On connections between the decomposition of an algebra into sums of direct systems of subalgebras

by

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Abstract. The paper considers the notion of the sum of a direct system of algebras defined in [1]. Another characterization of the partition function is given (see [1]) and it is proved that different decompositions of an algebra $\mathfrak{A}$ into the sums of direct systems form a meet semilattice which can be imbedded into a meet semilattice of congruences of $\mathfrak{A}$.

§ 0. In [1] the notion of the sum of a direct system of algebras and that of partition function were given. We shall use the results of [1]. Hence we consider here only algebras of a given type τ without nullary fundamental operations. If $\mathfrak{A}$ is an algebra, $\boldsymbol{L}$ is a direct system of subalgebras of $\mathfrak{A}$ and $\mathfrak{A} = S(\boldsymbol{L})$, $\boldsymbol{L}$ is called a *decomposition of* $\mathfrak{A}$. It was proved in [1] (Theorem 2) that there exists a 1-1 corespondence between the decompositions and the partition functions of $\mathfrak{A}$.

In this paper we describe the partition function of an algebra by identities (§ 1, Theorem 1).

In § 2 we consider connections between different decompositions of an algebra $\mathfrak{A}$; in particular, we look for a decomposition with the smallest components. The main result is contain in Theorem 2.

The identity $\varphi = \psi$ is called *regular* (see [3]) if the sets of variables on both sides are the same. Let $\boldsymbol{K}$ be the equational class of algebras defined by a set E of equations and $\boldsymbol{K}_{R(E)}$ — the equational class defined by the set $R(E)$, where $R(E)$ consists of all regular consequences of E. It was proved in [3] that if for some term $x \cdot y$ the equality $x \cdot y = x$ is a consequence of E, then any algebra from $\boldsymbol{K}_{R(E)}$ is the sum of a direct system of algebras from $\boldsymbol{K}_E$.

In § 3 we show that the converse is not true even if there exist non-regular equalities in E.

§ 1. We say (see [1]) that a binary function $x \circ y$ is a *partition function* or briefly a *p-function of an algebra* $\mathfrak{A}$ *of the type* τ if it satisfies the following conditions:

$$x \circ x = x, \tag{1}$$

(2) $$(x \circ y) \circ z = x \circ (y \circ z)\,,$$

(3) $$x \circ y \circ z = x \circ z \circ y$$

and if for any fundamental operation $f(x_1, \ldots, x_n)$ of $\mathfrak{A}$ we have:

(4) $$f(x_1, x_2, \ldots, x_n) \circ y = f(x_1 \circ y, x_2 \circ y, \ldots, x_n \circ y)\,,$$

(5) $$f(x_1, x_2, \ldots, x_n) \circ x_i = f(x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, n)\,,$$

(6) $$y \circ f(x_1, x_2, \ldots, x_n) = y \circ f(y \circ x_1, y \circ x_2, \ldots, y \circ x_n)\,,$$

(7) $$y \circ f(y, \ldots, y) = y\,.$$

It was proved in [2] (Lemma 1) that

(i) a binary function $x \circ y$ is a p-function of an algebra $\mathfrak{A}$ iff it satisfies (1)-(3) and

(8) $$f(x_1, x_2, \ldots, x_n) \circ y = f(x_1, x_2, \ldots, x_{i-1}, x_i \circ y, x_{i+1}, \ldots, x_n) \quad (i = 1, 2, \ldots, n)\,,$$

(9) $$x_1 \circ x_2 \circ \ldots \circ x_n \circ f(x_1, \ldots, x_n) = x_1 \circ x_2 \circ \ldots \circ x_n\,.$$

THEOREM 1. *A binary function $x \circ y$ is a p-function of an algebra $\mathfrak{A}$ iff it satisfies* (1)-(4) *and*

(10) $$y \circ f(x_1, x_2, \ldots, x_n) = y \circ x_1 \circ x_2 \circ \ldots \circ x_n\,.$$

