Form a direct system taking $\mathfrak{A}/S$ as the set of indices with the semilattice partial order $\leqslant$; take clases $[a]$ as components and for $x \in [a]$, $[a] \leqslant [b]$, $[a] \neq [b]$ put $h_{[a]}^{[b]}(x) = b^2$, $h_{[a]}^{[a]}(x) = x$. Obviously $\boldsymbol{L}$ is a well-defined direct system with the l.u.b. property. If $x \in [a]$, $y \in [b]$, then $x \cdot y \in [a \cdot b]$. Using (15) we have:

$$x \cdot y = x^2 \cdot y^2 = a^2 \cdot b^2 = (a \cdot b)^2 = (a \cdot b)^2 \cdot (a \cdot b)^2 = h_{[a]}^{[a \cdot b]}(x) \cdot h_{[b]}^{[a \cdot b]}(y) .$$

Thus $\mathfrak{A}$ is the sum of direct system $\boldsymbol{L}$. However, no identity of the form $f(x, y) = x$ holds in $\boldsymbol{K}_E$, because any binary term is equivalent to $x \cdot y$ and $x \cdot y$ is commutative. Hence if we had $x \cdot y = x$, then we would have $x \cdot y = y$ and $x = y$, which is a contradiction because $\boldsymbol{K}_E$ is not trivial (e.g. an algebra $(\{a, b\};\ x \cdot y)$ where $x \cdot y \equiv b$) belongs to $\boldsymbol{K}$.

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Non-planar embeddings of planar sets in E^3

by

D. R. McMillan, Jr. (Madison, Wis.)

Abstract. Let Z be a compact, 1-dimensional set in the plane, E^2. Now let X be a subset of E^3 homeomorphic to Z. It is shown that for each tame arc A in E^3 with $X \cap \partial A = \emptyset$, and for each $\varepsilon > 0$, there is a homeomorphism h of E^3 onto E^3 that moves each point less than ε, that is the identity off an ε-neighborhood of $X \cap A$, and is such that $X \cap h(A) = \emptyset$. An analog is also proven in the case in which Z is 2-dimensional.

1. Introduction. A standard technique in studying geometric properties of the embedding of a compactum X in Euclidean n-space, E^n, requires that (for certain values of k) it should be possible to move a k-simplex off X by a small homeomorphism of E^n. (See, e.g., [7] and [13].) If n is large, the conditions under which this can be done are fairly well understood. In the "simplest" nontrivial case, however, when $n = 3$, $k = 1$, and X is 1-dimensional, our intuition does not always serve us well. No obvious dimensional or algebraic obstructions suggest themselves, yet in general it cannot be done. References [5] and [11] give embeddings of Menger's universal 1-dimensional curve in E^3 that cannot be freed from certain 1-simplexes by any homeomorphism of E^3 onto E^3 that moves only points close to the universal curve. In [6], the ambient homeomorphism exists, but it cannot be close to the identity. (Caution: the term "tangled" is used with different meanings in [11] and [6].) It is easy to see from the constructions of these 1-dimensional sets that they are not locally embeddable in E^2.

We show here (Theorem 3) that if a closed set X in E^3 is a countable union of at-most-1-dimensional compact sets, each of which embeds in E^2, then tame arcs can be freed from X by a homeomorphism of E^3 that is close to the identity. On the other hand, if $X \subset E^3$ is compact and X embeds in E^2 (but perhaps $\dim X = 2$), then we can still do the best possible considering the circumstances: we can push a tame arc off with a homeomorphism of E^3, moving only points close to X (Lemma 4; cf. also Theorem 4). All of our theorems are stated for E^3, but the proofs work for arbitrary boundary less 3-manifolds. In particular, Theorem 4 gives a new (and simpler) proof of the result [10] that a 2-cell topologically embedded in any orientable 3-manifold-without-boundary has arbitrarily close neighborhoods that are cubes-with-handles.

