Classes of Dedekind finite cardinals

by

John Truss (Oxford)

Abstract. We discuss seven possible definitions of "finiteness" of cardinal numbers, and associate with each definition the class of cardinals "finite" in that sense. The results are extensions of those in Levy's paper "The independence of various definitions of finiteness" (Fund. Math. 46 (1958), pp. 1–13). We investigate the closure of the classes under addition, multiplication, unions, and disjoint unions. In the final section we give a wide variety of possible combinations of definitions and equalities between the classes. Also we give an affirmative answer to Tarski's question "Can there be exactly 2⁰ Dedekind finite cardinals?"

§ 1. The object of this paper is to investigate various properties of seven classes of Dedekind finite cardinals, and to discover what possible combinations of definitions and equalities can hold between them. The starting point is Levy's paper "The independence of various definitions of finiteness" [8]. All but two of the seven classes correspond to definitions of Levy (which were in turn taken from Tarski [18]), and the other two arise quite naturally.

The classes are defined in § 2, and some of their elementary properties given. In § 3 we discuss the closure of the classes under +, ×, unions, disjoint unions, and ⊂, and a model is given in which A is not closed under ×. In § 4 we show that if any two Dedekind finite cardinals are comparable, then any infinite set is the disjoint union of two infinite sets. § 5 contains five models which establish various possibilities of strict inclusion and equality between the classes. Various combinations of these models actually yield thirteen models in which the combination of definitions actually yield non-equality and equality of cardinals is different. It is shown that there can be no more than twenty-three possible combinations, using the results of § 2 and § 3. All of the ten unsolved cases involve the following situation: There is an infinite set with no infinite orderable subset, and if a set X has a countable partition, there is a map from X onto X which is not 1–1.

(1) This paper is part of the author's Ph. D. thesis at the University of Leeds. He would like to thank Dr. F. R. Drake and Prof. A. Levy for their supervision, and the Science Research Council for their financial support.
We do not know if this is possible, and suspect that it is.

One or two results are proved incidentally which are of interest apart from the immediate setting. One of them has been mentioned above, namely that if any two Dedekind finite cardinals are comparable, then \( \omega = \Delta \) (in the notation of \( \S 2 \)). This is a very small start to answering Tarski's question as to whether the comparability of any two members of \( \Delta \) implies that \( \omega = \Delta \). Another incidental result is a strengthening of L"{u}nscsl's result [6] that

\[ W: \text{any set of non-empty well-orderable sets has a choice function,} \]

does not imply the ordering principle.

We show that \( W \) does not even imply that any infinite set has an infinite ordered partition, and at the same time avoid any group-theoretical complications. (The model is due to Gauntt [3].)

Finally we show that in Halmos-Levy's model [4] (our \( \mathcal{R}_0 \) of Theorem 6), there are exactly \( 2^{2^\omega} \) Dedekind finite cardinals, thus answering another question of Tarski [17] (see [17]). We also show that \( \mathcal{R}_0 \) and \( \mathcal{R}_0 \) of Theorem 6 (which are Fraenkel-Mostowski models) satisfy \( |\mathcal{R}| = 2^{2^\omega} \), and hence \( \Delta \) can be well-ordered. In fact it follows easily that \( \mathcal{R}_0 \) and \( \mathcal{R}_0 \) satisfy "any set of cardinals can be well-ordered".

Most of our notation is standard. If \( x \) and \( y \) are cardinals, we write \( x \leq y \) to mean that whenever \( |X| = x \) and \( |Y| = y \), there is a mapping from a subset of \( X \) onto \( Y \). A class \( Q \) of cardinals is closed under unions if whenever \( |X| \in Q \) and \( \xi \in \mathcal{X} \to \{ \xi \} \in Q \), then \( \bigcup \mathcal{X} \in Q \). If whenever \( X \) is disjointed, then \( |\bigcup X| \in Q \). If \( \xi \in \mathcal{X} \to \{ \xi \} \in Q \), then \( \bigcup \mathcal{X} \in Q \). If \( \bigcup \mathcal{X} \in Q \) if whenever \( x, y \in Q \), then \( x \geq y \) respectively, and \( Q \) is closed under \( \leq \). If whenever \( x \in Q \), then \( y \in Q \).

\[ \S 2. \] The seven classes of Dedekind finite cardinals are as follows.

\[ \omega = \{ x: x \text{ a finite ordinal} \}. \]

(Proven now on "finite" means "has cardinal in \( \omega \)).

\[ A_1 = \{ x: x = y + z \text{ or } z \leq y \leq x \}, \]

\[ A_2 = \{ |X|: \text{any ordered partition of } X \text{ is finite} \}, \]

\[ A_3 = \{ |X|: \text{any ordered subset of } X \text{ is finite} \}, \]

\[ A_4 = \{ x: x \leq \n \}, \]

\[ A_5 = \{ x: x \neq \omega \}, \]

\[ A_6 = \{ x: x < \omega \}. \]

(*) This question was first brought to our attention by Dr J. E. Rubin. See her forthcoming joint paper with A. L. Rubin: "The cardinality of the set of Dedekind finite cardinals in Fraenkel-Mostowski models".