Necessity. By (5), (3), (9) we have

$$\begin{aligned} y \circ f(x_1, \ldots, x_n) &= y \circ f(x_1, \ldots, x_n) \circ x_1 \circ x_2 \circ \ldots \circ x_n \\ &= y \circ x_1 \circ x_2 \circ \ldots \circ x_n \circ f(x_1, \ldots, x_n) \\ &= y \circ x_1 \circ x_2 \circ \ldots \circ x_n\,. \end{aligned}$$

Sufficiency. We prove (5): by (1)-(3) and (10) we have

$$\begin{aligned} f(x_1, \ldots, x_n) \circ x_i &= f(x_1, \ldots, x_n) \circ f(x_1, \ldots, x_n) \circ x_i \\ &= f(x_1, \ldots, x_n) \circ x_1 \circ x_2 \circ \ldots \circ x_n \circ x_i \\ &= f(x_1, \ldots, x_n) \circ x_1 \circ x_2 \circ \ldots \circ x_n \\ &= f(x_1, \ldots, x_n) \circ f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n); \end{aligned}$$

(6) follows from

$$\begin{aligned} y \circ f(x_1, \ldots, x_n) &= \overbrace{y \circ y \circ \ldots \circ y}^{n+1 \text{ times}} \circ x_1 \circ x_2 \circ \ldots \circ x_n \\ &= y \circ (y \circ x_1) \circ (y \circ x_2) \circ \ldots \circ (y \circ x_n) \\ &= y \circ f(y \circ x_1, y \circ x_2, \ldots, y \circ x_n); \end{aligned}$$

(7) follows at once from (10) and (1).

§ 2. Let $x \circ y$ be a p-function of an algebra $\mathfrak{A} = \langle A; F\rangle$. Then the relation $\sim_\circ$ defined by the conditions $a \sim_\circ b$ iff $a \circ b = a$ and $b \circ a = b$ is a congruence of the algebra $\mathfrak{A}' \neq \langle A; \circ\rangle$, and $\mathfrak{A}'/\sim_\circ$ is a semilattice which yields the structure of poset ($[a]_\circ \leqslant [b]_\circ$ iff $b \circ a = b$) on A with l.u.b. $([a]_\circ, [b]_\circ) = [ab]$. It follows from [1] that the system

$L(\circ) = \langle I, \{i\}_{i \in I}, h^j_{i,\, i \leqslant j,\, i,j \in I}\rangle$, where $h^j_i(a) = ab$ for $a \in i$, $b \in j$, $i \leqslant j$, is a decomposition of $\mathfrak{A}$, and that each decomposition L of $\mathfrak{A}$ is of the form $L = L(\circ)$ for some p-function $\circ$ of $\mathfrak{A}$. We say that $L(\circ_1) = L(\circ_2)$ if $\circ_1 = \circ_2$.

We shall concern ourselves only with non-unary algebras, i.e. we assume that there exists an operation $f(x_1, \ldots, x_n)$, with $n > 1$, among the fundamental operations of $\mathfrak{A}$, because in unary algebras homomorphisms h^j_i have no influence on algebraic operations. In the set $\boldsymbol{D}(\mathfrak{A})$ of all decompositions of $\mathfrak{A}$ we introduce two orders as follows:

(i) $$\begin{aligned} &L(\circ_1) <_c L(\circ_2) &&\text{iff}\quad [a]_{\circ_1} \setminus^c [b]_{\circ_2} &&\text{for all } a, b \in A\,.\\ &L(\circ_1) <_h L(\circ_2) &&\text{iff}\quad a \circ_2 b = b \Rightarrow a \circ_1 b = b &&\text{for all } a, b \in A\,. \end{aligned}$$

It is clear that there exists a greatest element in $\boldsymbol{D}(\mathfrak{A})$ with respect to $<_c$ and with respect to $<_h$.

The following example shows that minimal elements need not exist.

EXAMPLE 1. Let

$$\mathfrak{A} = \left(\left\{1 - \frac{1}{n}\right\} \cup \{1, 2\};\ x \cdot y\,, \quad n = 1, 2, \ldots\right)$$

where $x \cdot y$ is defined as follows:

$$x \cdot x = x\,, \quad x \cdot y = y \cdot x\,, \quad \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{1}{m}\right) = 1 - \frac{1}{\max(m, n)}\,,$$

$$2 \cdot \left(1 - \frac{1}{n}\right) = 1\,, \quad 2 \cdot 1 = 2\,, \quad 1 \cdot \left(1 - \frac{1}{n}\right) = 1\,.$$

By (i) $\boldsymbol{D}(\mathfrak{A})$ is not empty. Let $\boldsymbol{L} \in \boldsymbol{D}(\mathfrak{A})$ and let X_0 be the component to which 2 belongs. We shall prove that $1 \in X_0$.

