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Set existence principles of Shoenfield, Ackermann, and Powell

by

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Abstract. The author proposes a formalization of an informal set existence principle of Shoenfield. Some consequences of the axioms are developed and comparisons are made with other axiomatic theories which have been proposed. The author also makes some general remarks about the problem of axiomatic principles in mathematics.

Introduction. Shoenfield has formulated the following principle $\mathcal{S}$ for the existence of sets. The principle assumes that sets are built up in cumulative stages, and that there is an ordering on the stages as they are built up.

$\mathcal{S}$ If P is a property of stages, and if we can *imagine* a situation in which all the stages having P have been built up, then there *exists* a stage s beyond all the stages which have P.

We remark at the outset that one can read $\mathcal{S}$ in either (i) a more or (ii) a less constructive way, namely (i) that the stage s exists mathematically because of (or in) the act of imagination, which is thus a sort of construction of s, or (ii) that what can be imagined is but an indication of what has mathematical existence, so that the latter can retain a certain changeless Platonic impregnability or Cantorian absoluteness. It is (ii) which seems appropriate to this author in the context of classical set theory. (Evidently this does not preclude consideration of processes, constructions, and the like, only they are not to be regarded as more primitive than existence.)

Although $\mathcal{S}$ is certainly vague, Shoenfield has used it rather convincingly to derive a number of the usual axioms of set theory [10]. The purpose of this paper is to propose a formalization of this principle, (§ 1, § 5) and to deduce some of its consequences, the most striking being the existence of measurable cardinals (see § 5, Theorem 5.12). The formalization proposed will bear a close relation to two other set theories, one due to Ackermann and one to Powell, (see § 2, § 5 Remark 5.13 and § 6). In a sense, adding arbitrary properties of sets to Ackermann's theory yields measurable cardinals (see 5.13).

In stating Shoenfield's principle I have taken the liberty of speaking of *properties* of stages rather than *collections* of stages. Quoting directly from notes of Shoenfield, "If C is a collection of stages, and if we can imagine a situation in which all of the stages in C have been completed, then there is to be a stage S after all the stages in C" (private communication). A similar (*) principle appears in Shoenfield [10, p. 234]: given a collection of stages... there is a stage which follows every stage in the collection... whenever we can visualize a situation in which all the stages in the collection are completed.

We remark that various authors have suggested set theory should in some sense accomodate everything that can be imagined. (This tendency in thought undoubtedly goes back to Leibniz, who formulated various maximality principles for existence. This in turn presumably goes back to medieval ideas of plenitude.) A recent expression of such an attitude may be found in Tarski [11], p. 134, last paragraph. The only such formulation I know of which seems sharp enough to suggest a precise formal axiom, however, is the principle S.

A number of results of this paper were obtained independently by Powell and will be included in his thesis.

§ 1. The theory S. The formal theory **S** is introduced to express more precisely (part of) the intuitive principle S. This theory is formalized in a first order language with the usual logical symbols (including $\exists$, $=$) together with one binary relation symbol ϵ, and one individual constant symbol V. It will also be convenient to have a unary predicate symbol U to indicate the range of the quantifier $\exists$ (the universal predicate).

We now indicate what the intuitive meanings of these symbols are, and how the principle S is to be expressed in terms of these. First, ϵ will as usual mean "belongs to", and U will mean "set". For our purposes the objects (situations and stages) referred to in **S** may be considered sets. Some of the terms occuring in the principle S can be expressed using ϵ and U. We interpret "x has been built up in situation (or stage) y" as "$x \epsilon y$", and "s_1 is beyond s_2" as "$s_2 \epsilon s_1$". It would be possible to take "stage" as primitive, or to define it from ϵ (see § 3), but it will not be necessary to distinguish stages from other objects in order to state the

(*) Only "similar" because we choose here to pursue the sense of "imagine" according to which Existing = Real $\subseteq$ Imaginary rather than that according to which Imaginable = Visualizable $\subseteq$ (mathematically) Existing. To choose the latter would turn S into a tautology (or an exhortation to visualize!) and require for its usefulness a criterion for visualizability. One can read the axioms for **S** as such criteria, taking V to be the visualizable sets. Then (S3) gives a condition under which φ will provide a "visualization" of a set. The axioms so read may be plausible for some kind of "visualizability" (such a concept of "visualizable" seems close to definable). However, I have trouble seeing that each $x \subseteq w$ is visualizable.

axioms (except for the regularity axiom in § 3). The principle S distinguishes between "imagine" and "exist". We shall do this formally for sets by treating "imagine" as the quantifier and "exist" as quantification relativized to a certain predicate. (Consequently we should not call $\exists$ the existential quantifier, but a generalized existential quantifier. We keep the usual logic for $\exists$, however. We do not consider the possibility of a non-classical logic for "imaginary" objects.) We suppose that we can imagine the set of all existing sets; V denotes this imaginary set. We have yet to explain the term "property". Generally, formulas correspond to properties of sets. Notice that the principle S only refers to existing properties, not imaginary ones (taking "is" not in the scope of "imagine" to have existential force). Formulas in which all parameters are existing sets will correspond to existing properties (we do not assume the correspondence is onto). Formulas involving merely imagined objects x, such as "$t \epsilon x$", will not in general correspond to existing properties. Note in particular that we must not assume that V exists, and consequently the property P such that $t \epsilon V \leftrightarrow P(t)$ must not be assumed to exist either.

Our formalization of "there exists a set x such that ..." is "$\exists x(x \epsilon V \& \ldots)$". (In other words, "$x$ exists and is a set" is formalized "$x \epsilon V$".) This differs from the usual formalization, which would be "$\exists x(U(x) \& \ldots)$" or "$\exists x(\ldots)$" (since we understand the quantifier to range over U). This usual formalization automatically gives the quantifier the meaning of existence. (In other words, "x exists and is a set" would usually be formalized simply by "$U(x)$", or, equivalently, "$\exists y(U(y) \& x = y)$" or "$\exists y(x = y)$"). The justification offered for the peculiar treatment of existence is 1). It works smoothly in formalizing S. 2) In section 5 we introduce for every $X \subseteq V$ a corresponding property Q (so that each imaginable set of existing sets corresponds to an existing property of imaginable sets). In the case $X = V$, this property will be expressed by $\exists y(y = x)$, not by $x \epsilon V$. (So that the property corresponding to the imagined set of all existing sets is expressed by the usual formalization of "x exists and is a set".)

After stating the axioms we will indicate some other ways of reading our $\exists x$ and $x \epsilon V$.

Although in § 5 we shall introduce (existing) properties Q corresponding to arbitrary $X \subseteq V$, no attempt is made in this paper to introduce such properties corresponding to arbitrary expressions involving V (as $x = V$, $x \epsilon \mathfrak{S}V$, etc.).

We now give the axioms.

We assume extensionality for all entities:

$$\text{(S0)} \qquad \forall t(t \epsilon x \leftrightarrow t \epsilon y) \rightarrow x = y\,.$$

This is in accord with the usual understanding of set.

About V we assume that

(S1) $$t \in x \wedge x \in V \to t \in V, \quad t \subseteq x \wedge x \in V \to t \in V,$$

where of course $t \subseteq x$ is $\forall u(u \in t \to u \in x)$. These express in part the completeness we expect of what exists when existence is conceived in the classical way rather than as what we can construct. It would also be appropriate here to include the Cantorian

$$x \in V \wedge t \subseteq V \wedge t \approx x \to t \in V.$$

(This says that mathematical existence is not affected when existing entities are replaced pointwise by existing entities. We shall be interested in approaching the replacement axiom in a different way, so omit this strengthening.) We employ the familiar Zermelo principle to give a supply of "imagined entities" z (including "situations")

(S2) $$\exists z \forall t(t \in z \leftrightarrow \theta \wedge t \in x)$$

where θ is any formula not involving z. Here we indicate further the decision to proceed in a perfectly classical manner with the treatment of sets, including merely "imaginable" sets.

Finally, for the principle $\mathcal{S}$ itself we take (S3) below. To make clearer the connection with $\mathcal{S}$ we give the following intermediate formulation of $\mathcal{S}$:

Suppose imaginable an x such that x is a situation and for any t, if t is a stage and t has property P, then t has been built up in situation x; then a stage s exists such that for every t, if t has P then s is beyond t.

Now replace "x can be imagined" by "$\exists x$", "x exists" by "$x \in V$", "t has P" by "$\varphi(a_1, \ldots, a_n, t)$" (where the a_i exist). Ignoring the distinction between stages and other objects, this yields

(S3) $$a_1, \ldots, a_n \in V \wedge \exists x \forall t(\varphi(a_1, \ldots, a_n, t) \to t \in x) \\ \to \exists s[s \in V \wedge \forall t(\varphi(a_1, \ldots, a_n, t) \to t \in s)],$$

where φ is any $\in$-formula (i.e. any formula of L) with free variables among $a_1, \ldots, a_n, t$, and x is distinct from all these.

Definition 1. **S** is the theory whose axioms are (S0), (S1), (S2), and (S3).

Remark. Regularity is discussed in § 3. Choice could be added to **S** but will play no role in our considerations. However, Powell pointed out that it could be used in lieu of regularity, for example in Theorem 4.1. Both can be avoided if (S3) is replaced by a suitable reflection principle (see Remark 4.3).

If the reader is unhappy with the quantifier "imaginable" and the subsequent distinction between "existing" and "imaginable" sets, he may prefer to read "imagine" and "exist" as follows. First suppose that all existing sets and properties of sets are split into cumulative levels

(say according to their degree of abstractness). Now take "exists" to mean "exists at a certain level", "imagine" to mean "exists, but possibly only at higher levels". This seems to amount to the same thing, but retains the usual interpretation of $\exists$. Since the plausibility of the axioms we consider rests (in the first interpretation) on the completeness or fullness in some sense of what exists, this second reading rests correspondingly on the existence of a "certain level" which incorporates the completeness (with respect to existence) which we expect mathematical existence to have with respect to "imaginable". This of course is an approach which is familiar from reflection principles. Read in this way, Shoenfield's principle says, of a certain level (indicated by V), that if it or a higher level situation collects all stages having a strictly lower level property, then already some strictly lower level stage is beyond all of them. Another possibility is to interpret "imaginable set" as "class" or "set of second type", and "existing set" as "set". This is like the interpretation Ackermann indicates for his theory [1]. "Existing" properties may then be interpreted as those properties of classes whose existence does not depend on the existence of the class of all sets (or any class of as great rank). For short we may call such properties *independent*. On this interpretation, Shoenfield's principle says that if P is an independent property of classes, and there exists a *class* x with $P \subseteq x$, then there exists a *set* x such that $P \subseteq x$. Again this seems to amount to the same thing. Here the plausibility of the axioms rests on type distinctions (notably set-class) and on the fullness of the independent properties. The plausibility of this approach is perhaps impaired by the fact that the concept of "independent property" does not seem to be uniquely determined.

