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A 3-dimensional irreducible compact absolute retract which contains no disc

by

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Abstract. R. H. Bing and K. Borsuk gave an example of a 3-dimensional compact absolute retract which contains no disc. In this paper, we construct a 3-dimensional irreducible compact absolute retract which contains no disc.

1. Introduction. Borsuk [8] described a 2-dimensional compact absolute retract which does not contain any proper 2-dimensional compact absolute retract. Following Borsuk, we say that an n-dimensional compact absolute retract A is *irreducible* if and only if A does not contain any proper n-dimensional compact absolute retracts. Molski [10] generalized Borsuk's example of [8] to obtain for each $n \geqslant 2$ an n-dimensional irreducible compact absolute retract. Bing and Borsuk [6] gave an example of a 3-dimensional compact absolute retract which does not contain any (2-dimensional) disc. The following is a natural question: Does there exist an irreducible n-dimensional compact absolute retract for $n \geqslant 2$ which does not contain any (2-dimensional) disc?

For $n = 2$, the answer is affirmative as proved by Borsuk [8]. The purpose of this note is to answer the question in the affirmative when $n = 3$. For $n > 3$, the answer is unknown.

By an AR we mean a compact absolute retract for metric spaces. For notation and terminology see [3], [6] and [7]. The techniques of construction are similar to those used in [3] and [6].

If G is an upper semi-continuous decomposition of a topological space X, we denote by X/G the associated decomposition space and $p\colon X \to X/G$ the canonical projection.

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2. Antoine's Necklaces. Let r be a fixed positive integer and Σ_r be an unknotted polyhedral solid torus in 3-dimensional Euclidean space E^3. All tori considered will be solid, unknotted and polyhedral. Let $\{T_{r1}, \ldots, T_{rm_{r0}}\}$ denote a chain of linked solid tori in $\operatorname{Int}(\Sigma_r)$ circling Σ_r exactly twice such that for $i = 1, 2, \ldots, m_{r0}$ the diameter of T_{ri} is less than one. For each i, with $i = 1, 2, \ldots$, or m_{r0}, let $\{T_{ri1}, T_{ri2}, \ldots, T_{rim_{ri}}\}$ be a chain of linked tori in $\mathrm{Int}(T_{ri})$ circling T_{ri} exactly twice, with the diameter of each T_{rij} less than $\frac{1}{2}$, where $1 \leqslant j \leqslant m_{ri}$. Let $\{T_{rij1}, T_{rij2}, \ldots, T_{rijm_{rij}}\}$ be a chain of linked tori in $\mathrm{Int}(T_{rij})$, each of diameter less than $\frac{1}{4}$, circling T_{rij} exactly twice for all $i = 1, 2, \ldots$, or m_{r0} and $j = 1, 2, \ldots$, or m_{ri}. We continue this construction to obtain the following sets:

$$M_{r1} = \bigcup_{i=1}^{m_{r0}} T_{ri},$$

$$M_{r2} = \bigcup_{i=1}^{m_{r0}} \bigcup_{j=1}^{m_{ri}} T_{rij},$$

$$M_{r3} = \bigcup_{i=1}^{m_{r0}} \bigcup_{j=1}^{m_{ri}} \bigcup_{k=1}^{m_{rij}} T_{rijk},$$

$$\ldots\ldots\ldots\ldots\ldots$$

Let N_r denote $\bigcap_{i=1}^{\infty} M_{ri}$; N_r will be called a *dyadic Antoine's necklace* circling Σ_r. Note that N_r is contained in $\mathrm{Int}(\Sigma_r)$.

An A-arc substituting for Σ_r. Consider the first stage torus T_{ri} for $i = 1, 2, \ldots$, or m_{r0} and the set $(N_r \cap T_{ri})$. It is well-known that for each i, there is an arc a_{ri} in $\mathrm{Int}(T_{ri})$ such that a_{ri} contains the set $(N_r \cap T_{ri})$. Construct arcs $b_{r1}, b_{r2}, \ldots, b_{r(m_{r0}-1)}$ as constructed in [3] such that $(\bigcup_{i=1}^{m_{r0}} a_{ri}) \cup (\bigcup_{j=1}^{m_{r0}-1} b_{rj})$ is an arc A_r. The arc A_r will be called an *A-arc* substituting for Σ_r.