Let $1 \in X_j$; since $2 \cdot 1 = 2$ and the value of the operation always belongs to the component with an index which is the least upper bound of the indices of components to which the arguments belong, we have $j \leqslant i_0$. Since $2 \cdot 0 = 1$, we have $i_0 \leqslant j$. Thus $j = i_0$.

Further we see that $2 \cdot x \in X_{i_0}$ for any x; hence X_{i_0} is the component with the maximal index in I, where I is the set of indices of $\boldsymbol{L}$. Similarly we observe that I is linearly ordered by $\leqslant$. We cannot have $X_{i_0} = \{1, 2\}$ because $2 \cdot (1 - 1/n) = 1$, and so, if $1 - 1/n \in X_i$, it is not possible to define the homomorphism $h_i^{i_0}$. In fact, $2\cdot(1-1/n)=1$ and $2\cdot h_i^{i_0}(1-1/n)=2$, whatever is the homomorphism $h_i^{i_0}(1-1/n)$.

Let $n_0=\min\{n\colon (1-1/n)\in X_{i_0}\}$. If $n>n_0$, then $(1-1/n)\in X_{i_0}$. In fact, let $(1-1/n)\in X_i$; since $(1-1/n_0)\cdot(1-1/n)=1-1/n$, $(1-1/n)\cdot 1=1$, we have $i\leqslant i_0$ and $i\geqslant i_0$. For $(1-1/n)\in X_i$ and $i\leqslant i_0$ we must have $h_i^{i_0}(1-1/n)=1-1/n_0$, because $(1-1/n)\cdot(1-1/n_0)=1-1/n_0$. Form a new decomposition L' dividing X_{i_0} into two components

$$X_{j_1}=\left\{1-\frac{1}{n_0}\right\},\qquad X_{j_2}=X_{i_0}\diagdown\left\{1-\frac{1}{n_0}\right\}.$$

Put for $i\leqslant i_0$, $i\neq i_0$, $i\leqslant j_1$ and put $j_1\leqslant j_2$. For $i\leqslant i_0$ and $i\neq i_0$, $(1-1/n)\in X_i$ put $h_i^{j_1}(1-1/n)=1-1/n_0$. Put

$$h_{j_1}^{j_2}\left(1-\frac{1}{n_0}\right)=1-\frac{1}{n_0+1}.$$

Do not change the other things fixed in L. We obtain a new decomposition L' and $L'<_c L$, $L'<_h L$. Since L is arbitrary, there exists no minimal element in $D(\mathfrak{A})$.

LEMMA 1. *For every non-unary algebra $\mathfrak{A}=\langle A;\ F\rangle$ the relation $L(\circ_1)<_c L(\circ_2)$ is equivalent to the implication*

$$a\circ_1 b=a\Rightarrow a\circ_2 b=a\quad \text{for all } a,b\in A.$$

Proof. Let $f(x_1,\dots,x_n)\in F$, $n>1$, and suppose that $a\circ_1 b=a$ for some $a,b\in A$. By (5) and (10) we have $a\circ_1 f(a,b,\dots,b)=(a\circ_1 b)=a$, $f(a,b,\dots,b)\circ_1 a=f(a,b,\dots,b)$. Thus $f(a,b,\dots,b)\in[a]_{\circ_1}$. By hypothesis $f(a,b,\dots,b)\in[a]_{\circ_2}$. This gives $a\circ_2 f(a,b,\dots,b)=a$ and also $a\circ_2\circ_2 f(a,b,\dots,b)=a\circ_2 b$. Hence $a\circ_2 b=a$. The converse implication is trivial.