There is an alternative interpretation of V which is compatible with $\mathcal{S}$, and even suggested by the terminology of $\mathcal{S}$ (which conveys the idea of process and change). Namely, we may conceive of V, the class of all sets which exist or have been built up, the "available" sets, having definite membership, as a variable in the old fashioned sense of a "quantity which varies", (i.e., a function on the ordering of stages taking sets as values). V itself does not have definite membership and is not available. Thus the extent of V varies along the ordering of stages. Similarly the properties of sets which are available becomes a variable (of suitable type). In this picture V contains the sets actually built up in the "process" of generating sets along the ordering of stages. Thus, thinking of the order "temporally", V changes from "moment to moment". It is clear in this picture that $t \in V$ should not be allowed as a definite property of sets, since the question whether $t \in V$ or not depends on the particular value which V assumes among many possible values. This picture can probably not be used to justify or render plausible the axioms of **S**. (E.g. V must grow by leaps and bounds, if the axioms of **S** are to hold, and these jumps will be difficult to describe.) Nevertheless it seems to be very suggestive. It is possible to regard V as the least R_k compatible with what can be truly expressed about the Cantorian universe by means of language L. Since the language at our disposal does not seem fixed (the simplest case is addition of a truth predicate to L), there seems to be an essential variation here. Moreover since we are always caught in some L, it appears impossible to distinguish R_k from the Cantorian universe $V = R_\Omega$ by any absolute means.

§ 2. Relation to ZF and to Ackermann's set theory. Ackermann [1, also see 7] has formulated a set theory **A** in the language L'. The schema (S3) is stronger than the main schema of Ackermann's set theory, which is

$$(1)\quad a_1, \ldots, a_n \in V \wedge \forall t(\varphi(a_1, \ldots, a_n, t) \to t \in V) \\ \to \exists s[s \in V \wedge \forall t(\varphi(a_1, \ldots, a_n, t) \to t \in s)] .$$

(same restrictions on φ as in (S3)). Thus the "imaginable sets" in the interpretation of **S** correspond to what Ackermann called classes, and V to his class of all sets. The other axioms of Ackermann are immediate from (S0)-(S2). Lévy and Vaught [8] have shown that **A** with regularity ($\mathbf{A}^*$) is consistent relative to **A**. Since it is known that $\mathbf{A}^*$ is as strong as **ZF** [9], assuming regularity we get (relativized to V) all the axioms of **ZF**. We will also see (§ 4) that the axioms of **ZF** hold outright, without relativization to V. In the other direction, Lévy [7] has given a (finitary) proof of the consistency of **A** relative to **ZF** which yields also the consistency of **S** relative to **ZF**. Indeed, we will see that **S** with regularity ($\mathbf{S}^*$) coincides with the theory Sb used in that proof (Theorem 4.1). However, the first ordinal $\varkappa$ which provides a (natural) model for **S** is larger than the first providing a model for **ZF** (see § 7).

The arguments in **S** or **A** for the axioms of Zermelo (excluding regularity) are easy, and are not so different from the informal arguments Shoenfield gives using $\mathcal{S}$. (Perhaps the chief difference is that in Shoenfield's arguments the emphasis is on asking the reader to *imagine* a suitable situation, whereas in the formal arguments, the emphasis is on seeing that the imagined situation is *suitable* — that is, all stages having P have indeed been built up.) On the other hand, the replacement schema presents technical difficulties. These suggest a strengthening of **S** which better captures the (relatively simple) intuitive argument for replacement which can be given using $\mathcal{S}$ (see § 5,4). In order to get the regularity axiom one must of course restrict to the regular sets. The only proofs I know of the replacement schema of **ZF** also require this restriction. Regularity can be avoided by exchanging (S3) for a suitable reflection schema (see 4.3).

The existence for $a, b \in V$ of the empty set O, unordered pair $\{a, b\}$, power set $\mathcal{P}a$, and union $\bigcup a$ are all proved as follows. In each case there is an ϵ-formula $\varphi(a, b, t)$ which expresses membership of t in the desired set. In each case one has (possibly with the help of (S1))

$$(1)\quad \forall t(\varphi(a, b, t) \to t \in V) .$$

Thus by (S2) one can at least imagine the desired set $z = \{t: \varphi(a, b, t)\}$. By (S3) and (1), there is $s \in V$ so $z \subseteq s$. Thus, by (S1), $z \in V$, and the desired set exists. The axiom of infinity is proved similarly by showing that

$$\omega = \bigcap\{t:\ O \in t \wedge \forall u(u \in t \to u \cup \{u\} \in t)\}$$

exists, this time using union and pairing axioms to see $\omega \subseteq V$.

§ 3. Regularity; the theory $\mathbf{S}^*$. Let us call x a stage, and write $S(x)$, if there is an ordinal α such that $t \in x$ iff t is of rank $< \alpha$. We show in this section that S may be expressed by an ϵ-formula in such a way that

$$(S4.1)\quad \begin{aligned} &S(x) \wedge t \in y \in x \to t \in x ,\\ &S(x) \wedge t \subseteq y \in x \to t \in x ,\\ &S(x) \wedge S(y) \to x \subseteq y \vee y \subseteq x ,\end{aligned}$$

$$(S4.2)\quad z \neq 0 \wedge \forall x(x \in z \to S(x)) \to (\exists x \in z)(\forall x' \in z) x \subseteq x'$$

are provable in **S**, and moreover that the theory $\mathbf{S}^*$ obtained from **S** by adding the regularity axiom

$$(S4.3)\quad \forall x \exists y(S(x) \wedge x \in y)$$

is interpretable in **S** by restricting the universe to x's such that $\exists y(S(y) \wedge x \in y)$. Therefore in discussing the replacement schema we will assume (S4.1), (S4.2), (S4.3). In any case these are natural assumptions in this setting. Indeed, we could take the notion of "stage" as primitive and assume (S4.1), (S4.2), (S4.3) as axioms, where S is a new unary predicate which is allowed in the schemata of the theory **S**. What we do with the theory $\mathbf{S}^*$ goes through for this theory also. It is not hard to see that so read, (S4.1), (S4.2), (S4.3) are compatible with a variety of failures of regularity. However, if S is a formula expressing "$\exists \alpha\, x \in R_\alpha$", then (S4.3) expresses regularity. By (S4) we understand this version of (S4.3). More precisely, let $\mathrm{Rn}(x, y)$ be a formula expressing "x is an ordinal and y is the sets of rank less than x". (For an explicit formulation of Rn, see e.g. Lévy and Vaught [8, p. 1047]). Then the regularity axiom can be given as

$$(S4)\quad \forall x \exists y, z(\mathrm{Rn}(y, z) \wedge x \in z) .$$

Of course, assuming replacement it is possible to show that (S4) is equivalent to the following axiom (frequently called the axiom of regularity)

$$(1)\quad x \neq 0 \to (\exists u \in x)(\forall t \in x)(t \notin u) .$$

However, in **S** we are only able to show (without use of (S4.1), (S4.2), (S4.3)) that (1) implies regularity for *sets*, i.e. (S4) holds for $x \in V$.

Let $\mathrm{Ord}(x)$ be an ϵ-formula saying that x is a transitive set and that x is well ordered by ϵ (i.e. that x is a Von Neumann ordinal). Let $\mathrm{Rn}(x, y)$ be as above. We need to know the following fact in **S**:

a) $\mathrm{Ord}(x) \wedge \mathrm{Ord}(y) \to x \in y \vee y \in x \vee x = y$.

Now a) is provable in **S** in the usual way, because the usual proof uses only (S0) and (S1). It is easy to see using a) and (S2) that any transitive set of ordinals is an ordinal, and that the ordinals are well ordered. Using this and (S0), (S2), induction on ordinals establishes

$$\mathrm{Rn}(x, y) \wedge \mathrm{Rn}(x, y') \to y = y'.$$

Combining these facts with the logical validities

$$\mathrm{Rn}(x, y) \wedge x' \in x \to \exists y'(y' \in y \wedge \mathrm{Rn}(x', y')),$$
$$\mathrm{Rn}(x, y) \wedge t \subseteq u \in y \to t \in y,$$
$$\mathrm{Rn}(x, y) \wedge t \in u \in y \to t \in y$$

we see that (S4) is satisfied if $S(x)$ is the formula $\exists y \mathrm{Rn}(y, x)$.

We need some lemmas for **S** analogous to facts known for **A**. Although the proofs are nearly the same the results for **S** do not seem to follow from those for **A**.

LEMMA 3.1. *Let $\theta(\beta, x)$ be an ϵ-formula with free variables β, x. Then in* **S** *we have*

$$x \in V \wedge \exists \beta_0(\mathrm{Ord}(\beta_0) \wedge \theta(\beta_0, x)) \to (\exists \beta_1 \in V)(\mathrm{Ord}(\beta_1) \wedge \theta(\beta_1, x)).$$

Proof. Let $\psi(\gamma)$ be the formula $\mathrm{Ord}(\gamma) \wedge (\forall \gamma' \leqslant \gamma) \neg \theta(\gamma, x)$. Let β_0 be such that $\mathrm{Ord}(\beta_0) \wedge \theta(\beta_0, x)$. Now we claim that $\forall \gamma(\psi(\gamma, x) \to \gamma \in \beta_0)$. For suppose that $\psi(\gamma, x)$. Then $\mathrm{Ord}(\gamma)$, and hence $\gamma \in \beta_0$ or $\beta_0 \in \gamma$ or $\beta_0 = \gamma$. The last two cases are excluded because $\theta(\beta_0, x)$. Therefore $\gamma \in \beta_0$ as desired. Since the formula ψ involves only x, and we assume $x \in V$, the schema (S3) gives $(\exists \beta \in V) \forall \gamma(\psi(\gamma, x) \to \gamma \in \beta)$.

Now let $\beta_1 = \{\gamma \in \beta: \psi(\gamma, x)\}$. Clearly $\beta_1 \in V$. Moreover β_1 is an ordinal: it is obviously transitive, and by remarks above any transitive set of ordinals is an ordinal. We have $\beta_1 \in \beta_0$ or $\beta_0 \in \beta_1$ or $\beta_1 = \beta_0$. In the last two cases, clearly $\neg\psi(\beta_1)$, since $\theta(\beta_0, x)$. In the first case, if $\psi(\beta_1)$ then $\beta_1 \in \beta$, which is impossible (by the definition of ordinal). Hence $\neg\psi(\beta_1)$. Since $\gamma \in \beta_1$ implies $\neg\theta(\gamma, x)$, this means we must have $\theta(\beta_1, x)$. Thus $\beta_1 \in V \wedge \mathrm{Ord}(\beta_1) \wedge \theta(\beta_1, x)$, as desired.

COROLLARY 3.2. a) $(\forall x \in V)(\exists y \in V)\mathrm{Rn}(x, y)$, b) $\forall x \exists y \mathrm{Rn}(x, y)$.

Proof. a) follows from the remarks of § 2, since it is a theorem of *A*. Alternatively, a) is easily proved by induction using the schema (S3). b) then follows by Lemma 3.1.