Of these, \( A_1 \) and \( A_4 \) are new, so far as we know, and the others appear in Levy [8]. The significance of \( A_1 \) is that it comprises all cardinals which can be cancelled additively from \( \leq \)-inequalities. \( A_4 \) is considered because of the obvious parallel with \( A_4 \), and that \( A_4 \) comprises just those cardinals finite in sense \( \Pi \) of Levy in shown now.

**Lemma 1.** \( |X| \in A_1 \)

\[ \leftrightarrow \text{any non-empty chain of subsets of } X \text{ has a maximal member} \]

\[ \leftrightarrow \text{any non-empty chain of subsets of } X \text{ has a minimal member}. \]

**Proof.** The equivalence of the last two is easy.

Suppose \( |X| \in A_1 \), and let \( U \) be any non-empty chain of subsets of \( X \).

For \( \xi \in X \) let \( A_6 = \{ x: x \in C \wedge \xi \in C \} \).

Define the partition \( \pi \) on \( X \) by \( \xi \eta \in \pi \) in the same manner as \( \pi \triangleright A_1 = A_\varnothing \).

Let \( \xi \) be the member of \( \pi \) containing \( \xi \).

\[ (\xi) \leq (\eta) \text{ if } A_4 \subseteq A_\varnothing. \]

Since \( A_1 \) is a chain, \( (\pi, <) \) is an ordered partition of \( X \). By definition of \( A_1 \), \( x \neq \pi \) is finite. If \( C = \{ \emptyset \} \), then \( \emptyset \) is maximal in \( C \). Otherwise, if \( C \neq \emptyset \), there is a \( (\xi) \in C \) such that \( A_4 \subseteq A_\varnothing \). Let \( \xi \) be the least such, and let \( \xi \in A_4 \).

It is clear that \( \xi \) is a maximal member of \( C \).

Conversely, suppose that any non-empty chain of subsets of \( X \) has a maximal element. Let \( (\pi, <) \) be an ordered partition of \( X \). To show that it is finite we need only show that \( < \) and \( \triangleright \) are well-orderings. Let \( P \) be a non-empty subset of \( \pi \), and let \( C = \{ \bigcup \mathcal{X}: \mathcal{X} \text{ a proper initial segment of } (P, <) \} \). Then \( C \) is a chain of subsets of \( X \), and is non-empty, as \( \emptyset \in C \). By our hypothesis, \( C \) has a maximal member. This gives a maximal proper initial segment of \( (P, <) \), and hence a maximal member of \( P \).

Similarly, by considering final segments, we see that \( P \) has a minimal member.

**Lemma 2.** \( x \in A_1 \leftrightarrow 2^x \in A. \)

This is due to Kuratowski, and is proved on pages 94, 95 of [18].

Thus \( A_1 \) comprises just those cardinals finite in sense \( \Pi \) of Levy. We shall like to have the analogue of this for \( A_4 \) and \( A_5 \), but suspect that it is in fact false (though it holds in all the models considered here).

**Lemma 3.** \( \omega \in C \iff A \subseteq A_4 \subseteq A \).

**Proof.** Levy showed in [8] Theorem 1 that \( \omega \in C \iff A \subseteq A_4 \subseteq A \), so it remains to show that \( A \subseteq A_4 \subseteq A \). The only point presenting any difficulty is \( A_4 \subseteq A_4 \).

Suppose that \( x \in A_1 \), \( |X| = x \), and let \( f \) map \( X \to X \). Then every \( X \) is non-empty, as \( f \) is "onto", and all the \( X \)'s are disjoint and contained in \( X \). Hence \( x \leq |X| \).
LEMMA 4 (Tarski [9], page 304, 24). If \( x + y \leq \omega + z \) there are \( a, b \) such that \( y = a + b \), \( b \leq \omega \), and \( x + a \leq \omega \).

LEMMA 5. \( x \in A \Rightarrow \) (for all \( p,q \), \( p \leq q \Rightarrow q \leq x + p = x + q \)).

Proof. Suppose \( x \in A \), and \( x + p \leq x + q \). By Lemma 4 there are \( a, b \) such that \( p = a + b \), \( b \leq \omega \), and \( x + a \leq \omega \). Since \( x + 1 \geq x \), \( a = 0 \). Therefore \( p = b \leq \omega \), as desired.

Conversely, if \( x \in A \), \( x + 1 \leq x \), so there are \( p, q \), namely \( p = 1 \), \( q = 0 \), such that \( p \leq \omega \) and \( x + p \leq x + q \).

LEMMA 6. Let \( g(X) = \) the set of all finite sequences of members of \( X \) with distinct entries. \( g(X) \Rightarrow g[\{X\}] \). If \( x \in A \), \( A \) respectively, then \( g(a) \in A \), \( A \), and if \( x \) is infinite, \( g(a) \notin A \).