LEMMA 2. *For each non-unary algebra $\mathfrak{A}=\langle A;\ F\rangle$ the relation $L(\circ_1)<_h L(\circ_2)$ implies $L(\circ_1)<_c L(\circ_2)$.*

Proof. Let $a\circ_1 b=a$ and $f(x_1,\dots,x_n)\in F$, $n>1$. Since $a\circ_2(a\circ_2 b)=a\circ_2 b$, we have by hypothesis $a\circ_1(a\circ_2 b)=a\circ_2 b$. Because of $a\circ_1 f(a,b,\dots,b)=a$, we get by (5) and (8)

$$\begin{aligned}a\circ_2 b&=a\circ_1(a\circ_2 b)=\big(a\circ_1 f(a,b,\dots,b)\big)\circ_1(a\circ_2 b)\\&=a\circ_1 f\big(a\circ_1(a\circ_2 b),b,\dots,b\big)=a\circ_1 f\big((a\circ_2 b),b,\dots,b\big)\\&=a\circ_1\big(f(a,b,\dots,b)\circ_2 b\big)=a\circ_1 f(a,b\circ_2 b,\dots,b\circ_2 b)\\&=a\circ_1 f(a,b,\dots,b)=a.\end{aligned}$$

COROLLARY. *The relation $<_h$ is partial order in $D(\mathfrak{A})$.*

LEMMA 3. *If $x\circ y$ is an algebraic p-function of $\mathfrak{A}=\langle A;\ F\rangle$ (i.e. $x\circ y$ is p-function and $x\circ y$ is a term in $\mathfrak{A}$), then $L(\circ)$ is the smallest element in $(D;<_h)$.*

Proof. Let $L(\circ_1)\in D(\mathfrak{A})$ and $a\circ_1 b=b$. Observe that (8) holds not only for the fundamental operations but also for all terms. Thus $a\circ b=a\circ(a\circ_1 b)=(a\circ a)\circ_1 b=a\circ_1 b=b$. Hence $L(\circ)<_h L(\circ_1)$.

LEMMA 4. *If $x\circ y$ is an idempotent term in $\mathfrak{A}$, then $<_c=<_h$.*

Proof. By Lemmas 1 and 2 it is enough to prove that $a\circ_2 b=b$ implies $a\circ_1 b=b$ provided $x\circ_1 y=x\Rightarrow x\circ_2 y=x$. Let $a\circ_2 b=b$. Since $(a\circ_1 b)\circ_1 b=a\circ_1 b$, we get $(a\circ_1 b)\circ_2 b=a\circ_1 b$, and therefore by (8) we have:

$$\begin{aligned}a\circ_1 b&=(a\circ_1 b)\circ_2 b=\big((a\circ a)\circ_1 b\big)\circ_2 b=\big(a\circ(a\circ_1 b)\big)\circ_2 b\\&=(a\circ_2 b)\circ(a\circ_1 b)=\big((a\circ a)\circ_2 b\big)\circ_1 b=(a\circ_2 b)\circ_1 b=b\circ_1 b=b.\end{aligned}$$

LEMMA 5. *If there exists in $\mathfrak{A}=\langle A;\ F\rangle$ an idempotent binary term $x\circ y$ and $L_1(\circ_1)$ and $L_2(\circ_2)$ are two decompositions of $\mathfrak{A}$, then there exists a decomposition $L(\circ_3)$ in $\mathfrak{A}$ such that $[a]_{\circ_3}=[a]_{\circ_1}\cap[a]_{\circ_2}$ for all $a\in A$.*

Proof. Define:

(11) $$x\circ_3 y=(x\circ_1 y)\circ_2 y.$$

By (8) we have

$$\begin{aligned}(x\circ_1 y)\circ_2 z&=\big((x\circ x)\circ_1 y)\circ_2 z\big)=\big(x\circ(x\circ_1 y)\big)\circ_2 z=(x\circ_2 z)\circ(x\circ_1 y)\\&=\big((x\circ_2 z)\circ x\big)\circ_1 y=\big((x\circ x)\circ_2 z\big)\circ_1 y=(x\circ_2 z)\circ_1 y.\end{aligned}$$