THEOREM 3.3. *Let $U(x)$ be the formula $\exists y(S(y) \wedge x \in y)$. Then* $\mathbf{S}^*$ *is interpretable in* **S** *by interpreting $\exists x \ldots$ as $\exists x(U(x) \wedge \ldots)$, ϵ as ϵ, and V as $\{x \in V: U(x)\}$.*

Proof. First we show that if $\alpha = \{x \in V: \mathrm{Ord}(x)\}$, then $R_\alpha = \{x \in V: U(x)\}$ (clearly α is an ordinal). Note that if $\beta \in \alpha$, then $R_\beta \in V$. Thus since $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$, we have $R_\alpha \subseteq V$. Clearly then, $R_\alpha \subseteq \{x \in V: U(x)\}$. (Recall that $S(x) \leftrightarrow \exists \beta\, x = R_\beta$). Now suppose that $x \in V$ and $\exists \beta\, x \in R_\beta$. By Lemma 3.1 (applied to the formula $x \in R_\beta$) there must be $\beta \in V$ so that $x \in R_\beta$. But then $x \in R_\alpha$, so $R_\alpha = \{x \in V: U(x)\}$, as desired.

Next we show that $U(R_\alpha)$. In fact, for all β, $\mathrm{Ord}(\beta) \to U(R_\beta)$: by 3.1 it is enough to check that $(\forall \beta \in \alpha)\, U(R_\beta)$, which is easy using Corollary 3.2a and § 2.

Now since $\mathbf{S} \vdash U(x) \wedge y \in x \to U(y)$, it is easy to see that the interpretation of (S0) is provable in **S**. (S1) is also easy. Use $\mathbf{S} \vdash U(x) \wedge y \subseteq x \to U(y)$ to get (S2). We now check (S3). Suppose

$$a_1, \ldots, a_n \in R_\alpha \wedge \exists^U x \forall^U t(\varphi^U(a_1, \ldots, a_n, t) \to t \in x),$$

where the superscript U indicates relativization of quantifiers to the formula $U(x)$. Clearly $a_1, \ldots, a_n \in V$. Also

$$\exists x \forall t(\varphi^U(a_1, \ldots, t) \wedge U(t) \to t \in x)$$

and U is an ϵ-formula, so (S3) gives

$$\exists x \in V \forall t(\varphi^U(a_1, \ldots) \wedge U(t) \to t \in x).$$

Evidently if $x' = \{t \in x: U(t)\}$, we still have

$$\forall t(\varphi^U(a_1, \ldots) \wedge U(t) \to t \in x').$$

Moreover, $x' \subseteq R_\alpha = \{t \in V: U(t)\}$, so that $x' \in R_{\alpha+1}$ and hence $U(x')$. Thus since $x' \subseteq x \in V$, $x' \in R_\alpha$ and

$$(\exists x' \in R_\alpha)\forall^U t(\varphi^U(a_1, \ldots) \to t \in x'),$$

which shows that the interpretation of (S3) is provable in **S**.

§ 4. Replacement and related topics.

THEOREM 4.1. *The theory* $\mathbf{S}^*$ *satisfies the reflection schema*

$$\mathbf{S}^* \vdash a_1, \ldots, a_n \in V \to (\varphi^V \leftrightarrow \varphi)$$

where φ is any ϵ-formula with free variables among $a_1, \ldots, a_n$, and φ^V is obtained from φ by relativizing all quantifiers to V.

COROLLARY 4.2. *If σ is any theorem of* **ZF**, *then* $\mathbf{S}^* \vdash \sigma$ *and* $\mathbf{S}^* \vdash \sigma^V$.

Proof of Corollary 4.2. See e.g. [7]. Because of its relevance to the next section we give here a simple argument for the replacement axiom (the technicalities mentioned in § 2 are taken care of by Theorem 4.1). This argument is due to Ackermann and is close to an intuitive argument given by Shoenfield using S. The situation is rather curious because one feels the intuitive argument shows that V satisfies the full replacement axiom (any function $F\subseteq V$ with domain in V is in V), but for technical reasons in $\mathbf{S}^*$ it only gives the replacement schema of **ZF**.

First suppose that $\varphi(x,y)$ gives y as a function of x, where φ is an ϵ-formula, and that if $x\in V$ and $\varphi(x,y)$ then $y\in V$. Then for all $a\in V$, $\{y\colon (\exists x\in a)\varphi(x,y)\}\subseteq V$, and hence the image of a under the function exists in V. (Evidently the argument still works if φ has parameters from V.) Clearly if every function $F\subseteq V$ were represented by such a formula, or even by a property to which Schoenfield's principle applies, this argument would give the replacement axiom. The replacement schema of **ZF** involves formulas in which (according to the interpretation of "x exists" as "$x\in V$") all quantifiers are relativized to V. Thus to make the argument yield the **ZF** replacement schema here, we must apply Theorem 4.1.

Proof of Theorem 4.1. It is enough to show that

$$x_1,\ldots,x_n\in V\wedge\exists x_0\varphi\to(\exists x_0\in V)\varphi\,, \tag{1}$$

for any ϵ-formula with free variables among $x_0,\ldots,x_n$. So suppose that $\exists x_0\varphi$. By regularity, (S4.3) we may choose y_1 so that $S(y_1)$, $x_0\in y_1$ and φ, and z so that $y_1\in z$ and $S(z)$. Let $z_0=\{y\in z\colon S(y)\wedge(\exists x_0\in y)\varphi\}$. Now $y_1\in z_0$, so by regularity, (S4.2) there is $y_0\in z_0$ so that $y\in z_0\to y_0\subseteq y$. For y_0 we have

$$t\in y_0\leftrightarrow\forall y\big(S(y)\wedge(\exists x_0\in y)\varphi\to t\in y\big)\,. \tag{2}$$

From left to right is because $y_0\in z_0$. For the other direction suppose $S(y)\wedge(\exists x_0\in y)\varphi$. Either $y\subseteq y_1$ or $y_1\subseteq y$. If $y\subseteq y_1\in z$, then $y\in z\wedge S(z)$, so $y\in z$. But then $y\in z_0$ and so $y_0\subseteq y$. If $y_1\subseteq y$, then since $y_1\in z_0$, $y_0\subseteq y_1\subseteq y$.

Now from (2) and (S3) it follows that there is $y\in V$ so that $y_0\subseteq y\in V$. Since $y_0\in z_0$, $(\exists x_0\in y_0)\varphi$ and thus by (S1), $(\exists x_0\in V)\varphi$ as desired.

Remark 4.3. Theorem 4.1 shows that the theory $\mathbf{S}^*$ is the same as the theory Sb of Lévy and Vaught [7]. Thus only the axiomatization is new. In [9] this theory is called **ZA**. Notice that the schema (1) is ostensibly stronger than (S3), but is a natural reflection principle. Exchanging (S3) for (1) yields an axiomatic theory $\mathbf{S}'$ which works more smoothly than **S** (regularity is no longer needed for Theorem 4.1). I do not know if $\mathbf{S}'$ is actually stronger than **S**.

§ 5. Axiomatizing properties; the theory $\mathbf{S}^+$. In this section we extend the theory $\mathbf{S}^*$ to a new theory $\mathbf{S}^+$ by enriching the notion of property. Moreover, this is done in such a way that the intuitive argument (§ 4) for the replacement axiom in V will work directly in $\mathbf{S}^+$ (Theorem 5.8). We also indicate how to make a similar extension of $\mathbf{A}^*$ (Remark 5.13). In Theorems 5.7, 5.11 and the discussion following Definition 5.3 we analyze the way in which $\mathbf{S}^+$ extends $\mathbf{S}^*$.

For the language of $\mathbf{S}^+$ it will be convenient (but not essential; see 5.5, 5.6) to introduce (in addition to ϵ, V) two new unary predicate symbols U and P. The interpretation of $U(x)$ (in $\mathbf{S}^+$) will be the same as the interpretation of the formula $U(x)$ of Theorem 3.3, namely "x is regular." We interpret $P(x)$ as "x is an (existing) property of sets". Whereas the quantifiers of $\mathbf{S}^*$ ranged over "imaginable" sets x satisfying $U(x)$, the quantifiers in $\mathbf{S}^+$ range over $U\cup P$. In stating the axioms, and in the sequel, we *assume the upper case letter Q* (with or without subscripts or superscripts) *is relativized to P*; that is, $\exists Q$ means $\exists Q\big(P(Q)\wedge\ldots\big)$. Relativization to U or V, respectively $\exists x\big(U(x)\wedge\ldots\big)$ or $\exists x(x\in V\wedge\ldots)$, is indicated by $\exists^U x$ or $\exists^V x$. All free variables are understood to be universally quantified.

The axioms of $\mathbf{S}^+$ are:

(S0) (extensionality)

$$\forall t(t\in x\leftrightarrow t\in y)\to x=y\,,$$

(S1) (closure of V)

$$t\in u\wedge u\in V\to t\in V\,,\qquad x\subseteq u\wedge u\in V\to x\in V$$

where $x\subseteq u$ is the formula $\forall t(t\in x\to t\in u)$.

(S2) (Zermelo principle)

$$\forall^U r\exists^U z\forall t(t\in z\leftrightarrow t\in r\wedge\theta)$$

where θ is any formula not involving z.

(S3.1) (construction of properties)

$$u_1,\ldots,u_n\in V\to\exists Q\forall^U r\big(r\in Q\leftrightarrow\theta^U(Q_1,\ldots,Q_n,u_1,\ldots,u_n,r)\big)$$

where θ is any ϵ-formula whose free variables are among $Q_1,\ldots,Q_n$, $u_1,\ldots,u_n,r$ and Q is distinct from all these. θ^U has all quantifiers relativized to U.

(S3.2) (Shoenfield principle)

$$\exists^U r\forall v(v\in Q\to v\in r)\to\exists^V u\forall v(v\in Q\to v\in u)\,.$$

Notice that (S3.2) generalizes (S3) (because of (S3.1)).

(S3.3) (non-constructive property existence principle)

$$w\subseteq V\to\exists Q\forall^V t(t\in w\leftrightarrow t\in Q)\,,$$

(S4) (regularity)

$$\forall x\exists y,u\big(\mathrm{Rn}(y,u)\wedge x\in u\big)$$

where $\mathrm{Rn}(y, u)$ is a formula expressing "y is an ordinal and u is the sets of rank less than y" (see § 3).

(S5) $U(V)$,

(S6) (normalization)

$$x \in y \to U(x), \quad P(x) \vee U(x).$$

DEFINITION 5.1. a) If L is any language including symbols ϵ, V, U, P, then $S^+(L)$ is the theory whose axioms are (S0)-(S6), where θ in the schema (S2) is any L-formula not involving z.

b) In case L contains only ϵ, V, to get $S^+(L)$ read $U(x)$ as $x = V \vee \vee \exists y(x \in y)$ and $P(x)$ as $x \in V \vee \forall y(x \notin y)$ (see 5.5, 5.6).