Proof. This lemma for \( A \) is due to Tarski. See [7], page 225, lines 16-29. Suppose that \( (a,c) \) is a well-ordered (ordered respectively) sub-
set of \( g(X) \), where \( |X| = \omega \).

Let \( A_\omega = \{ \varepsilon \in A : \varepsilon \equiv \text{length} \} \). Assume the result of Theorem 1 (vi) below (in whose proof we do not use this lemma) that \( A \) and \( A_\omega \) are closed under \( \varepsilon \). Then \( |X| \in A \), \( (|X| \in \omega) \), respectively. Therefore each \( A_\alpha \) is finite, and \( A \) can be well-ordered. The set of entries of members of \( A \) can be well-ordered by first occurrence, so as \( |X| \in A \), \( A \) is finite.

Therefore \( A \) too is finite, as desired.

If \( X \) is infinite, map \( g(X) \) onto \( g(X) \cup \{ \varepsilon_0 \} \) thus \( (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) \rightarrow (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) \) if \( n > 0 \), \( \varepsilon_0 \rightarrow (\varepsilon_0) \) (the empty sequence).

Since \( |X| \) is infinite, for any \( (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) \) there is a \( \varepsilon_0 \neq \varepsilon_1 \), any \( i \), and so the map is "onto".

LEMMA 7. If \( A, \in A \), then \( A \in A \).

Proof. Let \( x \in A \), \( A \), and suppose \( (\varepsilon, c) \) is an ordered partition of \( X \). Then \( |x| \in A \).

Hence \( x \in A \subseteq A \), by hypothesis. Now \( (\varepsilon, c) \) is an ordered subset of \( X \).

Therefore \( \varepsilon \) is finite, and \( x \in A \).

COROLLARY. If \( A, \in A \), then \( A, \in A \).

The converse of this corollary does not hold, for in Halpern-Levy's model [11], page 136), the ordering principle holds, so \( A, \in A \), \( A \neq \omega \), any infinite set has a countable partition, giving \( A, \in \omega \), but there is an infinite set with no countable subset, so \( \omega \neq A \).

§ 3. Closure under various operations. We now investigate the question of the closure of the classes under unions, disjoint unions, \( + \), \( \times \), and \( \omega \).

The positive results are set out in the following theorems.
Let $X = \bigcup \{X_i : i \in \omega\}$, $X_i$ disjoint and non-empty. If $\xi \in X_i$, let $X_i = X \cup \{i\}$.

For each $\xi \in X$, $|X_i| \leq |\Delta_1|$ and $|X| = |\Delta_2|$. But $\omega \subseteq \bigcup \{X_i : i \in X\}$.

Hence $\Delta_1, \Delta_2$ are not closed under unions.

(iii) We first prove the following lemma, which will be used in a number of cases where we wish to show that a particular cardinal is in $\Delta_1$.

**Lemma 5.** If $X = \bigcup \{X_i : i \in \omega\}$ where each $|X_i| \leq \Delta_1$, and $|X| \leq \Delta_2$, then there is a map $f$ from $X$ onto $X \cup \{0\}$ (a $\Delta_1$), and an increasing function $n$ from $0$ to $\omega$ such that for each $i$,

$$f(i+\alpha)(\alpha) \subseteq X_{n(\alpha)}.$$

**Proof.** As $|X| \leq \Delta_2$, there is a map $g$ from $X$ onto $X \cup \{0\}$. Define $A_i$, $n(i)$, $m(i)$ in the following way, so that $A_i \neq \emptyset$, each $i$, $n$ is increasing, $m(0) > 0$, each $i,

n(0)$ is the least $\xi$ such that $g^{-1}(\{\alpha\}) \cap X_i \neq \emptyset$,

$m(0) = 1$,

$A_0 = g(\{\alpha\}) \cap X_{n(0)}$.

Otherwise suppose that $A_i, n(i), m(i)$ are defined for $i < k$. Then

$$|\bigcup \{X_i : j \leq m(k-1)\}| \leq \Delta_2$$

(ub 1).

Hence

$$\bigcup \{X_i : j \leq m(k-1)\} \subseteq A_{k-1}.$$

So for some $n > m(k-1)$, $m > 0$,

$$g^{-1}(A_{k-1}) \cap X_n \neq \emptyset.$$

Let $n(k)$ be the least possible such $n$, and for that value of $n, m(k)$ the least possible $m$. Let $A_k = g^{-1}(A_{k-1}) \cap X_{n(k)}$.

Thus $A_i, n(i), m(i)$ are all defined, each $n$, and $g_\omega$ maps $A_i \subseteq A_{i-1}$ (not necessarily "onto").

Let $B_k = \{\xi \in A_k : \exists \eta \in \Delta_1 \bigcup \{A_i : i \in \omega\}, g_\omega(\eta) = \xi\}$.