Hence we get

(12) $$(x\circ_1 y)\circ_2 z=(x\circ_2 z)\circ_1 y\quad\text{and}\quad x\circ_3 y=(x\circ_2 y)\circ_1 y.$$

We check that $x\circ_3 y$ is a partition function in $\mathfrak{A}$. Therefore by Theorem 1 we prove that $x\circ_3 y$ satisfies (1)-(4) and (10). The proof of (1) is trivial because $\circ_1$ and $\circ_2$ are idempotent. The proof of (2): by (12) we have:

(13) $$(x\circ_3 y)\circ_3 z=\Big(\big((x\circ_1 y)\circ_2 y\big)\circ_1 z\Big)\circ_2 z=\big((x\circ_1 y\circ_1 z)\circ_2 y\big)\circ_2 z.$$

By (8), (10) and (12) we have:

(14) $$\begin{aligned}x\circ_3(y\circ_3 z)&=\big(x\circ_1\big((y\circ_1 z)\circ_2 z\big)\big)\circ_2(y\circ_1 z)\circ_2 z\\&=\big(x\circ_1(y\circ_2 z)\circ_1 z\big)\circ_2(y\circ_1 z)\circ_2 z\\&=\big(x\circ_1\big((y\circ_2 z)\circ z\big)\big)\circ_2\big((y\circ_1 z)\circ z\big)\\&=\big(x\circ_1\big((y\circ z)\circ_2 z\big)\big)\circ_2\big((y\circ z)\circ_1 z\big)\\&=\big(x\circ_1(y\circ z)\big)\circ_2(y\circ z)\\&=(x\circ_1 y\circ_1 z)\circ_2 y\circ_2 z.\end{aligned}$$

Hence we have (2). The proof of (3) follows from (12). The proof of (4) and (10) is trivial by (12). From (11) it follows that if $a \circ_1 b = a$ and $a \circ_2 b = a$ then $a \circ_3 b = a$. We prove the converse. Suppose $a \circ_3 b = a$. Then $a \circ_1 b = \big((a \circ_3 b) \circ_1 b\big) = \big((a \circ_2 b) \circ_1 b\big) \circ_1 b = (a \circ_2 b \circ_1 b) = a \circ_3 b = a$ by (1) and (12). Analogously, if $a \circ_3 b = a$ then $a \circ_2 b = a$. Thus $[a]_{\circ_3} = [a]_{\circ_1} \cap [a]_{\circ_2}$ for all $a \in A$, which comples the proof.

EXAMPLE 2. Shows that $(\boldsymbol{D}(\mathfrak{A}); \leqslant_c)$ need not be closed under the join, even if there exists an algebraic p-function $x \cdot y$ in $\mathfrak{A}$. Consider an algebra

$$\mathfrak{A} = \left(\left\{1-\frac{1}{n}\right\}_{n=1,2,\dots} \cup \left\{1+\frac{1}{n}\right\}_{n=1,2,\dots};\ \max(x, y)\right).$$

Let L_1 be the decomposition defined as follows: any of the numbers $1-1/n$ form one-element components. The set $\{1+1/n\}_{n=1,2,\dots}$ is divided into two-element components of the form $\left\{1+\frac{1}{2m-1}, 1+\frac{1}{2m}\right\}_{m=1,2,\dots}$. We define the relation $\leqslant$ for indices of components C_i and C_j as follows: $i \leqslant j$ if the minimal element in C_i is less or equal to the minimal element in C_j. If $a \in C_i$, $i \leqslant j$ and $i \neq j$, then $h_i^j(a) = b$ where b is the minimal element in C_j; further, h_i^i is the identity map. Let L_2 be defined similarly with this difference that the set $\{1+1/n\}$ is divided into components in the following way:

$$\left\{1+\frac{1}{1}\right\},\ \left\{1+\frac{1}{2m},\ 1+\frac{1}{2m+1}\right\}_{m=1,2,\dots}.$$

Observe that if $L \geqslant_c L_1$ and $L \geqslant L_2$, then the set $\bigcup_{n=1}^{\infty}\{1+1/n\}$ must be contained in one component C of L. That component cannot be $C = \bigcup_{n=1}^{\infty}\{1+1/n\}$ because it would not be possible to define the homomorphism $h\colon C_i \to C$ where C_i is a component such that i is less or equal to the index of the component C. In fact, the value of the homomorphism must be the smallest element in the given component in view of the definition of the operation $\max(x, y)$. If C contains some number $1-1/n$, then arguing as in Example 1, we conclude that C must contain any number $1-1/k$ $(k > n)$. But then, as in Example 1, we can show that there exist $L' \leqslant_c L$ and $L' \neq L$. Observe that $\max(x, y)$ is the algebraic p-function in $\mathfrak{A}$.