Remark 5.2. a) In (S3.1) the restriction to ϵ-formulas θ is essential: we cannot allow a property Q such that $\forall^U r\,(r \in Q \leftrightarrow r \in V)$, as this would give $V \in V$ by (S3.3), (S5) (contradicting regularity). Similarly, since $x \in V \leftrightarrow \exists Q \forall t(t \in Q \leftrightarrow t = x)$, all quantifiers in θ must be relativized to U.

b) It is essential in the above axioms that the variables r, z be relativized to U. If r is not, the theory becomes inconsistent; if z is not it is weakened.

c) The axioms of S^+ can be read treating lower case variables as having range U. (If this is done one can still assume (S6), by reading "$x \in y$" as "$x \in y \wedge U(x)$" and restricting all quantifiers to $P(x) \vee U(x)$; all the other axioms are preserved). This has the effect of dropping the following two assumptions:

1. extensionality for properties (S0),
2. $Q \subseteq t \in V$ is identified with a set in V (S1).

To clarify the role that properties play in the arguments of this section, we avoid these assumptions when possible (e.g. in Theorems 5.8, 5.12).

To facilitate comparison with other theories, we consider two subtheories of S^+.

DEFINITION 5.3. The theory S^1 is obtained from S^+ by weakening the Zermelo schema (S2). Namely, it is required that all quantifiers in θ be relativized to U; θ may however have free property variables. The theory S^0 in addition omits (S3.3).

Note. The weakened version of (S2) which S^0 has is redundant in the language with ϵ, V as it follows from (S3.1) and (S1).

The theories S^0, S^1 clarify the way in which S^+ strengthens S^*. S^0 introduces properties as objects, but is an inessential extension of S^* in the same way that Gödel-Bernays (GB) is an inessential extension of **ZF**. Namely, if M is any model of S^*, and P is taken to be the collection of all subsets of M definable using parameters from V, then $M \cup P$ is a model of S^0 (see Theorem 5.7). Indeed, the theories S^0 and S^1 are very

closely related to GB (Theorem 5.11). The (non constructive) principle (S3.3) for the existence of properties makes it possible to carry out in S^1 the intuitive argument (given in § 4) for the replacement axiom over V. (Moreover, the assumption (S3.3) seems to be exactly what is needed to make the argument work.) S^+ differs from S^1 only by having the full Zermelo schema rather than the weakened version necessary for Theorem 5.7. It turns out however that S^+ greatly strengthens S^1 (Theorems 5.12, 7.3). This circumstance perhaps makes it easier to judge the merits of (S3.3). One could suggest that (S3.3) rests on a *confusion* of two notions, that of "class" and that of "property" (the latter necessarily associated with formulas), rather than on an insight into the nature of properties. If this is so, one might (*) expect something to be wrong with S^1. The relative weakness of S^1 seems to indicate this is not the case. In this connection it should be mentioned that Solovay has shown that $\varkappa \to (\omega)^{<\omega}$ yields models of S^1 in which $V = R_\alpha$ and $\alpha < \varkappa$; from this and work of Silver it follows that S^1 is compatible with $V = L$.

Since S^+ is obtained from S^1 by a separation schema, a comment on Zermelo's principle seems appropriate here. In the Zermelo principle

$$\forall^V x\, \mathrm{E}^V z \forall t(t \in z \leftrightarrow t \in x \wedge \theta),$$

or even in

$$\exists^U z \forall t(t \in z \leftrightarrow t \subseteq V \wedge \theta),$$

any restriction whatever on the language in which θ is formulated is extremely unnatural (**). The only condition should be that for each $t \in V$ (resp. $t \subseteq V$), θ makes a meaningful assertion (true or false) about t. Of course if θ is indefinite in the sense that its meaning depends on some hidden parameter, the parameter should be fixed.

We observe that S^0 does extend S:

THEOREM 5.4. *If $S^* \vdash \sigma$, and σ^U is obtained from σ by relativizing all quantifiers to U, then $S^0 \vdash \sigma^U$.*

Proof. Notice that because of (S6), in the axioms of S^+ it does not matter whether the variables t, u, v are relativized to U or not. The axioms are now immediate; because of (S3.1) every formula occurring in the schema (S3) is represented by a property, so (S3.2) yields (S3).

THEOREM 5.5. *In S^0,*

i) $U(X) \leftrightarrow \exists Y(X \in Y)$,
ii) $U(X) \leftrightarrow X = V \vee \exists Y(X \in Y)$,
iii) $P(X) \leftrightarrow X \in V \vee \neg \exists Y(X \in Y)$.

(*) Caution is needed here; there are of course consistent set theories with no natural models.

(**) See Zermelo, *Über Grenzzahlen und Mengenbereiche*, Fund. Math. 16 (1930), p. 30, footnote.

Proof. i) is immediate from Theorem 5.3 and the assumption $X\in Y\to U(X)$. Thus ii) follows from S5. To see iii) first argue from left to right. Suppose $P(Q)$ and $\exists Y(Q\in Y)$. Then by i) $U(Q)$. But $\forall x(x\in Q\to x\in Q)$ so by (S3.2) there is $u\in V$ so that $\forall x(x\in Q\to x\in u)$. By (S2) there is $u'\in V$ so that $\forall x(x\in Q\leftrightarrow x\in u')$, so by (S0) $Q=u'$. Thus $Q\in V$. Next suppose the right hand side. If $\neg\exists Y(X\in Y)$, then by i) $\neg U(X)$, so since $P(X)\vee U(X)$, $P(X)$ as desired. If $X\in V$, then by (S3.1) $\exists Q\forall t(t\in X\leftrightarrow t\in Q)$, so by (S0), $X=Q$ and hence $P(X)$.

Remark 5.6. If U, P are not taken as new symbols, but defined by ii), iii) above, then $X\in Y\to U(X)$ and $P(X)\vee Q(X)$ follow immediately, as well as (S5).

THEOREM 5.7. *Let $M=\langle M_0,\in^M,V^M\rangle$ be any model of* $\mathbf{S}^*$ *(but for convenience assume $X\subseteq M_0\to X\notin M_0$). Let $P=\{X\subseteq M_0: X$ is definable in $(M,a)_{a\in A}\}$, where $A=\{a\in M_0: a\in^M V^M\}$. Identify $a\in V^M$ with $\{t\in M_0: t\in^M a\}\in P$. Suppose that $N=\langle N_0,\in^N,V^M,U^N,P^N\rangle$ is given by*

1) $N_0=M_0\cup P$; $U^N=M_0$, $P^N=P$,

2) *$x\in^N y$ iff either $x,y\in M_0$ and $x\in^M y$, or $x\in M$, $y\in P$ and $x\in y$.*

Then N is a model of $\mathbf{S}^0$.

Proof. Immediate from the axioms of $\mathbf{S}^0$ and $\mathbf{S}^*$.

In the next theorem we use the argument of Corollary 4.2 (and regularity) to show that the replacement axiom holds.

THEOREM 5.8. (Replacement axiom). *In* $\mathbf{S}^+$ *(indeed in S^1) we can prove: If $f\subseteq V$ is a function and $x\in V$, then $\{f(t): t\in x\}\in V$.*

Proof. First suppose that $f\colon V\to\varkappa$, where $\varkappa=\{a\in V: a$ is an ordinal$\}$. By (S3.3) we choose a property Q so that

$$\forall^V z(z\in f\leftrightarrow z\in Q'). \tag{1}$$

Now it is not immediate that Q is a function, i.e. that

$$\langle t,a\rangle,\langle t,a'\rangle\in Q\to a=a'.$$

However, by (S3.1) there is a property Q' so that

$$\langle t,a\rangle\in Q'\leftrightarrow \mathrm{Ord}(a)\wedge\langle t,a\rangle\in Q\wedge(\forall a'<a)\langle t,a'\rangle\notin Q',$$

and it is immediate that Q' is a function (using the comparability of ordinals). Moreover we still have (1) when Q is replaced by Q'. Now evidently for each $x\in V$,

$$(\exists t\in x)\langle t,a\rangle\in Q'\to a\in\varkappa\subseteq V,$$

because for $t\in V$, $\langle t,a\rangle\in Q'$ implies $a=f(t)$, and $f\colon V\to\varkappa$. Thus by (S3.1), (S3.2) there is $\lambda\in\varkappa$ such that

$$(\exists t\in x)\langle t,a\rangle\in Q'\to a\in\lambda,$$

and therefore

$$\{f(t): t\in x\}=\{a: (\exists t\in x)\langle t,a\rangle\in Q'\}\subseteq\lambda\in V.$$

By (S1) this implies $\{f(t): t\in x\}\in V$ as desired.

So far the proof is very close to the intuitive argument of § 4. However, to take care of arbitrary $g\colon V\to V$, we use regularity. For each $t\in V$ we can by regularity choose $a\in\varkappa$ so that $g(t)\in R_a$. Let $f(t)$ be the least such ordinal a. Now if $x\in V$, $\lambda=\bigcup\{f(t): t\in x\}\in V$, and clearly $\{g(t): t\in x\}\subseteq R_\lambda\in V$, so by (S1) the g image of x is in V. This completes the proof.

Remark. Theorem 5.8 shows that axiom (S3.3) is sufficient to make the intuitive argument for replacement work in the setting of $\mathbf{S}^0$. It is not easy to think of a reasonable weaker assumption with the same effect. (Although it can be done; D. Perlis recently formulated such assumptions.) However, with (S2), (S3.3) already leads to assumptions much stronger than inaccessible cardinals. In Theorem 5.12 we show it yields measurable cardinals.

Next we observe that a reflection principle holds in $\mathbf{S}^+$.

THEOREM 5.9. *In* $\mathbf{S}^+$ *(or* $\mathbf{S}^0$*) we have*

a) $\exists^U x(x\in Q)\to\exists^V x(x\in Q)$,

b) $V\in Q\to\exists^V x(x\in Q)$.

Proof. Suppose $x\in Q$ and $U(x)$. Then by regularity there is a stage s (or ordinal a, and put $s=R_a$) such that $x\in s$ and $U(s)$. There is in fact a least such stage s, and this s corresponds to a property Q_0. Indeed,

$$u\in s\leftrightarrow\forall s[S(s)\wedge(\exists x\in s)(x\in Q)\to u\in x]$$

so by (S3.1) there is a property Q_0 such that

$$\forall u(u\in s\leftrightarrow u\in Q_0).$$

Thus since $U(s)$, (S3.2) gives an $s'\in V$ such that $\forall u(u\in Q_0\to u\in s')$. Hence $s\subseteq s'\in V$, and so $s\in V$. But by choice of s, $(\exists x\in s)(x\in Q)$, so $\exists^V x(x\in Q)$ as desired. Part b) is immediate from a) since $U(V)$.

Remark. As in Remark 4.3, exchanging (S3.2) for the reflection principle 5.9a yields an alternative axiomatization of $\mathbf{S}^+$ (or $\mathbf{S}^0$).