An A-wreath substituting for Σ_r. For each i, $i = 1, 2, \ldots$, or m_{r0}, let $\{T_{ri1}, T_{ri2}, \ldots, T_{rim_{ri}}\}$ be the chain of linked tori in $\mathrm{Int}(T_{ri})$ exactly twice. Consider T_{rij} for $j = 1, 2, \ldots$, or m_{ri} and the set $(N_r \cap T_{rij})$, for all j. As before, there are arcs $b_{ri1}, b_{ri2}, \ldots, b_{ri(m_{ri}-1)}$ such that $(\bigcup_{j=1}^{m_{ri}} a_{rij}) \cup (\bigcup_{k=1}^{m_{ri(m_{ri}-1)}} b_{rik})$ is an A-arc A_{ri} contained in the interior of T_{ri}. The union W_r of $A_{r1}, A_{r2}, \ldots$, and $A_{rm_{r0}}$ will be called an *A-wreath* substituting for Σ_r and the A-arcs $A_{r1}, A_{r2}, \ldots, A_{rm_{r0}}$ will be called *links* of W_r.

3. Cantor-manifolds.

DEFINITION. Let X be a metric space of dimension $\leqslant n$. We say X is *dimensionally uniform* if for each point $p \in X$ and $\delta > 0$ there is an open ball $B_\varepsilon(P)$ with $0 < \varepsilon < \delta$ such that the boundaries of uncountably many open balls contained in $B_\varepsilon(P)$ and centered at P have dimension $\leqslant n-1$.

Let X be a separable metric space with $\dim(X) = n$. Given an upper semi-continuous decomposition G of X into closed subsets of X such that $\dim(g) \leqslant K$ for each $g \in G$. The following is the main lemma of this section:

LEMMA 3.1. *If Y is a subset of X/G such that $0 < \dim(Y) \leqslant K$ then $\dim[P^{-1}(Y)] \leqslant K$ provided X/G is metrizable, G contains at most countably many non-degenerate elements and Y is dimensionally uniform.*

Proof. Let P_1: $P^{-1}(Y) \to Y$ denote the restriction of P: $X \to X/G$. It is easy to see that the collection $\{P^{-1}(y)\colon y \in Y\}$ is upper semi-continuous and Y can be thought of as the decomposition space. We shall show that the family $\{P^{-1}(y)\colon y \in Y\}$ of closed subsets of $P^{-1}(Y)$ satisfies the hypotheses of the following proposition of Hurewicz and Wallman: If a separable metric space X is the sum of a family of closed sets $\{K_\lambda\}$ with the properties: each K_λ has dimension $\leqslant n$, and given any K_λ and open set U containing K_λ there is an open set V, $K_\lambda \subset V \subset U$, with $\dim[\mathrm{Bd}(V)] \leqslant n-1$.

Then X has dimension $\leqslant n$.

We proceed with the proof. For each $y \in Y$, there is an open ball $B_\varepsilon(y)$ with $\varepsilon > 0$ such that $\mathrm{Bd}[B_\varepsilon(y)]$ does not contain any element which is the image of a non-degenerate element of the decomposition and $\dim[\mathrm{Bd}[B_\varepsilon(y)]] \leqslant K-1$. This can be done since the family of non-degenerate elements is at most countable and Y is dimensionally uniform.

Now

$$\dim\Big[P_1^{-1}[\mathrm{Bd}[B(y)]]\Big] \leqslant K-1,$$

and hence

$$\dim\Big[\mathrm{Bd}[P_1^{-1}[B(y)]]\Big] \leqslant K-1.$$

Let U be an arbitrary open subset of $P^{-1}(Y)$ containing $P_1^{-1}(y)$ for some $y \in Y$. Since Y is a decomposition space of the upper semi-continuous decomposition $\{P^{-1}(y)\colon y \in Y\}$, there exists a saturated open set W such that $P_1^{-1}(y) \subset W \subset U$. Now $P_1(W)$ is an open subset of Y containing y. There exists an open ball $B_\varepsilon(y) \subset P_1(W)$ such that the dimension of $\mathrm{Bd}\big[P_1^{-1}[B_\varepsilon(y)]\big] \leqslant (K-1)$. Clearly, $P_1^{-1}[B_\varepsilon(y)] \subset W \subset U$ since W is saturated. Since this can be done for each set $P^{-1}(y)$ and $P^{-1}(Y)$ is the union of $P^{-1}(y)$'s the proof of the lemma is finished. q.e.d.

Remark. The condition that "Y dimensionally uniform" can be omitted. This follows from a remark in [9], page 107.

We have the following theorem:

THEOREM 3.1. *Let X be a Cantor-manifold of dimension n, with $n \geqslant 3$. If G is an upper semi-continuous decomposition of X such that G contains at most countably many non-degenerate elements and $\dim(g) \leqslant 1$ for each $g \in G$ then X/G is a Cantor-manifold of dimension K provided $K = \dim(X/G) \leqslant n$.* (For a definition of a Cantor-manifold see [7].)

Proof. Let $Y \subset X/G$ such that $\dim(Y) \leqslant K-2$.