THEOREM 2. *If there exist in $\mathfrak{A}$ the algebraic partition function, then the relational system $(\boldsymbol{D}(\mathfrak{A}); \leqslant_c)$ is a meet semilattice with 0 and 1 which is isomorphic to some subsemilattice of the semilattice of congruences of $\mathfrak{A}$ with the meet operation.*

Proof. For $L(\circ) \in \boldsymbol{D}(\mathfrak{A})$ let $\varphi(L(\circ)) = R$, where $a R b$ if $[a]_\circ = [b]_\circ$. We check that $R \in \operatorname{Congr}\mathfrak{A}$.

Let $f_k(x_1, \dots, x_{n_k})$ be a fundamental operation in $\mathfrak{A}$ and let $a_s R b_s$ for $s = 1, 2, \dots, n_k$. Then by (10) and (8)

$$\begin{aligned} f_k(a_1, \dots, a_{n_k}) \circ f_k(b_1, \dots, b_{n_k}) &= f_k(a_1, \dots, a_{n_k}) \circ b_1 \circ b_2 \circ \dots \circ b_{n_k} \\ &= f_k(a_1 \circ b_1, a_2 \circ b_2, \dots, a_{n_k} \circ b_{n_k}) \\ &= f_k(a_1, \dots, a_{n_k}), \end{aligned}$$

and similarly

$$f_k(b_1, \dots, b_{n_k}) \circ f_k(a_1, \dots, a_{n_k}) = f_k(b_1, \dots, b_{n_k}).$$

Consequently $R \in \operatorname{Congr}\mathfrak{A}$. Now our theorem follows from the Lemmas, q.e.d.

§ 3. Let $\boldsymbol{K}_E$ be an equational class of non-unary algebras without nullary algebraic operations, defined by the set E of equations. An equation $\varphi = \psi$ is called *regular* if the sets of variables in φ and ψ are the same.

Let $\boldsymbol{K}_{R(E)}$ be the equational class of the same type as the algebras from $\boldsymbol{K}$-defined by the set $R(E)$ of all regular consequences of E. It was proved in [3] that

(ii) If there exists a term $f(x, y)$ such that the equation $f(x, y) = x$ is a consequence of E, then any algebra $\mathfrak{A} \in \boldsymbol{K}_{R(E)}$ is the sum of a direct system of algebras from $\boldsymbol{K}$ and $f(x, y)$ is a p-function in $\mathfrak{A}$.

The converse of (ii) is not true in the trivial case where all consequences of E are regular, because then any algebra from $\boldsymbol{K}$ has a trivial decomposition and no identity $f(x, y) = x$ is true in $\boldsymbol{K}$.

However, the problem arose whether this converse is true when there are non-regular equations in E. The following Example 3 shows that this is not the case either.

EXAMPLE 3. Let $\boldsymbol{K}_E$ be the equational class of groupoids defined by the equation $x \cdot y = u \cdot v$, and let $\mathfrak{A} = (A; \cdot) \in \boldsymbol{K}_{R(E)}$. Observe first that we have a lot of regular consequences of $xy = uv$, for example

$$(15) \qquad \begin{gathered} x \cdot y = y \cdot x, \quad (x \cdot y) \cdot z = x \cdot (y \cdot z), \quad x^2 \cdot y = x \cdot y, \\ (x \cdot y)^2 = x^2 \cdot y^2, \quad x^2 = (x^2)^2. \end{gathered}$$

Define in A the relation S as follows $a S b \Leftrightarrow a^2 = b^2$; obviously S is an equivalence and by (15) it is a congruence of $\mathfrak{A}$, since $x^2 S (x^2)^2 - \mathfrak{A}/S$ is a semilattice.