Define $f$ on $X$ thus. If $\xi \in B_k$, $f(\xi) = g_\omega(\xi)$; $f(\xi) = \xi$ otherwise.

Then all the required conditions are satisfied, except possibly that $f$ is "onto".

If $\xi \notin B_k$, each $h$, then $\xi = f(\xi)$, so $\xi \not\in \text{image } f$.

Suppose that $\xi \not\in B_k$, or that $\xi = o$, and $\xi \not\in \text{image } f$. Then for each $\eta \in A_{k+1}$ (or $A_o$ if $\xi = o$) such that $f(\eta) = \xi$, $\eta \not\in B_k$.

That is, for every such $\eta \in A_{k+1}$ there is a least $h(\mu)$ $\in \omega$ such that $\xi \in \bigcup \{A_i : i \in \omega\}$.

Arbitrarily large values of $h(\mu)$ must appear, by definition of $B_k$.

Hence $\kappa < |A_{k+1}|$. This is contrary to $|A_{k+1}| \leq \Delta_1$.

We now give a Fraenkel-Mostowski model in which $\Delta_1$ is not closed under $\times$. The $\text{ZF}$ consistency result then follows by the Jech-Sochor Theorem [5], since the statement theorem is

$$\mathfrak{A}(X, Y, x, f) \subseteq \bigcup (X \times Y) \cup \{X \times Y\} \subseteq (X \times Y)^3$$

and for all $\gamma \subseteq (X \times Y)^3$. If $g$ is a mapping from $X \cup Y$ onto $X \cup Y$, then $g$ is $1 \times 1$.

We suppose then that we have a model $\mathfrak{M}$ of $\text{ZF}$ modified so as to allow the existence of "urelements"; one only needs to alter the axiom of extensionality in which $\mathcal{U}$, the class of all urelements, is a set of cardinal $\kappa$. We suppose that the axiom of choice holds in $\mathfrak{M}$.

Since $\mathcal{U}$ is countable, it may be indexed by $\Pi$, the set of all finite sequences of $0$'s and $1$'s, so we let $\mathcal{U} = \{u_{\pi} : \pi \in \Pi\}$. For convenience we write $[a_0, a_1, \ldots, a_n]$ instead of $u_{\pi(0) \ldots \pi(n)}$.

Let

$$V = \{u_{\pi} : \pi \text{ begins with a } 0\} \quad \text{and} \quad W = \{u_{\pi} : \pi \text{ begins with a } 1\}.$$

$\mathcal{G}$ is the group of all permutations of $\mathcal{U}$ which preserve $V$, $W$ and the lengths of the sequences.

Define $f : \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{V} \times \mathcal{W} \cup \{0\}$ thus

$$f([0a_0, 1\beta_1, \ldots, 1\beta_n], [1\beta_1, \ldots, 1\beta_n]) = ([0a_0, a_1, \ldots, a_n], [1\beta_1, \ldots, 1\beta_n])$$

and

$$f((\pi_0, \pi_1), (\pi_0, \pi_2)) \quad \text{otherwise}.$$

Thus for members of $\mathcal{V} \times \mathcal{W}$ where each half is of equal length $\geq 2$, $f$ deletes the last elements, and interchanges the last but one.

Clearly $f$ maps $\mathcal{V} \times \mathcal{W}$ onto $\mathcal{V} \times \mathcal{W} \cup \{0\}$.

If $\sigma \in \mathcal{G}$ and $\xi \in \mathcal{M}, \sigma \xi$ is defined by transfinite induction on rank $\xi$ thus:

$$\sigma \xi = (\sigma, \xi) \quad \text{if} \quad \xi \in \mathcal{G}.$$
in $\mathbb{R}$. We show that $v$ (and similarly $w$) $\notin D_q$ in $\mathbb{R}$. It follows that $D_q$ is not closed under $\times$ in $\mathbb{R}$.

Since each member of $G$ preserves the lengths of members of $\Pi$, the sequence $(U_1, U_2, \ldots)$ is in $\mathbb{R}$, where $U_i$ is the set of members of $V$ of length $i$. Of course each $U_i$ is finite.

Suppose that $v \notin D_q$. By Lemma 8 there is a map $g$ in $\mathbb{R}$ from $V$ onto $V \cup \{0\}$, and an increasing map $\eta : \omega \rightarrow \omega$ such that for each $i$, every member of $f^{i+1}(\eta(0))$ has length $\eta(i)$.

Since $g \in \mathbb{R}$ there is a finite $A \subseteq S$ such that

$$H(g) \supset H(f) \cap K(A).$$

Let $i$ be the least integer $> 0$ such that every member of $A$ has length $\leq i$. Let $g(a) = \beta$, where length $a = \kappa$, length $\beta = \eta$, and $k > \eta > i$.