COROLLARY 5.10. *In* $\mathbf{S}^+$ *(or* $\mathbf{S}^0$*) we have the schema*

$$x_1,\ldots,x_n\in V\to[\theta^V(Q_1,\ldots,Q_n,x_1,\ldots,x_n)\leftrightarrow\theta^U(Q_1,\ldots,Q_n,x_1,\ldots,x_n)]$$

where θ is an $\in$-formula, and the Q_i, x_i exhaust the free variables of θ.

Proof. It is enough to show that

$$\exists^U x_1\,\theta^U(Q_1,\ldots,Q_n,x_1,\ldots,x_n)\to\exists^V x_1\,\theta^U(Q_1,\ldots,Q_n,x_1,\ldots,x_n)$$

for $x_i \in V$. But $\theta^U(Q_1 \ldots)$ is represented by a property Q, and the hypothesis asserts $\exists^U x_1(x_1 \in Q)$, so by Theorem 5.9, $\exists^V x_1(x_1 \in Q)$, and the conclusion holds.

Recall that Gödel-Bernays set theory **GB** (as presented in [3]) is formulated in a language with similarity type (i.e. nonlogical symbols) $t = \{\epsilon, \mathfrak{M}, \mathrm{Cls}\}$, where $\mathfrak{M}(x)$ means "x is a set", and $\mathrm{Cls}(x)$ means "x is a class". In the interpretation in [3] (see p. 2), classes are what appear in Zermelo's formulation as "definite Eigenschaften". The type of $\mathbf{S}^+$ is $s = \{\epsilon, V, U, P\}$. In $\mathbf{S}^+$, V may be considered as a unary predicate $(V(x) \leftrightarrow x \in V)$ rather than an individual constant; this will facilitate comparisons of structures of types t and s. The role of Cls in **GB** is played by P in $\mathbf{S}^+$. However, the role of $\mathfrak{M}$ in **GB** splits into two roles in $\mathbf{S}^+$: the role of V and the role of U. This is made precise in the next theorem.

Given a structure M of type t, it induces a structure M_s of type s (with universe Cls^M) by the rule $(\epsilon, V, U, P) \to (\epsilon, \mathfrak{M}, \mathfrak{M}, \mathrm{Cls})$. A structure N of type s induces structures $N_{t,1}, N_{t,2}$ of type t (with the same universe as N) by the rules $(\epsilon, \mathfrak{M}, \mathrm{Cls}) \to (\epsilon, V, P)$, $\to (\epsilon, U, P)$ respectively.

In the next theorem we suppose that N, N_0 are structures of type s, and M_1, M_2 are of type t. Also we assume $N_0 = M_s$, $M_1 = N_{t,1}$, $M_2 = N_{t,2}$ and that in N, $P(x) \vee V(x) \vee U(x)$. Recall that **GB** has among its axioms

(A1) $$\mathfrak{M}(x) \to \mathrm{Cls}(x),$$

(A2) $$\mathrm{Cls}(x) \wedge \mathrm{Cls}(y) \wedge x \in y \to \mathfrak{M}(x),$$

and also an axiom of choice (axiom E). The theory $\mathbf{GB}_1$ has the axioms of **GB** except choice, and has also a modification of (A2).

(A2′) $$x \in y \wedge \mathfrak{M}(y) \to \mathfrak{M}(x).$$

$\mathbf{GB}_2$ omits both E and (A1), but has

(A2″) $$x \in y \wedge [\mathfrak{M}(y) \vee \mathrm{Cls}(y)] \to \mathfrak{M}(x).$$

(It is not assumed in **GB** that $\forall x[\mathrm{Cls}(x) \vee \mathfrak{M}(x)]$.)

THEOREM 5.11. a) *Suppose M is a model of* **GB**. *Then N_0 (see above) is a model of the axioms of $\mathbf{S}^1$ except for* (S5).

b) *If N is a model of $\mathbf{S}^1$, then M_1 is a model of $\mathbf{GB}_1$, and M_2 of $\mathbf{GB}_2$.*

c) *If N satisfies* (S5) *and also M_1, M_2 are models of $\mathbf{GB}_1$, $\mathbf{GB}_2$ respectively, then N is a model of $\mathbf{S}^0$. Explicitly,* (S5) *is*

$$\exists^U x \forall t(t \in x \leftrightarrow V(t)).$$

Proof of a), b). Part a) is immediate from the axioms. Part b) follows from Theorem 5.8 (replacement) and Corollary 5.10 (reflection) as follows. For M_2, the axioms of group B (class existence axioms) are immediate from (S3.1), (A2″) from (S6), (A3) (extensionality) from (S0) and (S6), and D (regularity) from (S4) and (S6). For M_1, the axioms of group C (set existence axioms) are immediate by Theorem 5.4 and Theorem 5.8, as is (A4) (pairing); A1 follows from (S3.1) and (S0) or (S1). Now by Corollary 5.10, (A4) and the axioms of C go up to M_2, and (A3) and D go down to M_1. The axioms of group B go down to M_1 because by (S6), $x \in V \to U(x)$. (A2′) is contained in (S1).

Proof of c). First we check (S6). By (S5) there is v so that $U(v)$ and $V(t) \to t \in v$. Thus by (A2″), $V(t) \to U(t)$. Now since the universe of N is $P^N \cup U^N \cup V^N$, we clearly have $P(x) \vee U(x)$ in N. Now suppose $y \in x$. By (A2″) we must have $U(y)$. Using (S6), (S0) follows immediately from the extensionality axiom (A3) for M_2.

Next we observe that (S3.1) holds. The axioms of group B for M_2 yield

(1) $$\exists Q \forall^U r(r \in Q \leftrightarrow \theta^U(Q_1, \ldots, Q_n)),$$

where the Q_i are in P. (This can be seen by checking through the proof in [3] of M1, p. 8.) Since (A1) for M_2 assures $V \subseteq P$, (S3.1) follows. Of course, (1) holds also with U replaced by V.

Because of the extensionality axiom for M_1, we have for each property Q that

$$\exists^U x(x \in Q) \to \exists^V x(x \in Q).$$

Using this and (1) for M_1, M_2 it is easy to check that

(2) $$\theta^V(Q_1, \ldots, Q_n) \leftrightarrow \theta^U(Q_1, \ldots, Q_n).$$

(Use the transitivity of V given by (A2′) for M, and the pairing axioms for M_1 and M_2).

We have observed the transitivity of V, $x \in y \in V \to x \in V$. To see (S1) we must check that $x \subseteq y \in V \to x \in V$. By the power set axiom for M_1, there is $p \in V$ so that

(3) $$\forall t(t \in p \leftrightarrow t \subseteq y)$$

holds in V. Since p is uniquely determined by (3), and (2) holds, the same p must satisfy (3) in U. Thus

$$x \subseteq y \wedge U(x) \to x \in p.$$

But since $p \in V$, the transitivity of V gives $x \in V$. If $x \subseteq y \wedge P(x)$, then by the replacement axiom for M_1 (or the weaker Aussonderungs) $x \in V$. This shows (S1) holds. Combining (S3.1) with this gives the weak version of (S2) needed for $\mathbf{S}^0$.

To prove (S3.2) we must show

$$Q \subseteq x \wedge U(x) \to (\exists y \in V)(Q \subseteq y).$$

Let Q' be the property defined (using (S3.1)) by

$$x \in Q' \leftrightarrow \forall^U t(t \in Q \to t \in x).$$

Since $\exists^U x\, x \in Q'$, we have $(\exists x \in V)(x \in Q')$, which shows (S3.2). In particular, if $U(x) \wedge P(x)$, then $x \in V$.

Regularity now follows easily from axiom D of **GB**.

THEOREM 5.12 (Proved in $\mathbf{S}^+$). *Let $\varkappa = \{\alpha \in V\colon \alpha$ is an ordinal$\}$. Then $\varkappa$ is a measurable cardinal, and moreover $\{\alpha \in \varkappa\colon \alpha$ is measurable$\}$ has normal measure* 1.

Proof. We define a $\varkappa$-complete nonprinciple ultrafilter $D \subseteq \mathfrak{P}\varkappa$ as follows. For each $X \subseteq \varkappa$, there is by (S3.2) a property Q such that

(1) $$\forall^V u(u \in X \leftrightarrow u \in Q)\,.$$

We write $X = Q^V$ if (1) holds; by extensionality, X is determined by Q. Put $X \in D$ if and only if $\varkappa \in Q$ for some Q such that $X = Q^V$. This defines D.

First we note that if $Q_1^V = Q_2^V$, then $\varkappa \in Q_1 \leftrightarrow \varkappa \in Q_2$. If not, then the symmetric difference $Q_1 \Delta Q_2$ is non empty, and so by the reflection principle, $(\exists u \in V)(u \in Q_1 \Delta Q_2)$. This means $u \in (Q_1 \Delta Q_2)^V = Q_1^V \Delta Q_2^V = 0$, a contradiction.

Now we show that D is an ultrafilter. Evidently for each Q with $Q^V \subseteq \varkappa$, $(\sim Q)^V = \varkappa \sim Q^V$. Thus for each $X \subseteq \varkappa$, $X \in D$ or $(\varkappa \sim X) \in D$. Suppose that $\lambda < \varkappa$, and X_ν $(\nu < \lambda)$ are in D. Define F by

$$F = \{\langle \nu, x\rangle \in V\colon\ \nu < \lambda \text{ and } x \in X_\nu\}\,.$$

Clearly for $\nu < \lambda$, $F^*(\nu) = X_\nu$. Choose Q_u so that $F = Q^V$. Consider a property Q_λ given by

$$u \in Q_\lambda \leftrightarrow (\forall \nu < \lambda)(\langle \nu, u\rangle \in Q)\,.$$

Clearly $Q_\lambda^V = \bigcap_{\nu<\lambda} X_\nu$. For $\nu < \lambda$, consider a property Q_ν given by

$$u \in Q_\nu \leftrightarrow \langle \nu, u\rangle \in Q\,.$$

Clearly for such Q_ν, $Q_\nu^V = X_\nu$. Since $X_\nu \in D$, this means $\varkappa \in X_\nu$, and hence $\langle \nu, \varkappa\rangle \in Q$. Consequently $(\forall \nu < \lambda)(\langle \nu, \varkappa\rangle \in Q)$, so $\varkappa \in Q_\lambda$. Since $Q_\lambda^V = \bigcap_{\nu<\lambda} X_\nu$, $\bigcap_{\nu<\lambda} X_\nu \in D$. This shows that D is $\varkappa$-complete.

To see D is non-principle, let $\nu \in \varkappa$ and suppose that $Q^V = \{\nu\}$. Now by the reflection principle, $\varkappa \in Q \wedge \varkappa \neq \nu \rightarrow (\exists \alpha < \varkappa)(\alpha \in Q \wedge \alpha \neq \nu)$, which means that $Q^V = \{\nu\}$. Thus $\varkappa \notin Q$, so $\{\nu\} \notin D$.