If $K-2 = 0$, then $\dim[P^{-1}(Y)] \leqslant 1$ ([9], page 92). Therefore $P^{-1}(Y)$ does not separate X and hence Y does not separate X/G.

If $0 < \dim(Y) \leqslant K-2$, then $\dim[P^{-1}(Y)] \leqslant K-2$. This follows from Lemma 3.1. If $K \leqslant n$, we have $K-2 \leqslant n-2$ and therefore $P^{-1}(Y)$ does not separate X and hence Y does not separate X/G. This shows that X/G is a Cantor-manifold.

COROLLARY 3.1. *Bing and Borsuk's example* [16] *of a 3-dimensional compact absolute retract is a Cantor-manifold.*

Remark. It is useful to know if a given compact AR is a Cantor-manifold because of the following:

Each Cantor-manifold of dimension n is n-dimensional at each of its points, and hence its open subsets are n-dimensional at every point. Since, Bing and Borsuk's example of a 3-dimensional compact AR [6] is a 3-dimensional Cantor-manifold it follows that at each point it contains arbitrarily small 3-dimensional open sets which are 3-dimensional non-compact absolute neighborhood retracts. The same holds for our example and for that of [10].

4. An upper semi-continuous decomposition. Let $\{A_i\}$ be a sequence of polyhedral solid tori in E^3. The sequence $\{A_i\}$ is A-*dense* in E^3 if for each simple closed curve $C \subset E^3$ and open subset U of E^3, there is an index i such that

(1) $A_i \subset E^3 - C$,

(2) the core C_i of A_i is homologically linked with C, and

(3) C_i meets U.

For the definition of core and matters related to linking see [6] where other references will be found.

We organize the rest of this section in parts (A) to (D).

(A) There exists in E^3 a countable family F of disjoint polygonal simple closed curves such that for any simple closed curve C in E^3 and open subset U of E^3, there exists an element P of F satisfying the following:

(1) P and C are homologically linked, and

(2) P meets U.

It is apparent that one can construct an A-dense sequence $\{A_i\}$ of solid polyhedral tori by taking the family F of simple closed curves as the cores of the tori. The above assertions follow from [6] by making suitable changes.

(B) Let B^3 be the closed unit ball in E^3 with boundary S^2. There exists a countable family of disjoint segments $\{K_i\}$ satisfying the following:

(1) For each i the end points of K_i lie on S^2.

(2) The diameters of the K_i's converge to zero.

(3) For each non-empty open subset G of S^2, there is an index j such that both the end points of the segment K_j lie in G. This is a result of [6].

(C) Let $\{K_j\}$ be a countable family of segments as in (B). There exists a sequence $\{A_i\}$ of solid polyhedral tori contained in $B^3 - S^2 - \bigcup_j K_j$ such that for each j:

(1) The inner radius of A_j is less than $1/j$.

(2) There exists in A_j an A-wreath W_j substituting for A_j. Also $W_k \cap W_j = \emptyset$ for $j \neq K$ and the diameter of each link of W_j is less than $1/j$, for $j = 1, 2, 3, \ldots$ For a definition of the term "inner radius" and other related matters see [4].

(D) Let $\{K_j\}$ be a countable family of disjoint segments as provided in part (B) and $\{A_i\}$ be a sequence of polyhedral solid tori described in part (C). Also in (C) we described a sequence $\{W_i\}$ of A-wreaths with the properties:

(1) W_j is an A-wreath substituting for A_j,

(2) W_j and W_k are disjoint if $j \neq k$, and

(3) the diameter of each link of W_j is less than $1/j$. For each j, $j = 1, 2, 3, \ldots, W_j$ has only finitely many links, say $\{a_{j1}, a_{j2}, \ldots, a_{jm_j}\}$. Put $S_j = \{a_{j1}, a_{j2}, \ldots, a_{jm_j}\}$ and define a set $S = \bigcup_{j=1}^{\infty} S_j$. Clearly S is a countable set of disjoint arcs. Take the countable family of disjoint segments $\{K_j\}$ as in (B) and form a set S' by taking its union with S.

We define a decomposition G of B^3 whose non-degenerate elements are precisely the elements of S'. G is an upper semi-continuous decomposition since the non-degenerate elements form a null collection. It follows directly from [4] that B^3/G is a compact AR of dimension 3.

5. The main theorem.

THEOREM 5.1. *The decomposition space B^3/G is an irreducible AR of dimension 3 such that B^3/G does not contain any (2-dimensional) disc.*

Proof. The fact that B^3/G is an AR of dimension three follows from [4] and results quoted in [6]. The proof that B^3/G does not contain any 2-dimensional disc is similar to the proof in [6] and hence will be omitted.