Let us recall some definitions and conventions. A "manifold" is always connected. A *closed* manifold is compact and without boundary. A *surface* is a closed 2-manifold. An *n-cell* is a space homeomorphic to the standard n-simplex Δ^n. An *n-sphere* is a space homeomorphic to $\partial\Delta^{n+1}$, where "∂" denotes "boundary". Homeomorphism is symbolized by "$\approx$", and $\Delta^1 = I$. Let $Y \subset X$, and $x \in X$. We say that Y is *locally p-connected at x* if for each open set U of X containing x there is an open set V such that $x \in V \subset U$ and each mapping of $\partial\Delta^{p+1}$ into $V \cap Y$ is homotopic to a constant in $U \cap Y$. (Cf. [8].)

2. Sierpiński curves in E^3. A *continuum* is a compact, connected Hausdorff space. If P and Q are disjoint closed sets in a space X, then a closed $F \subset X-(P \cup Q)$ *disrupts* P *from* Q *in* X if some continuum in X intersects each of P and Q but each such continuum intersects F. A point set has a property *irreducibly* if it has the property, but none of its proper closed subsets has the property.

Lemma 1. *Let X be a compact set in E^n $(n > 1)$. Let U be a component of $E^n - X$, and J a continuum in $\overline{U} - U$. Let $p \in J$ and let $Q \subset X$ be a closed set that intersects J but does not contain p. Suppose that $\{X_t \mid t \in T\}$ is an uncountable, disjoint collection of compact sets in $X - (\{p\} \cup Q)$ such that each X_t irreducibly disrupts p from Q in X but such that no continuum in X_t separates two points of Q in E^n. Then, there are only countably many $t \in T$ for which X_t contains a continuum that separates E^n.*

Proof. Suppose that for uncountably many $t \in T$, X_t contains a continuum C_t that separates E^n. Then we replace our original indexing set T by the collection consisting of these troublesome values of t, and seek a contradiction. We continue to denote our new indexing set by T.

Suppose that $s, t \in T$ $(s \neq t)$ and that U_s, U_t are components of $E^n - C_s$, $E^n - C_t$, respectively, such that

$$X \cap U_s = \emptyset = X \cap U_t .$$

It follows that $U_s \cap U_t = \emptyset$. Hence, for only countably many $t \in T$ can it happen that some component of $E^n - C_t$ fails to intersect X. In particular, for all but countably many $t \in T$, C_t separates in E^n some pair of points of X. Since X is separable, we can in fact assume without loss of generality that for each $t \in T$, C_t separates in E^n the point $p \in X$ from some fixed point $r \in X$.

Let $s, t \in T$, $s \neq t$. Now one of C_s and C_t, say C_s, separates the other, C_t, from p in E^n. In fact, C_s separates C_t from each point of $J - C_s$ in E^n. Write $E^n - C_s = E \cup F$, where E, F are disjoint, nonempty open sets and $C_t \subset E$, $J - C_s \subset F$. Since X_t disrupts p from Q in X and $J \cap Q \neq \emptyset$, it follows that $X_t \cap J \neq \emptyset$. Thus, X_t has as a separation

$$X_t = (X_t \cap E) \cup (X_t \cap F) .$$

Since $X_t \cap F \neq X_t$, there is a continuum $Z \subset X - X_t \cap F$ such that $p \in Z$ and $Z \cap Q \neq \emptyset$. Then $C_s \cup (Z \cap F)$ is a continuum in X that contains p, intersects Q, and misses X_t. This contradiction completes the proof.

Let D be a 2-cell, and X a compact set in $\operatorname{Int} D$. Let $D_1, D_2, \ldots$, be any sequence of disjoint 2-cells in $(\operatorname{Int} D) - X$ whose diameters converge to zero and whose union is dense in $D - X$. Then $Z = D - \bigcup \operatorname{Int} D_i$ is called a *generalized Sierpiński curve*, with *accessible simple closed curves* $\partial D, \partial D_1, \partial D_2, \ldots$ Each point of Z that belongs to no accessible simple closed curve is said to be *inaccessible*. Clearly, Z is a continuum. It can be shown that Z is locally connected and that $Z - X$ has dimension one. In the case that X contains no open set in D, we obtain the classical *Sierpiński curve* (see, e.g., [2], [3], and [14]).