Next we observe that D is normal. Suppose that $f\colon \varkappa \to \varkappa$, and that

$$X = \{\nu \in \varkappa\colon f(\nu) < \nu\} \in D\,.$$

Choose Q_1, Q_2 so that $Q_1^V = f$, and

$$\forall u, t(\langle u, t\rangle \in Q_2 \leftrightarrow u = t)\,.$$

Now let Q satisfy

$$\forall \nu\big(\nu \in Q \leftrightarrow \exists x(\langle \nu, x\rangle \in Q_1 \wedge x < \nu)\big)\,.$$

Clearly $Q^V = X$. Thus since $X \in D$,

$$\exists x(\langle \varkappa, x\rangle \in Q_1 \wedge x < \varkappa)\,,$$

say $\lambda < \varkappa$ and $\langle \varkappa, \lambda\rangle \in Q_1$. Now let Q_2 satisfy

$$x \in Q_2 \leftrightarrow \langle x, \lambda\rangle \in Q_1\,.$$

Clearly $\varkappa \in Q_2$, so $Q_2^V \in D$. This means that

$$\{\nu \in \varkappa\colon \langle \nu, \lambda\rangle \in Q_1\} = \{\nu \in \varkappa\colon f(\nu) = \lambda\} \in D\,,$$

and hence that D is normal.

Finally we show that $X = \{\nu < \varkappa\colon \nu \text{ is measurable}\} \in D$. To see this it suffices to observe that if

$$x \in Q \leftrightarrow x \text{ is a measurable cardinal}\,,$$

then $Q^V = X$, and moreover $\varkappa \in Q$. This completes the proof.

Remark 5.13. It is easy to formulate a theory **T** bearing the same relation to Ackermann's theory $\mathbf{A}^*$ (with regularity) that $\mathbf{S}^+$ does to $\mathbf{S}^*$, in particular so that an analogue of Theorem 5.7 holds. It turns out that most of the results of this section hold for **T**. In particular, Theorems 5.8 and 5.9b hold, and the proofs are essentially the same. When working in such a theory one must use the special definition of ordinal devised by Lévy and Vaught [8] in order that $\varkappa = \{\alpha \in V\colon \mathrm{Ord}(\alpha)\}$ will be an ordinal comparable with every ordinal. In the proof of 5.8, one must write

$$\mathrm{Ord}(\alpha)\ \&\ \forall \beta\big(\mathrm{Ord}(\beta) \rightarrow \alpha \in \beta \vee \beta \in \alpha \vee \alpha = \beta\big)$$

instead of $\mathrm{Ord}(\alpha)$.

Explicitly, the axioms of **T** are those of S^+, but (S2) and (S3.2) are replaced by

(1) $$\exists^U z \forall t(t \in z \leftrightarrow t \in V \wedge \theta)$$

(where θ is any formula not involving z) and

$$\forall v(v \in Q \rightarrow v \in V) \rightarrow \exists^V u \forall v(v \in Q \rightarrow v \in u)$$

respectively.

To obtain Theorem 5.12 one must replace (1) by

$$\exists^U z \forall t(t \in z \leftrightarrow t \subseteq V \wedge \theta)\,,$$

θ any formula not involving z.

§ 6. The set theory of Powell. A very elegant axiomatization of set theory has been introduced recently by W. Powell (see [4]). Although the axioms for this theory appear rather different from those of $\mathbf{S}^+$, it can conveniently be described in terms of $\mathbf{S}^+$. Indeed, it is essentially equivalent to $\mathbf{S}^1$ (Theorem 6.1).

Because of (S3.3), for each $X \subseteq V$ there is a property Q with $X = \{x \in V\colon x \in Q\} = Q^V$. Moreover, because of extensionality and the reflection principle

$$Q = Q' \leftrightarrow \forall^V t(t \in Q \leftrightarrow t \in Q')\,.$$

Thus the correspondence $Q \leftrightarrow Q^V$ is 1-1 and onto. If we identify properties with subsets of V via this correspondence, then the universe of discourse is again U rather than $U \cup P$. To express $t \in Q$ for $t \notin V$, however, we must introduce a new binary relation (called predication, and written $\ni$ in Powell's theory) such that

$$t \in Q \leftrightarrow Q^V \ni t\,.$$

We can then define properties to be subsets of V:

$$P(x) \leftrightarrow \forall t(t \in x \to t \in V)\,.$$

When this is done we get the following (using Q, Q_i etc. as before to range over properties). From extensionality:

(P0) $$\forall^V t(t \in Q \leftrightarrow t \in Q') \to Q = Q'.$$

From (S1),

(P1) $$x \in y \wedge y \in V \to x \in V.$$

From (S2) we get the Zermelo schema

(P2) $$\forall x \exists z \forall t(t \in z \leftrightarrow t \in x \wedge \theta)$$

for any formula θ not involving z. Let us call an ϵ-formula θ normalized provided that i) if x is free in θ, $x \in y$ is not a subformula of θ, and ii) equality does not occur in θ. Any ϵ-formula may be normalized by replacing subformulas $x \in y$ by $\exists u(u = x \wedge u \in y)$, and then replacing subformulas $x = y$ by $\forall t(t \in x \leftrightarrow t \in y)$. Now if θ is any normalized ϵ-formula with free variables $Q_1, \ldots, Q_n, x_1, \ldots, x_n, t$ (distinct from Q), then (S3.1) translates as

(P3) $$x_1, \ldots, x_n \in V \to \exists Q \forall t\big(Q \ni t \leftrightarrow \bar{\theta}(Q_1, \ldots, Q_n, x_1, \ldots, x_n t)\big),$$

where $\bar{\theta}$ is obtained from θ by replacing $X \in Q_i$ by $Q_i \ni X$. We also have regularity, (P4) = (S4), and

(P5) $$\forall^V t(P \ni t \leftrightarrow t \in P)$$

which is immediate from the definition of $\ni$.

The axioms of Powell's theory **P** are (P0), (P1), (P3), (P4), and (P5) (together with choice, which we do not consider here). We now have the following.

THEOREM 6.1. a) *Under the interpretation of $\ni$ described above, any model of* $\mathbf{S}^1$ *becomes a model of* **P**.

b) *Conversely, any model of* **P** *becomes a model of* $\mathbf{S}^1$ *under the inverse process.*

c) *Similarly,* $\mathbf{S}^+$ *is equivalent to* **P** + (P2).

Proof. We have just observed a). We prove the converse. If M is any model of **P**, and $A = \{Q \in M\colon M \models Q \subseteq V\}$, let $P = \{Q'\colon Q \in A\}$ be any copy of A disjoint from M, and put $t \in Q' \leftrightarrow Q \ni t$ (and $t \in M$). Now (S3.1), (S3.3), (S4), (S5), and (S6) are immediate. Powell has shown that $\mathbf{ZF} \subseteq \mathbf{P}$, and moreover that the translation of Corollary 5.10 is provable in **P**. Corollary 5.10 gives (S0) immediately (from (P0)). To see (S1), note that by the power set axiom of **ZF**; $x \subseteq u \in V \to x \in u' \in V$, so by (P1), $x \in V$. (Of course $x \subseteq u \in V \to x \in V$ is needed in proving $\mathbf{ZF} \subseteq \mathbf{P}$, but that is not our concern here). Thus we have (S1). Now (S2) (the weak version which $\mathbf{S}^1$ has) follows immediately from (S1) and (S3.1).

The axiom (S2) is immediate from (P2).

As a corollary of Theorem 5.11, we observe that

THEOREM 6.2. *The theories* $\mathbf{S}^1$, **P** *are finitely axiomatizable.*

After seeing Powell's axioms, I observed the formal simplification of the axiomatization of $\mathbf{S}^1$ given in the next theorem. The simplified axioms no longer correspond directly to the ideas of Shoenfield or Ackermann.

THEOREM 6.3. *The three axioms* (S0), $x \subseteq y \in V \to x \in V$, *and* (S3.2) *of the theory* $\mathbf{S}^1$ (*formulated in the language with* ϵ, V *only*) *can be replaced by the single axiom*

(S0′) $$\forall^V x(x \in Q \leftrightarrow x \in Q') \to Q = Q'.$$

Proof. First we observe that (S0′) is a theorem of $\mathbf{S}^1$ (indeed of $\mathbf{S}^0$). This is immediate from (S0), (S6), and Corollary 5.10.

Now assume (S0′). We show that

(1) $$\exists^U x(x \in Q) \to \exists^V x(x \in Q)\,.$$

(It is easy to see, using (S3.1), that this is stronger than (S3.3).) By (S3.1) choose Q_0 so that

$$\forall t(t \in Q_0 \leftrightarrow t \neq t)\,.$$

Now if $\exists^U x(x \in Q)$, clearly $Q \neq Q_0$. Thus by (S0′), $\exists^V x(x \in Q)$. This proves (1). Now an easy induction shows that

(2) $$\theta^U(Q_1, \ldots, Q_n) \leftrightarrow \theta^V(Q_1, \ldots, Q_n)\,.$$

From the definition of P, $x \in V \to P(x)$, so (S0′) yields (for $x, y \in V$)

$$\forall^V t(t \in x \leftrightarrow t \in y) \to x = y$$

and
$$\forall^V t(t\in x \leftrightarrow t\in Q)\to x = Q\ .$$
By (2) we can replace V by U here, and since $P(x)\vee U(x)$ it follows that (S0) holds.

It remains to show that $x\subseteq y\in V\to x\in V$. Suppose $x\subseteq y\in V$. By the transitivity of V, $x\subseteq V$. Thus by (S3.3) there is Q so that
$$\forall^V t(t\in Q\leftrightarrow t\in x)\ .$$
Thus by (S0′), $Q = x$, so $P(x)$. We must now use (the weakened form of) (S2) to see that also $U(x)$: $y\in V$, so $\exists^U z\forall t(t\in z\leftrightarrow t\in y\wedge t\in x)$, therefore $z = x$ so $U(x)$. But now by (2), $\exists^U u(u = Q)\to \exists^V u(u = Q)$; thus since $x = Q$, $x\in V$.

§ 7. Natural models of S^1, S^+. In [4], Jech and Powell have discussed standard models $\langle M, \epsilon, \ni^*, V^*\rangle$ of P in which M is transitive (and ϵ is membership). Because of the discussion of § 6 this carries over to S^1.

Here we briefly discuss natural models of S^1.

Definition 7.1. A model $\langle M, \epsilon^*, V^*\rangle$ of S^1 is called *natural* if

i) there is an ordinal $\varkappa$ such that $V^* = R_\varkappa$,

ii) on $R_\varkappa$, ϵ^* is ϵ: $x\in R_\varkappa$ implies $\forall t(t\,\epsilon^*\, x\leftrightarrow t\in x)$,

iii) for all $X\subseteq V^*$, there is $Q\in M$ such that
$$(\forall t\in V^*)(t\in X\leftrightarrow t\,\epsilon^*\, Q)\ .$$

Definition 7.2. A cardinal $\varkappa$ is called *n-extendable* iff there is a cardinal λ, $\lambda > \varkappa$, and a function $f\colon R_{\varkappa+n}\to R_{\lambda+n}$, such that

i) $f(x) = x$ for $x\in R_\varkappa$,

ii) if $P = \{f(x)\colon x\in R_{\varkappa+n}\}$, then $\langle P, \epsilon\rangle \prec \langle R_{\lambda+n}, \epsilon\rangle$

(where $\prec$ means elementary substructure).