We proceed to show that B^3/G is irreducible. Let $A \subset B^3/G$ be a proper 3-dimensional compact AR. By [3; Lemma 7], there is a sequence $U_0, U_1, \ldots, U_n, \ldots$ of open subsets of B^3/G each containing A such that for each i, $U_{i+1} \subset U_i$ and each loop in U_{i+1} is null homotopic in U_i. Also U_0 can be chosen such that the interior of $B^3/G - U_0$ relative to B^3/G is non-empty. We assume that there is a point $y \in B^3/G - U_0$ such that y belongs to an open subset W' and W' is a subset of $B^3/G - U_0$. Let $V_i = P^{-1}(U_i)$ for $i = 0, 1, 2, \ldots$, and $W = P^{-1}(W')$. Now $V_{i+1} \subset V_i$, $P^{-1}(A)$

$\subset V_i$ and $V_i \cap W = \emptyset$ for each $i = 0, 1, 2, \ldots$ By [3], Lemma 9, we have that for each i, every loop in V_{i+1} is nullhomotopic in V_i.

By Lemma 3.1 of this paper it follows that the dimension of the set $P^{-1}(\Delta)$ is three. By [9] we conclude that the interior of $P^{-1}(\Delta)$ relative to B^3 is non-empty. Since the decomposition G is upper semi-continuous, there is a saturated open set O contained in $P^{-1}(\Delta)$. Now O is contained in the set $\bigcap_{i=0}^{\infty} V_i$. Since O is open, and non-empty we may choose a point x_0 in O. Since $\{x_0\}$ is a compact AR, by [3], Lemma 6, there is an open set O' such that O' contains x_0, O' is contained in O and each loop in O' is nullhomotopic in O. By choosing the point x_0 such that $\{x_0\}$ is a degenerate element of G we may assume that O' is saturated.

Let $D^2 = \{(x, y)\colon x^2+y^2 \leqslant 1$ and x, y real number$\}$ and $S^1 = \{(x, y) \in D^2\colon x^2+y^2 = 1\}$. Since O' is open, it follows that there is a simple closed curve C contained in O'. Let $f\colon S^1 \to C$ be some fixed homeomorphism of S^1 onto C. The simple closed curve C is nullhomotopic in O and hence there is a continuous map $\psi\colon D^2 \to O$ such that $f = \psi|S^1$. We consider $\Delta = \psi(D^2)$ and the open subset W of B^3. W and Δ are disjoint and there exists a polyhedral solid torus A_i, for some index i, such that the core C_i of A_i is homologically linked with $C = \psi(S^1)$ and A_i contains a meridional disc D such that $D \subset (A_i \cap W)$. The meridional disc D can be taken to be polyhedral. Let W_i be the A-wreath substituting for A_i. One of the links of W_i lies completely in the interior of $P^{-1}(\Delta)$. The above assertion follows from the fact that some link of W_i meets Δ and hence the link must be completely contained in O', since O' is a saturated open set contained in the interior of $P^{-1}(\Delta)$. Without loss of generality we may assume that a_{i1} is contained in the interior of $P^{-1}(\Delta)$.

By [3], Lemma 4, we obtain a polygonal simple closed curve γ in $V_0 \cap A_{i1}$ such that γ is not nullhomotopic in A_{i1}. This can be seen by setting $A_i = \Sigma_i$ and $A_{i1} = T_{i1}$ in the above mentioned lemma and keeping in mind that A_{i1} is a second stage torus in the construction of the dyadic Antoine's necklace. By applying the arguments of the proof of Lemma 5 of [3], we conclude that there exists a loop γ' in $V_0 \cap A_1$ such that γ' is not nullhomotopic in A_i. Hence the loop γ' must meet the meridional disc D, where $D \subset W \cap A_i$. This is a contradiction, since $W \cap V_0 = \emptyset$ by our construction. Hence B^3/G is an irreducible AR of dimension 3 and B^3/G contains no (2-dimensional) disc.

As an application of Theorem 5.1 we have the following generalization of Corollary 3.2:

COROLLARY 5.1. *There exists a non-compact absolute neighborhood retract which contains neither a 3-dimensional compact* AR *nor a (2-dimensional) disc.*

COROLLARY 5.2. *B^3/G has the singularity of Mazurkiewicz.*

In [5] Steve Armentrout announced that one could construct a cellular decomposition of E^3 whose decomposition space is neither strongly locally simply connected, locally peripherially spherical nor locally nice in dimension one. For the definitions of the terms involved see [5]. We have the following:

COROLLARY 5.3. *The space B^3/G is neither strongly locally simply connected, locally peripherially spherical nor locally nice in dimension* 1 *at every point.*

A proof can be constructed by using the techniques [3], [4] and this paper.

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