Then if $a \in H(g) \cap A(a)$, $\beta = g(a) = g(\eta(a)) = \eta(a) = \eta(\beta)$. This shows that $H(g) \cap H(a) \subseteq H(\beta)$, so

$$K(\beta) \cap H(f) \cap H(a) \subseteq H(\beta).$$

To obtain the desired contradiction, we find a $\sigma \in H(f)$ which fixes all members of $V$ of length $\neq j$, all members of $W$ of length $i$, and moves every member of $W$ of length $j$.

Such a $\sigma$ will be in $K(\beta) \cap H(f) \cap H(a)$ but not in $H(\beta)$. Let

$$\sigma[0, \gamma_1, \gamma_2, \ldots, \gamma_{j-1}] = 0, \gamma_1, \gamma_2, \ldots, \gamma_{j-1},$$

$$\sigma[1, \delta_0, \ldots, \delta_i] = [1, \delta_0, \ldots, \delta_{i-1}, \delta_i, \ldots],$$

if $i > j - 1$,

and $\sigma$ fixes everything else.

Then $\sigma$ fixes lengths of sequences, and $V, W$, so lies in $G$. $\sigma$ also fixes every member of $V$ of length $\neq j$ and every member of $W$ of length $\leq i$ (as $i < j$), and moves every member of $V$ of length $j$.

It remains to show that $\sigma \in H(f)$. That is the same as saying that $f(\eta(a)) = \sigma(\eta(a))$ for each $\eta \in \mathbb{R}$, with $\eta(a)$.

Let $\eta = (\eta_1, \eta_2)$. If $\eta_1$ and $\eta_2$ have different lengths, $f(\eta) = \eta$, and so $f(\eta) = \sigma(\eta)$ as well.

Suppose therefore that they have equal lengths, $l + 1$. Let

$$\eta = (0, \gamma_1, \gamma_2, \ldots, \gamma_j), \quad \eta = (1, \delta_0, \ldots, \delta_i).$$

(1) $l + 1 < j$. Then $\sigma$ fixes $\eta$ and $f(\eta)$, so $\sigma(\eta) = f(\eta) = f(\eta).$

(II) $l + 1 = j$. $f(\eta) = (0, \gamma_1, \gamma_2, \ldots, \gamma_{j-1}), (1, \delta_0, \ldots, \delta_{j-1}, \delta_j)$ as and

$$\sigma \eta = (0, \gamma_1, \gamma_2, \ldots, \gamma_{j-1}, (1, \delta_0, \ldots, \delta_{j-1}), \delta_j).$$

and so

$$f(\sigma \eta) = ((0, \gamma_1, \gamma_2, \ldots, \gamma_{j-1}, \delta_j), (1, \delta_0, \ldots, \delta_{j-1}, \delta_j)) = \sigma f(\eta).$$

(III) $l + 1 > j$.

$$f(\sigma \eta) = (0, \gamma_1, \gamma_2, \ldots, \gamma_{j-1}, \delta_j) = \sigma f(\eta),$$

and

$$f(\sigma f(\eta)) = ((0, \gamma_1, \gamma_2, \ldots, \gamma_{j-1}, \delta_j), (1, \delta_0, \ldots, \delta_{j-1}, \delta_j)).$$

Thus

$$f(\sigma f(\eta)) = (0, \gamma_1, \gamma_2, \ldots, \gamma_{j-1}, \delta_j, (1, \delta_0, \ldots, \delta_{j-1}, \delta_j)).$$

(IV) $l + 1 > j + 1$.

$$f(\sigma f(\eta)) = (0, \gamma_1, \gamma_2, \ldots, \gamma_{j-1}, \delta_j) = \sigma f(\eta),$$

and

$$f(\sigma f(\eta)) = (0, \gamma_1, \gamma_2, \ldots, \gamma_{j-1}, \delta_j, (1, \delta_0, \ldots, \delta_{j-1}, \delta_j)).$$

Thus

$$f(\sigma f(\eta)) = (0, \gamma_1, \gamma_2, \ldots, \gamma_{j-1}, \delta_j, (1, \delta_0, \ldots, \delta_{j-1}, \delta_j), (1, \delta_0, \ldots, \delta_{j-1}, \delta_j)).$$

To obtain the ZF consistency result as stated we use the Jech-Sochor Theorem [5], as mentioned above. Pincus [14] 24B could also be used.

§ 4. Now suppose that $|X| > D_q$. We define the 2-valued function $\mu_\eta$ from $P(X^\eta)$ to $\{0, 1\}$, by induction on $\eta$.

$$\eta = 0 \quad \mu_\eta(\emptyset) = 0,$$

$$\mu_\eta((\eta)) = 1 \quad ((\eta) \text{ is the empty sequence}).$$

$$\eta = \emptyset \quad \mu_\eta(A) = 0 \quad \text{if } (\exists \xi \in X^\eta: \mu_\eta(A) = 1) \text{ is finite},$$

$$\mu_\eta(A) = 1 \quad \text{otherwise},$$

where $A = \{\xi \in X^\eta: (\xi, \eta) \in \mathcal{A}\}$.