If a, b, c, d belong to the same congruence class $[a]$, then $a^2, c^2 \in [a]$; hence $a \cdot b = a^2 = c^2 = c \cdot d$. Thus $[a]$ is a subalgebra and belongs to $\boldsymbol{K}_E$.

Form a direct system taking $\mathfrak{A}/S$ as the set of indices with the semilattice partial order $\leqslant$; take clases $[a]$ as components and for $x \in [a]$, $[a] \leqslant [b]$, $[a] \neq [b]$ put $h_{[a]}^{[b]}(x) = b^2$, $h_{[a]}^{[a]}(x) = x$. Obviously $\boldsymbol{L}$ is a well-defined direct system with the l.u.b. property. If $x \in [a]$, $y \in [b]$, then $x \cdot y \in [a \cdot b]$. Using (15) we have:

$$x \cdot y = x^2 \cdot y^2 = a^2 \cdot b^2 = (a \cdot b)^2 = (a \cdot b)^2 \cdot (a \cdot b)^2 = h_{[a]}^{[a \cdot b]}(x) \cdot h_{[b]}^{[a \cdot b]}(y) .$$

Thus $\mathfrak{A}$ is the sum of direct system $\boldsymbol{L}$. However, no identity of the form $f(x, y) = x$ holds in $\boldsymbol{K}_E$, because any binary term is equivalent to $x \cdot y$ and $x \cdot y$ is commutative. Hence if we had $x \cdot y = x$, then we would have $x \cdot y = y$ and $x = y$, which is a contradiction because $\boldsymbol{K}_E$ is not trivial (e.g. an algebra $(\{a, b\};\, x \cdot y)$ where $x \cdot y \equiv b$) belongs to $\boldsymbol{K}$.

References

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Reçu par la Rédaction le 1. 1. 1973

Non-planar embeddings of planar sets in E^3

by

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Abstract. Let Z be a compact, 1-dimensional set in the plane, E^2. Now let X be a subset of E^3 homeomorphic to Z. It is shown that for each tame arc A in E^3 with $X \cap \partial A = \emptyset$, and for each $\varepsilon > 0$, there is a homeomorphism h of E^3 onto E^3 that moves each point less than ε, that is the identity off an ε-neighborhood of $X \cap A$, and is such that $X \cap h(A) = \emptyset$. An analog is also proven in the case in which Z is 2-dimensional.

1. Introduction. A standard technique in studying geometric properties of the embedding of a compactum X in Euclidean n-space, E^n, requires that (for certain values of k) it should be possible to move a k-simplex off X by a small homeomorphism of E^n. (See, e.g., [7] and [13].) If n is large, the conditions under which this can be done are fairly well understood. In the "simplest" nontrivial case, however, when $n = 3$, $k = 1$, and X is 1-dimensional, our intuition does not always serve us well. No obvious dimensional or algebraic obstructions suggest themselves, yet in general it cannot be done. References [5] and [11] give embeddings of Menger's universal 1-dimensional curve in E^3 that cannot be freed from certain 1-simplexes by any homeomorphism of E^3 onto E^3 that moves only points close to the universal curve. In [6], the ambient homeomorphism exists, but it cannot be close to the identity. (Caution: the term "tangled" is used with different meanings in [11] and [6].) It is easy to see from the constructions of these 1-dimensional sets that they are not locally embeddable in E^2.

We show here (Theorem 3) that if a closed set X in E^3 is a countable union of at-most-1-dimensional compact sets, each of which embeds in E^2, then tame arcs can be freed from X by a homeomorphism of E^3 that is close to the identity. On the other hand, if $X \subset E^3$ is compact and X embeds in E^2 (but perhaps $\dim X = 2$), then we can still do the best possible considering the circumstances: we can push a tame arc off with a homeomorphism of E^3, moving only points close to X (Lemma 4; cf. also Theorem 4). All of our theorems are stated for E^3, but the proofs work for arbitrary boundary less 3-manifolds. In particular, Theorem 4 gives a new (and simpler) proof of the result [10] that a 2-cell topologically