A compact set X in the interior of an n-manifold M^n is *definable-by-cells in* M^n if there is a sequence $N_1, N_2, \ldots$, such that each N_i is a finite, disjoint union of n-cells, $N_{i+1} \subset \operatorname{Int} N_i$, and $X = \bigcap_{i=1}^{\infty} N_i$. If M^n has a piecewise-linear structure and $n \neq 4$, then each component of N_i can be taken to be a polyhedron.

Theorem 1. *Let X be a Sierpiński curve embedded in E^3. Let ε be a positive number. Then, there is a compact set $C \subset X$, such that C is definable-by-cells in E^3, each component of C has diameter less than ε, and each component of $X - C$ has diameter less than ε.*

Proof. From examining a standard model of the Sierpiński curve, we find that X contains a finite, disjoint collection of arcs $A_1, \ldots, A_n$, such that: (i) each A_k has diameter less than $\varepsilon/9$, (ii) each component of $X - \bigcup A_k$ has diameter less than $\varepsilon/9$, and (iii) each A_k has its endpoints in distinct accessible simple closed curves of X and otherwise A_k lies in the inaccessible part of X (and hence $X - A_k$ is connected).

Let i and j be fixed for the moment $(1 \leqslant i < j \leqslant n)$. Then, for each sufficiently small $\delta > 0$, the δ-neighborhood of A_i in E^3 contains uncountably many disjoint, polyhedral surfaces $\{S_t \mid t \in I\}$, each of which separates A_i from A_j in E^3. (Specifically, in order to do what follows, we require that $0 < \delta < \varepsilon/9$ and that the δ-neighborhood of A_i in E^3 should miss not only each A_k, $k \neq i$, but also should miss some arc in $X - A_i$ that joins a point of A_j to a point in some accessible simple closed curve of X that meets A_i. Note that the δ-neighborhood of A_i in E^3 has diameter less than $\varepsilon/3$.) There is a compact set $X_t \subset S_t \cap X$ such that X_t irreducibly separates A_i from A_j in X. By Lemma 1, there exists $t \in I$ such that each component of X_t is a continuum and its homeomorphic image in E^2 fails to separate E^2. Since X_t lies in the polyhedral surface S_t, it follows that X_t is definable-by-cells in E^3. Hence: For each fixed $i<j$ and each sufficiently small $\delta>0$, the δ-neighborhood of A_i in E^3 contains a compact set

$$X_{i,j} \subset X - \bigcup A_k$$

such that $X_{i,j}$ separates A_i from A_j in X, diameter $X_{i,j}$ is less than $\varepsilon/3$, and $X_{i,j}$ is definable-by-cells in E^3.

Repeated application of the result of the previous paragraph yields disjoint compact sets

$$X_{1,2}, X_{1,3}, \ldots, X_{1,n}, X_{2,3}, X_{2,4}, \ldots, X_{n-1,n},$$

whose union

$$C \subset X - \bigcup A_k$$

has the following properties: C is definable-by-cells in E^3; each component of C has diameter less than $\varepsilon/3$; and no component of $X-C$ contains more than one A_i. Suppose now that K is a continuum in $X-C$. If K meets no A_i, then K has diameter less than $\varepsilon/9$, as desired. Suppose then that K meets exactly one A_i, say A_1. Then K is contained in the union of A_1 with those components of $X-\bigcup A_k$ whose closures meet A_1. Hence, K has diameter less than $\varepsilon/3$. Further, by the above properties of C, K meets no more than one A_i. Thus, each component of $X-C$ has diameter less than ε, and the proof is complete.