The proof of the following lemma is very simple, and is omitted.

LEMA 9. $\mu_\eta$ is a fixedly additive measure on $X^\eta$, vanishing on singletons when $\eta > 0$, and satisfying $\mu_\eta(X^\eta) = 1$, $\mu_\eta(X_\emptyset) = 0$.

Let $\sigma$ be a permutation of $\{0, 1, \ldots, n-1\}$. If $A \subseteq X^\eta$ we let

$$A_\emptyset = \{\xi \in A: (\xi, \emptyset) \in \mathcal{A}\} \quad \text{where } (\xi_\emptyset, \xi_{\eta_1}, \ldots, \xi_{\eta_n}) = (\xi_0, \xi_1, \ldots, \xi_{\eta_0})$$

LEMA 10. For any $A \subseteq X^\eta$ and permutation $\sigma_\emptyset: \mu_\emptyset(A_\emptyset) = \mu_\emptyset(A)$.

Sketch of Proof. Firstly we may suppose that $\sigma$ is a transposition of adjacent elements, as such transpositions generate the symmetric group. One performs the proof in three stages, corresponding respectively to the possibilities $\eta = 2, \sigma = (01); \eta > 2, \sigma = (01); \sigma = (i, i+1)$. 

}\[15pt]\[15pt]
THEOREM 3. If $X$ is infinite and $f$ is a function on $[X]^n$ such that

$\emptyset \neq f(A) \subseteq A$, $f(A) \neq A$, for each $A \in [X]^n$, then $|X| \neq \mathcal{A}_1$.

Proof. Suppose $|X| \neq \mathcal{A}_1$. Then $\mu$ can be defined. Let

$C = \{(s_1, \xi_1, \ldots, \xi_{n-1}) : \xi_1 \text{ all distinct, and for some } i < n, (s_i, \xi_1, \ldots, \xi_{n-1}) = f((s_1, \xi_1, \ldots, \xi_{n-1}))\}$.

Then $\mu(C_{\infty}) = \mu(C)$ by Lemma 10. It is clear that $C \cap C_{\infty} = \emptyset$ as $i \neq 0$. Hence $\mu(C_{\infty}) = \mu(C) = 0$.

Let $X' = \{(\xi, \xi, \ldots, \xi_{n-1}) : \text{ for some } i \neq j, \xi_i = \xi_j\}$. By Lemmas 9 and 10,

$\mu(\bigcup \{X'_\xi : \xi \text{ a permutation of } n\}) = 0$.

Therefore

$\mu(X^n) = \mu(\bigcup \{O_{\xi} : \xi \text{ a permutation of } n\}) + \mu(\bigcup X'_{\xi}) = 0 + 0 = 0$, a contradiction.

COROLLARY. If for some $n > 1$, $[X]^n$ has a choice function, then $X$ is finite or $|X| = \mathcal{A}_1$.

LEMMA 11. If two members of $\mathcal{A}$ are comparable, and $|e(X)| \neq \mathcal{A}_1$, then $e(X) = \emptyset$ has a choice function, where $e(X) = \{A \subseteq X : A$ finite\}.

Proof. Let $e'(X) = \{(A, n) : A \in e(X), n < |A!|\}$. $e'(X)$ is designed to be the same "size" as $q(X)$ (the set of finite 1-sequences of members of $X$; see Lemma 6) at each stage. For this reason we put in $|A!|$ copies of each $A$ corresponding to the $|A!|$ different orderings of $A$. It is easily seen that

$|A| \in e(X)$ $\implies$ $|e'(X)| \neq \mathcal{A}_1$.

By the comparability of any two members of $\mathcal{A}$,

$|e'(X)| < |q(X)|$ or $|q(X)| < |e'(X)|$.

Let $f$ map $e'(X)$ 1-1 into $q(X)$ (or vice versa), and let $A \in e(X)$. We show how to choose effectively in terms of $f$ a $B \supseteq A$ such that $f$ maps $e'(B)$ 1-1 onto $q(B)$ (or vice versa), and $B$ is finite.

Let $A_0 = A, A_{n+1}$ the union of $A_n$ and $n$: $n$ is the set of entries of a member of $f'(\{A_n\})$, or $n$: for some $i$, $(e, f') = (f'(A_n))$ if $f$ goes the other way round. As $|e(X)| \neq \mathcal{A}_1$, $A_n = A_{n+1}$ for some $n$, and this is our $B$. $f$ is then 1-1 from $e'(B)$ into $q(B)$ (or vice versa), and as $|e'(B)| = |q(B)|$ it is also "onto".

Thus by taking $f^{-1}$ in place of $f$ in the second case, we have an effectively determined finite $B \supseteq A$, and a 1-1 map $f$ from $e'(B)$ onto $q(B)$. We show that when $A \neq \emptyset$ this gives an effectively determined $g(A) \in \mathcal{A}$, which is what is required.