The n-extendable cardinals, and related topics, are to be discussed in a forthcoming paper by R. M. Solovay and the author, to be titled "Strong axioms of infinity and elementary embeddings".

Theorem 7.3. a) *There is a natural model of S^1 with $V^* = R_\varkappa$ iff $\varkappa$ is measurable.*

b) *Let $\varkappa$ be the first cardinal which admits a natural model of S^+, λ the first 1-extendable cardinal, μ the first measurable cardinal. Then $\mu < \varkappa < \lambda$.*

Proof. a) First suppose $\varkappa$ is measurable, and consider the structure $\langle R_{\varkappa+1}, \epsilon\rangle$ (which we write simply $R_{\varkappa+1}$, suppressing the ϵ). Let D be a $\varkappa$-complete, normal, nonprinciple, ultrafilter $D\subseteq \mathfrak{P}\varkappa$, and consider the ultraproduct $(R_{\varkappa+1})^\varkappa_D$. It is well-founded, so we assume it is realized as a transitive set. We have $R_{\varkappa+1}\cong_f M\prec (R_{\varkappa+1})^\varkappa_D$ by a cannonical embedding (such that for $x\in R_\varkappa$, $fx = x$). Moreover, $X\subseteq R_\varkappa$ implies $X\in M$ (because $X = \{t\in R_\varkappa\colon t\in f(X)\}$). Let $P = \{f(X)\colon X\in R_\varkappa\}$, $N = f(R_\varkappa)\cup P$, and consider the structure $\mathfrak{N} = \langle N, \epsilon, V^*\rangle$ where $V^* = R_\varkappa$. This structure is easily seen to be a model of S^1.

We can show moreover that $\mathfrak{N}$ is never a model of S^+ (By Theorem 5.12 it is obviously not a model of S^+ if $\varkappa$ is the first measurable cardinal). Let $D = \{X\subseteq \varkappa\colon \varkappa\in f(X)\}$. Clearly $D\subseteq R_{\varkappa+1}\in N$, and D is a definable subset of $\mathfrak{N}$. Thus to show that (S2) fails we need only show that $D\notin N$. Let $\varkappa$ be the first $\varkappa$ such that $D\in (R_\varkappa)^\varkappa_D$ for some D, as above. Now in $(R_\varkappa)^\varkappa_D$, $\varkappa$ is still the first such $\varkappa$, since the functions $g\colon \varkappa\to R_\varkappa$ from which the ultraproduct is constructed are in $R_{\varkappa+1}\subseteq (R_\varkappa)^\varkappa_D$. Now by the elementary embedding, $f(\varkappa)$ is also the first such cardinal, but since $f(\varkappa) > \varkappa$ this gives a contradiction.

Next suppose S^1 has a natural model $\mathfrak{N}$ with $V^* = R_\varkappa$. Because each $X\subseteq R_\varkappa$ is represented by a property of $\mathfrak{N}$, we have $\langle R_\varkappa, X\rangle_{X\subseteq R_\varkappa} \prec \langle U^{\mathfrak{N}}, Q_X\rangle_{X\subseteq R_\varkappa}$, where Q_X is the property representing X. Clearly $t\,\epsilon^{\mathfrak{N}}\, x\in R_\varkappa\to t\in R_\varkappa$, and thus by a characterization of measurability due to Keisler [5, Cor 3.8], $\varkappa$ is measurable. Alternatively, it is easy to check that $D = \{X\subseteq\varkappa\colon \varkappa\in Q_X\}$ is an ultrafilter of the required sort (e.g. follow the argument of Theorem 5.12).

b) Remarks in a) show that $\mu < \varkappa$. To see that $\varkappa < \lambda$, suppose $R_{\lambda+1}\to_f R_{\lambda'+1}$ as in Definition 7.2. It is easy to see that if $P = \{f(x)\colon x\subseteq R_\lambda\}$, then $\langle R_{\lambda'}\cup P, \epsilon, R_\lambda\rangle$ is a model of S^+. Moreover, the assertion that there are $\lambda < \lambda'$, $P\subseteq R_{\lambda'+1}$ which form such a model can be expressed in $R_{\lambda'+1}$, and hence is true in $R_{\lambda+1}$. It follows that $\varkappa$ is much smaller than λ.

The next two theorems state conditions on α, β, P under which $R_\beta\cup P$ is a model of S^+ (taking R_α for V, R_β for V, P for P, and ϵ for ϵ). If $\mathfrak{A}$ is a structure with underlying set A, and $P\subseteq \mathfrak{P}A$, we say that P is *closed under logical operations on* $\mathfrak{A}$ iff whenever Q' is definable (by a first order formula) in the structure $(\mathfrak{A}, Q)_{Q\in P}$ then $Q'\in P$.

Theorem 7.4. *Suppose $\alpha < \beta$ and $P\subseteq R_{\beta+1}$. Then $R_\beta\cup P$ is a model of S^+ iff*

i) *P is closed under logical operations on $\langle R_\beta, \epsilon\rangle$,*

ii) *$P\cap R_\beta = R_\alpha$,*

iii) *$R_{\alpha+1} = \{Q\cap R_\alpha\colon Q\in P\}$.*

When these conditions are satisfied, α is measurable, with many measurable cardinals below α.

Proof. This is simply a model theoretic restatement of the axioms of S^+, and of Theorem 5.12.

Theorem 7.5. *Let $\alpha < \beta$ and $P\subseteq R_{\beta+1}$. Suppose that $R_\alpha\cup P$, $R_\beta\cup P$ are (respectively) models of* **GB** *and* **GB** *except for* (A1) *(taking M to be R_α, resp. R_β, Cls to be P, and ϵ to be ϵ). Then $R_\beta\cup P$ is a model of S^0. Moreover,* *if* $\{Q \cap R_\alpha : Q \in P\} = R_{\alpha+1}$, *then* $R_\beta \cup P$ *is a model of* $\mathbf{S}^*$, *and* α *is measurable with many measurable cardinals below* α.

Proof. Theorems 5.11, 5.12. Observe that for α fixed, the least β providing a model of $\mathbf{S}^+$ (for some $P \subseteq R_{\beta+1}$) will have cofinality ω (by the downward Lowenheim-Skolem theorem). Thus β will not in general be measurable.

Intuitively, **S** appears stronger than **ZF**, since proper classes are allowed in **S** but not in **ZF**. Nevertheless, **S** and **ZF** are equiconsistent (by a finitary argument). However, **S** is stronger than **ZF** in a precise sense given by the next theorem. Of course **S** is also stronger than **ZF** in the (vague) sense that it more readily suggests strong theories such as $\mathbf{S}^+$.

Let us say a model $\langle M, \epsilon^*, V^* \rangle$ of **S** has standard sets in case

i) ϵ^* is ϵ on $V^*(t \epsilon^* x \leftrightarrow t \in x$ for all $x \epsilon^* V^*)$ and

ii) $\{x: x \epsilon^* V^*\}$ is transitive and equal to V^*.

Then $\{x \in V^*: x$ is an ordinal$\}$ is an ordinal, called the ordinal of the model.

THEOREM 7.6. *The first* α *such that there is a model of* **S** *with standard sets and ordinal* α *is greater than the first ordinal providing a standard model for* **ZF**.

Proof. In $\mathbf{S}^*$ it is easily proved that there is $\beta < V$ such that $R_\beta \equiv V$. Indeed, if T is $\mathrm{Th}\,V$, the first order theory of $\langle V, \epsilon\rangle$, then $T \in V$ and $\exists\alpha(T = \mathrm{Th}\,R_\alpha)$ and so $(\exists\beta \in V)(T = \mathrm{Th}\,R_\beta)$. Thus $\langle R_\beta \cap V^*, \epsilon\rangle$ is a standard model of **ZF** provided by $\beta < \alpha$.

§ 8. Philosophical remarks. The Zermelo-Fraenkel axioms express self evident principles in a natural and reasonably simple manner. We can say that the axioms successfully cannonize principles already accepted without further justification by mathematicians. These principles are in fact treated as self evident in mathematical practice, and were (with a qualification explained in the next paragraph) when the axioms were introduced. Moreover the axioms are adequate for classical mathematics, including classical set theory. Zermelo argued for his principles on just such grounds [12, especially § 2a]. One might hope for some more philosophical considerations, but what could be more persuasive to the working mathematician than to point out that he is already persuaded?

Three rather parenthetical remarks on "self-evident principles" may be helpful here. (1) Some contemporaries of Zermelo felt that principles countenancing a universal set were self evident; Zermelo saw that in practice only the separation axiom was used, and that it implies there is no universal set. (2) As every mathematician knows, one can have a good or bad intuition about something, or no intuition at all. Also one can have a good intuition about A which is misleading when brought to bear on B. It makes sense to speak of developing intuition about something. All this remains true when one deals with the basic (and

frequently tacit) presuppositions of mathematical thinking. These of course appear as axiomatic or self evident principles. (3) It is obvious that in discussions about the *philosophy* of mathematics basic mathematical principles were not (nor are they now) always treated as evident. Rather, there is a tendency to shift the grounds of discussion to presuppositions (usually still mathematical) which seem either even more secure, or even more basic, or even more general. Ideally, philosophy should provide a correct analysis of the relevant concept (or concepts) of evidence. The evidence for the correctness of this analysis, however, would presumably make appeal to some basic abstract (logical or mathematical) principles. So philosophical considerations might increase our understanding of the evidence of (or for) some principles without making it any more evident. Such considerations also might develop intuition and suggest stronger axioms, and this might lead eventually to increased confidence in the validity of the principles.

In formulating fundamental principles today we are still concerned with mathematical practice. The necessity for seeking such principles does not lie in the inadequacy of known principles for formalizing extant proofs, but in their incompleteness. Many questions of set theory arising in current practice are independent of the old axioms, including some from algebra and topology. The incompleteness extends also to number theory. (Although the questions known to be independent are more considered by logicians than by number theorists, the existence of such concrete problems makes incompleteness significant on nearly any philosophy of mathematics.) The task before us is not to reproduce mathematical practice but to extend it by finding true fundamental principles which it can utilize. Nevertheless we seek as much as possible to find self evident principles with antecedents in mathematical practice.

This section contains suggestions on the way in which our intuitive mathematical ideas can provide a source for new basic assumptions in mathematics. These suggestions are unfortunately neither very systematic nor very precise, but they may be appropriate in the present state of our knowledge. Also in this section are further remarks on the intuitions behind $\mathbf{S}^+$ and antecedents of these in mathematical practice.

The axioms of Zermelo are true in Cantor's universe: more specifically, in the cumulative hierarchy generated using Cantor's "absolutely limitless" system of ordinals. Here it is obvious that there is no universal set. This specification of "set" is implicit in Zermelo [12, 13]; he banished the universal set and clarified the operations used in forming the cumulative hierarchy. With this meaning of "set", the axioms of **ZFC** are intuitively evident.