THEOREM 2. *Let X be a Sierpiński curve embedded in E^3. Let A be a tame arc in E^3 with $X \cap \partial A = \emptyset$. Let ε be a positive number. Then there is a homeomorphism h of E^3 onto E^3 such that h moves each point less than ε, h reduces to the identity outside the ε-neighborhood U of $A \cap X$ in E^3, and $X \cap h(A) = \emptyset$. (That is, X has the "strong arc-pushing property".)*

Proof. We can assume that A is a straight line interval in the z-axis (which we picture as being vertical) in E^3, with $\partial A \subset E^3 - U$. Further, from the fact that X is nowhere-dense in E^3, we see that for some homeomorphism z of E^3 onto E^3 that moves each point less than $\varepsilon/2$ and reduces to the identity on E^3-U, $z(X)\cap A$ is 0-dimensional. If we can then find a homeomorphism g of E^3 onto E^3 that moves each point less than $\varepsilon/2$, reduces to the identity on E^3-U, and satisfies $z(X)\cap g(A)=\emptyset$, then we can take $h = z^{-1}\circ g$. Thus, it clearly suffices to take up our original problem with the added hypothesis that $A\cap X$ is 0-dimensional. This we do, with no change in notation.

Let $C_1, \ldots, C_n$ be a disjoint collection of solid cylinders in U whose interiors cover $A \cap X$, such that each solid cylinder has the z-axis as an axis, each has diameter less than $\varepsilon/2$, and each intersects $A\cap X$ but has its ends in E^3-X. Let E_i denote the union of the two ends of C_i. Let

$$\partial C_i \cap (z\text{-axis}) = \{a_i, b_i\}.$$

There is a δ $(0<\delta<\varepsilon/2)$ such that if K is a continuum of diameter less than 3δ in some $(\partial C_i)-E_i$, then K fails to separate a_i from b_i in ∂C_i. We also require that δ be less than the distance from X to

$$(A - \bigcup \operatorname{Int} C_i) \cup \bigcup E_i.$$

By Theorem 1, there is a disjoint collection $B_1, \ldots, B_k$ of polyhedral 3-cells of diameter less than δ in E^3, such that each B_j intersects X, and each component of $X-\bigcup \operatorname{Int} B_j$ has diameter less than δ. Some homeomorphism p of E^3 onto E^3 causes each $B_j \cap p(\partial C_i)$ to be empty, and is the identity outside a neighborhood of

$$\bigcup \{B_j \mid B_j \cap \bigcup \partial C_i \neq \emptyset\}$$

that has components of diameter less than δ and misses

$$\bigcup \{B_j \mid B_j \cap \bigcup \partial C_i = \emptyset\}.$$

(This neighborhood lies in U.) Note that each component of each $X \cap p(\partial C_i)$ has diameter less than δ. Further, each component of each $p^{-1}(X)\cap \partial C_i$ misses E_i and has diameter less than 3δ.

By the result of the previous paragraph, there is an arc

$$A_i \subset (\partial C_i) - p^{-1}(X)$$

from a_i to b_i, for each i. Let q be a homeomorphism of E^3 onto E^3 which moves each point less than $\varepsilon/2$, is the identity on E^3-U and on $A-\bigcup \operatorname{Int} C_i$, and which results in

$$q(A \cap C_i) = A_i, \quad \text{for each } i.$$

Thus, $p^{-1}(X)\cap q(A)=\emptyset$. Finally, the desired homeomorphism is $h = p\circ q$.

3. Extensions and generalizations. We state the following result without proof. It involves straightforward changes (mainly the introduction of locally finite covers) in the usual proof (see [9]) that a compact, n-dimensional metric space embeds in E^{2n+1}. The chief point to note is that the mapping f given in the Lemma can be replaced by an approximation g that extends $f|C$ and satisfies $g(C)\cap g(X-C)=\emptyset$ (see Theorem 2 of [8]). The inability to accomplish this step is the only sticking point in the attempt to extend the result to higher dimensions. In fact, if X is the disjoint union of a 2-cell D and a Cantor set C, then X is 2-dimensional, and for each $n \geqslant 3$ there is a mapping $f\colon X \to E^n$ such that $f|C$ is an embedding, $f|D$ is an embedding, and $f|\partial D$ represents a non-trivial element of $\pi_1(E^n - f(C))$. Clearly, there is no embedding of X that closely approximates f and extends $f|C$. See [1] and [4] for the existence of the wild Cantor set $f(C)$. (I am indebted to Michael P. Starbird for pointing out this example.)