Let $C$ be the closure of $\{A, 0\}$ under the operations

(i) if $(\xi, i) \in C$, so is $(\xi, j)$, any $j < |i|$; and

(ii) if $(\xi, i) \in C, P(\eta) \times C \subseteq C$ where $\eta$ is the set of entries of $f((\xi, i))$.

It is clear that $C$ can be effectively ordered, and that $C \cap \mathcal{A}_1(B)$. Also $f$ maps $C \setminus \{0\}$ onto $\bigcup \{q(\eta) : (\eta, 0) \in C, g(A)$ is the first entry of the first member of $g(A)$ in the ordering of the image of $C$. Thus $g$ is a choice function for $e(X)$, as desired.

COROLLARY. If any two members of $\mathcal{A}$ are comparable, then $\omega = \mathcal{A}_1$.

Proof. Use Theorem 3 and Lemma 11.

We mention another way in which this corollary can be derived, without giving details. First one proves the following.

If any two members of $\mathcal{A}$ are comparable, and $|X| \neq \mathcal{A}_1$, $X$ infinite, there is a permutation of $X$ moving infinitely many points.

To derive the corollary from this, suppose $X$ is infinite, $|X| \neq \mathcal{A}_1$. Then $|X| \neq \mathcal{A}_1$, so there is a permutation $\sigma$ of $X$ moving infinitely many points. Since $|X| \neq \mathcal{A}_1$, each cycle of $X$ under $\sigma$ is finite. There are thus infinitely many finite cycles with at least two members. Let $X$ be a subset of $X$ which chooses a member from each such cycle (using Lemma 11). Then $X$ and $X' = X$ are infinite disjoint subsets of $X$, contrary to $|X| \neq \mathcal{A}_1$.

It seems likely that an argument along the lines of Theorem 3, using Lemma 8, will show that if $x \in \mathcal{A}_1$, $e(x) \in \mathcal{A}_1$, $e(X)$ is the set of finite subsets of $X$). At present the best we have is the following.

THEOREM 4. If $x \in \mathcal{A}_1$, $e(x) \neq \mathcal{A}_1$.

Proof. Let $x \in |X|$. Of course $e(x) \neq \mathcal{A}_1$, as $e(X) = \bigcup \{[X]^n : n \in \omega\}$. Suppose $(\mathcal{A}_1, \mathcal{E})$ is an infinite ordered subset of $e(X)$, Then $|A| \mathcal{E} \mathcal{A}_1$ is infinite.

For $x \subseteq \bigcup \mathcal{A}_1$, let $n(x)$ be the least $n$ such that for some $B \in \mathcal{A}_1, x \subseteq B$, and $|B| = n$. As $x \in \mathcal{A}_1$, $n(x)$ is bounded for $x \in \bigcup \mathcal{A}_1$. Thus there are many infinite $B \subseteq \mathcal{A}_1, B \in [X]^n$. Hence $X^n$ is an infinite ordered partition, as it can be mapped onto $[X]^n$. This is contrary to Theorem 1 (vi) that $\mathcal{A}_1$ is closed under $\times$.

§ 5. Five models. The models we use in this section are already known; but we shall be concerned with properties of them not previously studied. For example it was not known before that in three of them $|A| = 2^n$, $\mathcal{N}_n$ is due to Halpern and Levy ([11], page 136), $\mathcal{N}_5$ and $\mathcal{N}_n$ are due to Mostowski ([10] and [11] respectively), and $\mathcal{N}_5$ is due to Gauntt [3]. $\mathcal{N}_5$ is only a slight modification of Levy’s model $\mathcal{M}_5$ [8] page 11. When the model we are using contains urelements, we appeal to Pincus [14] 2B7 to give the infinite ZF result.

"Combinations" of the models give all but ten of the possible combinations of equality and strict inclusion in Lemma 3 (subject to the restrictions of § 3 and § 3). For example, $\omega = \mathcal{A}_1 = \ldots = \mathcal{A}_1$ occurs whenever
we have DC. To obtain a model for \( w = A_4 \neq A_1 \neq A_4 \neq A_1 \neq A_1 \neq A_3 \neq A_4 \neq A_1 \), combine the methods of (i), (iii), and (iv).

**Theorem 5.** If \( ZF \) is consistent, then so are each of the following

(i) \( ZF + \omega = A_2 = A_3 = A_4 = A_1 \neq A_2 = A_3 \).

(ii) \( ZF + \omega = A_2 = A_3 = A_4 = A_1 \neq A_2 = A_3 = A_4 = A_1 \).

(iii) \( ZF + \omega = A_2 = A_3 = A_4 = A_1 \neq A_2 = A_3 = A_4 = A_1 \).

(iv) \( ZF + \omega = A_2 = A_3 = A_4 = A_1 \neq A_2 = A_3 = A_4 = A_1 \).

