. Then by Lemma 4, there is a homeomorphism \( h \) of \( B^p \) onto \( B^p \) that is the identity off the \( \varepsilon \)-neighborhood \( U \) of \( X \), and such that \( X_\alpha \cap h(A) = \emptyset \). Let \( W \) be a neighborhood of \( X_\alpha \) in \( B^p \) that is both open and closed in \( X_\alpha \) is a union of components of \( X_\alpha \), and misses \( h(A) \). Then \( X_\alpha \cap h(A) = \emptyset \) satisfies our hypotheses, and has degree less than \( \alpha \). Hence there is a homeomorphism \( g \) of \( B^p \) onto \( B^p \) such that \( X_\alpha \cap h(A) = \emptyset \) and \( g \) is the identity off a small neighborhood \( V \) of \( X_\alpha \). Then \( X_\alpha \cap h(A) = \emptyset \) and \( gh \) is the identity on \( B^p - U \), as desired.

**Question.** Is there a version of Theorem 4 when \( X \) has uncountably many components? For example, what if \( X \) is the product of a Cantor set and a Sierpiński curve?

**References**


On internal component of indecomposable plane continua

by

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Abstract. In [3] the author introduced the concept of internal component and proved that the union of internal components of an indecomposable continuum X is a second category subset of X. In the present paper we obtain some new results on internal components. The main theorem states that the union of internal components is a Gδ-subset of X. If X denotes the simplest indecomposable continuum defined by Knaster, then all components of X, except one, are internal.

1. Introduction. Throughout this paper all sets are assumed to be subsets of the sphere S². Let X be a continuum lying in this sphere. The union of all proper subcontinua of X containing a fixed point x ∈ X is denoted by C(x) and is called a component of X. If X is an indecomposable continuum, then the collection of all components of X constitute a partition of X into connected dense and pairwise disjoint sets, with c denoting the cardinal of the continuum. At first sight there is no difference between two distinct components of X. However, as we shall see in the sequel, one can distinguish several important classes of components.

The process of distinguishing components in an indecomposable plane continuum was initiated by S. Mazurkiewicz in 1929, when he showed that the union of accessible components of an indecomposable plane continuum X is a first category subset of X (answering a question of Kuratowski). Let us recall that a component C of X is said to be accessible provided there exist a point p ∈ C and a non-degenerate continuum L such that L ∩ X = {p}. Otherwise it is inaccessible. The above theorem found some applications in plane topology (see for example [2] and [3]).

In the same year K. Kuratowski [7] defined a class of components larger than that of accessible ones. Namely a component C of X is called a K-component [5] provided that there exist a continuum D ∩ C and a continuum L such that L ∩ X = D and D ∩ X = ∅. K. Kuratowski proved an analogue of the Mazurkiewicz theorem for the class of K-components. Precisely, the union of K-components is a first category subset of X. Up to now, these results have been the best known. In a conversation with the author, A. Lelek raised the question whether or not there

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