LEMMA 2. *Let f be a mapping of a compact metric space X into E^3 such that for some closed $C \subset X$, $f|C$ is an embedding. Suppose that $X - C$ is one-dimensional, $\dim C \leqslant 2$, and that $E^3 - f(C)$ is locally 0-connected at each point of $f(C \cap \overline{X - C})$. Then, for each $\varepsilon > 0$, there is an embedding $F: X \to E^3$ such that F extends $f|C$ and $d(f(x), F(x)) < \varepsilon$ for each $x \in X$.*

Remark. The hypotheses stated in the second sentence of Lemma 2 will automatically hold if (1) X is 1-dimensional, or (2) if $X - C$ is 1-dimensional and X embeds in E^2. That (1) suffices, follows from the fact that no compact, 1-dimensional space can separate an open, connected set in E^3. That (2) suffices, follows from the next lemma.

LEMMA 3. *Let Z be a closed set in E^2, and let $z \in Z$ be a limit point of $E^2 - Z$. Suppose that $h: Z \to E^3$ is an embedding with $h(Z) = X$ closed in E^3. Then $E^3 - X$ is locally 0-connected at $h(z) = x$.*

Proof. Let $U \subset E^3$ be the interior of a closed 3-cell B, where $x \in U$. Let $D \subset E^2$ be a (round) closed 2-cell such that $z \in \operatorname{Int} D$, $(E^2 - Z) \cap \partial D \neq \emptyset$, and $D \cap Z \subset h^{-1}(U)$. (Note that each component of $Z \cap \partial D$ is a point or an arc.) Let $V \subset U$ be an open, connected set in E^3 with $x \in V$ and

$$h^{-1}(\overline{V} \cap X) \subset \operatorname{Int} D .$$

We claim that each pair of points $p, q \in V - X$ can be joined by a path in $U - X$. If this is not so, then $K = X \cap B$ separates p from q in B. But K is the union of

$$K_1 = K \cap h(Z - \operatorname{Int} D) \quad \text{and} \quad K_2 = K \cap h(Z \cap D) .$$

Let P be a path in V from p to q. Note that $P \cap K_1 = \emptyset$, since $K_1 \cap \overline{V} = \emptyset$. Since $K_2 \subset U$, there is also a path Q from p to q in $U - K_2$. (No compact set that embeds in E^2 can separate $U \approx E^3$.) Note that $K_1 \cap K_2 = h(Z \cap \partial D)$ is a compact set in U, and each component of $K_1 \cap K_2$ is a point or an arc. Hence, $P \cup Q$ represents a 1-cycle in U that bounds in $U - K_1 \cap K_2$. By the Alexander Addition Theorem ([15], page 60), there is a path from p to q in $U - K$. The proof is complete.

THEOREM 3. *Let X be a closed set in E^3. Suppose that X can be expressed as the union of a countable number of compact sets of dimension at most one, each of which can be embedded in E^2. Then X has the strong arc-pushing property.*

Proof. We consider only the case where X is compact, 1-dimensional, and embeds in E^2. Once this has been done, we will leave to the reader

the easy argument that the strong arc-pushing property is preserved under countable unions. Let A be a tame arc in E^3 with $X \cap \partial A = \emptyset$. Let ε be a positive number, and U be the ε-neighborhood of $A \cap X$ in E^3.

For some compact, nowhere-dense set $Y \subset E^2$, there is a homeomorphism f of Y onto X. Let V be an open set in E^3 such that $X \subset V$ and $V \cap A \subset U$. Then for some finite disjoint collection $Z_1, \ldots, Z_n$ of Sierpiński curves in E^2, each point of Y belongs to the inaccessible part of some Z_i, and there is an extension F of f to a continuous mapping of $Z = \bigcup Z_i$ into V. By Lemma 2, F can in fact be chosen to be an embedding of Z into V. Let δ $(0 < \delta < \varepsilon)$ be a number so small that the δ-neighborhood in E^3 of $A \cap F(Z)$ is contained in U. By Theorem 2, there is a homeomorphism h of E^3 onto E^3 such that h moves each point less than δ, h reduces to the identity outside the δ-neighborhood in E^3 of $A \cap F(Z)$, and $F(Z) \cap h(A) = \emptyset$. Then h is the required homeomorphism.