It would be pleasant if "self evident" or "intuitively evident" principles were a) easy to find, b) a priori c) certain (especially in the sense of containing their own justification, as the term "self evident" suggests). They need not be easy to find, or Zermelo's principles would have been formulated by the early Greeks. The sense in which fundamental principles in mathematics can be certain and or a priori is somewhat problematic. A lack of absolute certainty in our basic presuppositions is at least suggested by the fact that it is possible to make mistakes in formulating such principles. (Is the Fregean "ideal set comprehension schema" a correct formulation of a bad presupposition, or a bad formulation of a correct principle?) Dedekind apparently felt that the principle of definition by recursion on integers was more certain after it was justified set theoretically [2, points (7) and (8)]. This may seem strange today, since there seems hardly anything more secure than such definitions. (Dedekind was perhaps impressed by the greater generality or richness of the set theoretic ideas. Or perhaps, for Dedekind, the evidence of the principle was not immediate, but mediate in something like the way indicated by his analysis.) Amplifying on an earlier discussion of Russell, Gödel suggested in [3a] that the justification for fundamental principles may lie (as in physics) in their consequences more than in their self evidence, and that this circumstance would affect the "absolute certainty" of such principles. To the extent that the consequences in question must be "looked for" in experience, it would also seem to affect the a priori character of mathematical principles. Although we may not have absolute certainty, we rightly have considerable confidence in the correctness of say Zermelo's principles or, if we adopt a reserved attitude about their ultimate meaningfulness, in the correctness of their meaningful — e.g. numerical-consequences. Moreover this confidence ordinarily appears very soon after one grasps the meaning of the principles (so that their justification seems implicit in their statement). This circumstance gives the principles an a priori flavor. They appear evident (as mentioned above) on the basis of intuitions which arise from no specific experience and can be brought into play by a description (of the cumulative hierarchy); this indicates that they are grounded in experience in a rather indirect (and perhaps subtle) manner (*). It may be that new principles will be found by expressing intuitions developed on the basis of experience with explicit formulations of old principles (**). Something like this appears to be the case with axioms of infinity in set theory (including, I think, those discussed in this paper). Here it appears to make sense to seek general principles which are either evident on mathematical

(*) There is a penetrating discussion of mathematical intuition in Gödel [3b].

(**) Gödel has observed that "we understand abstract terms more and more precisely as we go on using them, and that more and more abstract terms enter the sphere of our understanding". Cohen [1a, p. 14, 15] also discusses the development of intuition (emphasizing however "syntactic" intuitions developed considering formal axioms).

intuition or implicit in the meanings of the concepts involved. It should not be necessary to emphasize that the danger in too free reliance on appeals to intuition or meaning is that it will lead to arbitrary results rather than scientific ones. We may be able to extract mathematical knowledge from our intuition, but exactly how is not clear, and certainly it cannot be done in a scatter-brained manner. This is much less apparent with axioms concerning the richness of the power set operation, or with physical principles. In addition to seeking general principles which are themselves evident on mathematical intuition, one can also seek general principles with consequences in accord with particular intuitions (*). This is one way of implementing Gödel's suggestion above, is like what is done in physics, and may be appropriate for axioms on the power set. Many mathematicians believe in the sort of constructions made possible by the Hausdorff maximal principle, and therefore accept it, rather than finding it convincing in itself or believing it because of seeing the proof that it follows from the axiom of choice (which, despite all that has been said about it, most mathematicians do find self evident, when it is understood that the choice set need not be definable). Similarly one might accept Zermelo's separation principle because it unifies various more special principles such as definition by recursion. Stronger axioms of infinity can be obtained from weaker ones by answering the question: What general principle accounts for the particular axioms already presupposed? Concerning axioms on the power set operation, there is some tendency (independent of large cardinal considerations) to see $\exists x(x \subset \omega \,\&\, x \notin L)$ as intuitively correct. This may arise from the specific knowledge that it is a consistent assumption, or from some characteristic of Cohen's proof (which certainly develops some intuitions). The latter case would perhaps be more similar to the possibility mentioned above, although no suitably evident general principle based on these intuitions has yet been formulated. Concerning physics, the constraints imposed by experience arise more specifically from certain experiments. Although many physical intuitions do have an a priori or "unspecific" character (especially those appealed to in imaginary experiments), a good bit of physical intuition is quite self consiously a posteriori.

The axioms for properties expressed in $\mathbf{S}^+$ are supposed to be true in Cantor's set theory, and I hope the reader will agree they have been introduced in a natural way. Although properties and proper classes (as considered here) are not entrenched in mathematical practice in the way sets are, there are some antecedents. Constructive existence principles for properties (such as (S3.1)) are of course found in **GB**, and (implicitly) in Ackermann's set theory. Also, reflection principles and the Shoenfield

(*) The two approaches would be very close if it should be the case that our intuition tends to develop in the direction of unifying and generalizing its own contents.

principle (S3.2) seem very close to intuition. In order to see that a nonconstructive existence principle (such as (S3.3)) holds for properties, one must be clear what is meant by a property. (Just as with sets; for example the axiom of choice is not obvious for definable sets. Incidentally, since choice is true for sets, by reflection it is true also for classes or "imaginary" sets.) The discussion following Definition 5.3 may be helpful here. It may also be helpful to point out that properties correspond to what Cantor called systems or multiplicities (Vielheiten). Cantor distinguished between multiplicities whose extension can be comprehended in a unity (Einheit), and those which are essentially incomplete or unfinished. The former he called "sets", the latter "absolutely limitless". (Cantor also used the terminology "inconsistent multiplicity", since the assumption that a limitless multiplicity is a set leads to a contradiction.) The distinction can be given very elegantly in Powell's system: a property P is incomplete provided that no matter what entity E one takes to represent the extension of P, there will be an object $x \notin E$ which has the property P.

The impredicative **GB** (or Morse-Kelley-Tarski) theory in common use incorporates nonconstructive existence principles for proper classes. However it is frequently regarded philosophically not as a theory about classes, but about subsets of some R_θ where θ is inaccessible. Presumably any extensional theory of properties should allow such a natural model. Moreover, if it is the elements of R_θ which the model takes for sets, and we are dealing with a theory of arbitrary properties of sets (or classes of sets), then we expect that every subset of R_θ should correspond to a property. On the other hand, if the model takes something other than (all) elements of R_θ for sets, or if there is some restriction on the properties considered, then one need not expect that every subset of R_θ will correspond to a property. In the case of $\mathbf{S}^+$, such a natural model would have $V = R_\alpha$ for some $\alpha < \theta$, and $V \subseteq P \subseteq R_{\theta+1}$. The model takes for (existing) sets the elements of R_α, and properties of existing sets are unrestricted, but properties of (imaginable) sets (i.e., of elements of R_θ) are restricted to existing properties. Although every subset of R_α corresponds to a property, $P \neq R_{\theta+1}$. More generally, a theory with properties, properties of properties, etc. to level n would have natural models with $V \subseteq P \subseteq R_{\theta+n}$. In this connection notice that a correct form for the reflection principle 5.9a in case there is quantification over P is

$$\exists^U u\, \theta^{U,P}(u) \to \exists^V u\, (\exists P' \subseteq \mathfrak{P} u)\, \theta^{V,P'}(u)\,.$$

Concerning theories allowing properties of properties, etc. something more can be said. One way of formulating such theories leads to theories whose natural models are given by α, θ as above which admit an elementary embedding j of $R_{\alpha+\xi}$ into $R_{\theta+\xi'}$, where $ja = \theta$ and for all $x \in R_\alpha$,

$jx = x$. An α satisfying this last condition is called ξ-extendible to θ. Results of Kunen [6] show that one cannot fix α and consistently suppose there is an embedding which works simultaneously for every ordinal ξ. Thus $\xi = \xi' = \mathrm{OR}$ is impossible; however, if $\mathrm{cf}\,\lambda = \omega$, it is open whether $\xi = \xi' = \lambda + 1$ is possible.

Although today measurable cardinals are widely considered in set theory, most readers will not find the assumption "there exists a measurable cardinal" particularly natural as an axiom for Cantor's set theory. I know of no more natural way to introduce measurable cardinals than via properties as in Theorem 5.12. Of course the reader who is surprised that a proposed fundamental principle leads to measurable cardinals may either accept measurable cardinals or doubt the correctness of the fundamental principle. It seems that the acceptability of assumptions in mathematics, as in physics, is more influenced by the consequences derived from them than by their derivability from first principles. Here I refer to the opportunism mentioned by Cohen [1a], as much as to the circumstance mentioned above by Gödel. This opportunism may not be altogether bad, of course. Nevertheless I hope that considerations which base the existence of measurable cardinals on more fundamental principles may help clarify the proper place of measurable cardinals and other large cardinals in Cantor's set theory.

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Lusin density and Ceder's differentiable restrictions of arbitrary real functions

by

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Abstract. J. G. Ceder recently proved a theorem from which it follows that if A is an uncountable subset of the reals R, then for every $f\colon A\to R$, there exists a bilaterally dense in itself set $B\subset A$ such that $f|B$ is differentiable (infinite derivatives are allowed). Uncountability of A is necessary, and B cannot be made to have cardinality c (the cardinality of R). The main purpose of this paper is to characterize those sets $A\subset R$ for which it is true that for every $f\colon A\to R$, there exists a bilaterally c-dense in itself set $W\subset A$ and a dense in W set B such that $f|W$ is differentiable on B. A new notion of density results, and this notion is compared to known types of categoric density in metric spaces.

1. Introduction. A set B *is bilaterally dense* (c-*dense*) *in itself* if every closed interval containing an element of B contains points (c-many points) of B. A real function f *is differentiable at* x if and only if f is continuous at x, x is a limit point of the domain D_f of f, and it is true that there is an extended number m (possibly $+\infty$ or $-\infty$) such that if $\{x_n\}$ is a sequence of elements of $D_f-(x)$ converging to x, then $\{(f(x)-f(x_n))/(x-x_n)\}$ converges to m.

In [4] Ceder gives the following:

Theorem C. *If A is an uncountable number set, then for every $f\colon A\to R$, there exists a countable set $C\subset A$ such that for each $x\in A-C$ there exists a bilaterally dense in itself set $B\subset A-C$ containing x such that $f|B$ is differentiable and monotonic.*

B cannot be made to have cardinality c. Ceder's argument for the monotonicity part of Theorem C has a mistake in it, but a correction is given in [7], and a short alternative correction is given in this paper. It is easy to show that *if $A\subset R$ is countable, there exists $f\colon A\to R$ which has no continuous restriction to any dense in itself subset of A.*

The primary purpose here is to prove the following two theorems:

Theorem 1. *If A is an L_2 set, then for every $f\colon A\to R$, there exists an L_1 set $C\subset A$ such that for each $x\in A-C$ there exists a bilaterally c-dense in itself set $W\subset A-C$ and a dense in W set B containing x such that $f|W$ is differentiable on B.*