COROLLARY 3.1. *Suppose X satisfies the hypotheses of Theorem 3. Let M be a 2-manifold embedded as a closed subset of E^3. Then, for each $\varepsilon > 0$, there is a homeomorphism h of E^3 onto E^3 that moves each point less than ε, is the identity off a preassigned neighborhood of $M \cap X$, and is such that $X \cap h(M)$ is zero-dimensional.*

Proof. It follows from [2] that M has a sequence of triangulations $T_1, T_2, \ldots$, whose meshes converge to zero, and each of whose one-skeleta is tame. The result of Theorem 3 is used to construct a suitable sequence of homeomorphisms $h_1, h_2, \ldots$, such that the limit of the h_i's is a homeomorphism h of E^3 onto E^3 with each $h(|T_i^{(1)}|)$ missing X. Details are left to the reader.

LEMMA 4. *Let $Z \subset E^2$ be compact, and let $g: Z \to E^3$ be an embedding with $g(Z) = X$. Let A be a tame arc in E^3 with $X \cap \partial A = \emptyset$. Then, for each $\varepsilon > 0$, there is a homeomorphism h of E^3 onto E^3 such that h is the identity off the ε-neighborhood U of X, and $X \cap h(A) = \emptyset$. That is, X has the "arc-pushing property".)*

Proof. We only sketch the proof, since many of the ideas involved have already been introduced. Consider first the case that Z is a generalized Sierpiński curve. Let V be the interior of Z relative to E^2, and let $B = Z - V$. Since $g(B)$ is an at-most-1-dimensional, compact set that embeds in E^2, Theorem 3 allows us to assume without loss of generality that $A \cap g(B) = \emptyset$.

Let T be a nice tubular neighborhood of A that misses $g(B)$, and whose ends miss X. We assume that T is so thin that it can be partitioned into a finite number of smaller tubes (3-cells) $C_1, \ldots, C_n$ such that each $C_i \cap C_{i+1}$ is a 2-cell in $\partial C_i \cap \partial C_{i+1}$, $C_i \cap C_j = \emptyset$ if $|i-j| > 1$, and each C_i

either lies in U or misses X. We assume also that consecutive C_i's do not both intersect X. Let K be the union of all the C_i's that miss X. Let L and R be disjoint arcs in ∂T, each joining the ends of T. Then there are disjoint arcs P, Q in $X-A$ such that $X \cap L \subset P$, $X \cap R \subset Q$, each of P and Q meets the accessible part of X precisely in its endpoints, which are on the same accessible simple closed curve of X, and otherwise each of P and Q misses the accessible part of X.

Using Lemma 1 as we did in the proof of Theorem 1, some compact set

$$C \subset X-(A \cup P \cup Q)$$

is definable-by cells in E^3, and separates P from Q in X. Hence, there is a finite, disjoint collection $B_1, \ldots, B_k$ of polyhedral 3-cells in $U-(A \cup K \cup P \cup Q \cup L \cup R)$ such that each B_i meets X and $X \cap \bigcup B_i$ separates P from Q in X. Some homeomorphism p of E^3 onto E^3 is then identity on

$$A \cup K \cup P \cup Q \cup L \cup R \cup (E^3-U)$$

and causes each $B_i \cap p(\partial T)$ to be empty. Thus, no continuum in $X \cap \cap p(\partial T)$ meets both $X \cap L$ $\big(= X \cap p(L)\big)$ and $X \cap R$ $\big(= X \cap p(R)\big)$. It is now easy to find an arc A' that misses X and joins the ends of ∂A, in: the union of $A \cap K$ with the boundaries of those $p(C_i)$'s whose corresponding C_i's meet X. Finally, some homeomorphism of E^3 onto E^3 throws A onto A' as required.

For the general case, use Lemmas 2 and 3 to extend g to an embedding of the union of a finite, disjoint collection of generalized Sierpiński curves into U. Then apply the results of the special case to each of these.

We state the next theorem in what we consider a convenient form. More general hypotheses are possible. For example, one need only assume that X has an upper semicontinuous decomposition into a countable number of compact sets, each embeddable in E^2.

THEOREM 4. *Let $X \subset E^3$ be compact. Suppose that X has only countably many components, each embeddable in E^2. Then X has the arc-pushing property.*

Proof. There is a transfinite, properly nested "sequence" of compact sets

$$X = X_1 \supset X_2 \supset \ldots X_\alpha \supset \ldots,$$

such that $X_{\alpha+1}$ is obtained from X_α by removing the components of X_α that are open in X_α; and for each limit ordinal α,

$$X_\alpha = \bigcap \{X_\beta|\ \beta < \alpha\}.$$

It follows that for some countable ordinal Δ, X_Δ has a finite (nonzero) number of components. We call Δ the *degree of* X, and prove the theorem by induction on Δ.

If $\Delta = 1$, the result is immediate from Lemma 4. Suppose then that the theorem is proven for all compact $X \subset E^3$ that satisfy our hypotheses and have degree less than Δ. Let A be a tame arc in E^3 with $X \cap \partial A = \emptyset$, and let ε be a given positive number. Then by Lemma 4, there is a homeomorphism h of E^3 onto E^3 that is the identity off the ε-neighborhood U of X, and such that $X_\Delta \cap h(A) = \emptyset$. Let W be a neighborhood of X_Δ in X that is both open and closed in X, is a union of components of X, and misses $h(A)$. Then $X_0 = X-W$ satisfies our hypotheses, and has degree less than Δ. Hence there is a homeomorphism g of E^3 onto E^3 such that $X_0 \cap gh(A) = \emptyset$, and g is the identity off a small neighborhood V of X_0 $(V \subset U-W)$. Then $X \cap gh(A) = \emptyset$ and gh is the identity on E^3-U, as desired.

QUESTION. Is there a version of Theorem 4 when X has uncountably many components? For example, what if X is the product of a Cantor set and a Sierpiński curve?

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UNIVERSITY OF WISCONSIN-MADISON
Madison, Wisconsin

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On internal composants of indecomposable plane continua

by

J. Krasinkiewicz (Warszawa)

Abstract. In [3] the author introduced the concept of internal composant and proved that the union of internal composants of an indecomposable continuum X is a second category subset of X. In the present paper we obtain some new results on internal composants. The main theorem states that the union of internal composants is a G_δ-subset of X. If X denotes the simplest indecomposable continuum defined by Knaster, then all composants of X, except one, are internal.

1. Introduction. Throughout this paper all sets are assumed to be subsets of the sphere S^2. Let X be a continuum lying in this sphere. The union of all proper subcontinua of X containing a fixed point $x \in X$ is denoted by $C(x)$ and is called a *composant* of X. If X is an indecomposable continuum, then the collection of all composants of X constitute a partition of X into $\mathfrak{c}$ connected dense and pairwise disjoint sets, with $\mathfrak{c}$ denoting the cardinal of the continuum. At first sight there is no difference between two distinct composants of X. However, as we shall see in the sequel, one can distinguish several important classes of composants.

The process of distinguishing composants in an indecomposable plane continuum was initiated by S. Mazurkiewicz in 1929, when he showed that the union of accessible composants of an indecomposable plane continuum X is a first category subsets of X (answering a question of Kuratowski). Let us recall that a composant C of X is said to be *accessible* provided there exist a point $p \in C$ and a non-degenerate continuum L such that $L \cap X = \{p\}$. Otherwise it is *inaccessible*. The above theorem found some applications in plane topology (see for example [2] and [3]). In the same year K. Kuratowski [7] defined a class of composants larger than that of accessible ones. Namely a composant C of X is called a K-*composant* [5] provided that there exist a continuum $D \subset C$ and a continuum L such that $L \cap X = D$ and $L \setminus X \neq \emptyset$. K. Kuratowski proved an analogue of the Mazurkiewicz theorem for the class of K-composants. Precisely, the union of K-composants is a first category subset of X. Up to now, these results have been the best known. In a conversation with the author, A. Lelek raised the question whether or not there