These restrictions leave us with only the following to prove in Theorem 5:

- \( \mathfrak{L}_1 \models \omega = A_1 \neq A_4 \neq A_1 \).
- \( \mathfrak{L}_1 \models \omega = A_4 \neq A_1 \neq A_4 \).
- \( \mathfrak{L}_1 \models \omega = A_1 = A_4 \).
- \( \mathfrak{L}_1 \models \omega = A_1 \neq A_4 \neq A_4 \).
- \( \mathfrak{L}_1 \models \omega = A_1 \neq A_4 \neq A_4 \).
- \( \mathfrak{L}_1 \models \omega = A_1 \neq A_4 \).

When the ordering principle holds, we clearly have \( w = A_2 \). (See Levy [8] Theorem 3.) Hence \( w = A_3 \) holds in \( \mathfrak{L}_1 \) and \( \mathfrak{L}_1 \). Levy proves that \( w \neq A_4 \) in \( \mathfrak{L}_1 \) [8], Theorem 4.

It is known that in \( \mathfrak{L}_1 \) any set can be put in 1-1 correspondence with a subset of \( \kappa \times \kappa \), some ordinal \( \kappa \), and hence with a set of sets of ordinals (Cohen [1], page 139). So if \( x \in \mathfrak{L}_1 \) is infinite, \( x \) has a countable partition (which is in fact effectively determined by this 1-1 correspondence). This shows that \( w = A_1 \) in \( \mathfrak{L}_1 \).

We now show that \( w = A_3 \neq A_1 \) in \( \mathfrak{L}_1 \). Suppose \( b \neq w \). Then there is a function \( f \in \mathfrak{L}_1 \) from \( b \) onto \( b \cup \{0 \} \). Let \( b \) be a condition in \( \mathfrak{B} \) forcing "if a is a function, f must move infinitely many points," so there is \( i \neq j \) such that \( f(a_i) = a_1 \), \( i \neq j \), and \( a_1 \) is not involved in the definition of \( f \).

Let \( q \) be a condition in \( \mathfrak{B} \) extending \( p \) and forcing \( q(f(a)) = a_2 \). Then there is a permutation \( \sigma \) of \( w \) such that \( \sigma \) fixes each \( k \) such that \( k = i \), or \( a_1 \) is involved in the definition of \( f \), but not \( a_1 \), and such that \( q \) is compatible with \( q \). So \( \sigma \cup q \) forces \( f(a) = a_1 \) and \( f(a) = a_2 \), and this contradicts \( q \) forces \( f \) a function.

Therefore \( b = A_3 \neq A_1 \) in \( \mathfrak{L}_1 \), so \( \mathfrak{L}_1 \models A_4 \neq A_1 \).

Now to show the same for \( \mathfrak{L}_2 \). Certainly \( \{ \} \in A_2 \). Suppose that \( \{ \} \notin A_2 \). Then by Lemma 8, there is a function \( f \) from \( U \) onto \( U \cup \{0\} \), a finite \( n \) such that \( H(f) \supseteq X(V_n) \) where \( V_n = \{x_0 : i < \alpha \} \), and \( i, j, k \in I \) with \( i, k > n, i \neq k \), and \( f(u) = u_{a_3} \). Let \( \sigma \) be the permutation of \( U \) interchanging just \( u_{a_3} \) and \( u_{a_2} \). Then \( \sigma \in X(V_n) \) as \( k > n \), so \( \sigma(a_3) = f(x) \).

Therefore \( a_{a_2} = q(u_{a_3}) = f(u_{a_3}) = f(a_3) = u_{a_3} \), a contradiction. So \( \{ \} \models A_4 \neq A_1 \) and \( \mathfrak{L}_2 \models A_1 = A_2 \).

Now let \( x \in \mathfrak{L}_1 \) be infinite. Let \( f(\xi) \) be the least \( \xi \) such that \( H(\xi) \supseteq X(V_\alpha) \), for \( \xi \in X \). If the image of \( f \) is finite, let \( \xi \) be a well-ordering of \( X \) in \( \mathfrak{L}_1 \) and \( x \) an upper bound for \( f(\xi) \), \( \xi \in X \). Then if \( x \in X(V_n) \), \( f \) fixes each \( \xi \in X \), and hence the well-ordering of \( X \). Therefore \( x \notin X \).

If the image of \( f \) is infinite, then \( x \notin X \). But this follows from the fact that \( H(\xi) \supseteq X(V_n) \) if and only if \( H(\xi) \supseteq X(V_n) \), any \( \sigma \). (We omit the simple proof.)

Hence \( \mathfrak{L}_1 \models A_2 = A_3 \).
Now we show that in $\mathbb{R}_I$ and $\mathbb{R}_I$, $A = A$. In fact we show rather
more, namely that any ordered set can be well-ordered. (This also holds
in $\mathbb{R}_I$, however the proof is rather different as there is an "straight"
FM model). The important point here is that each member of $G$ has finite
order (for $\mathbb{R}_I$, this is 1 or 2; for $\mathbb{R}_I$ this is one reason why we took only
permutations of $U$ moving finitely many points). Any ordered set can
be well-ordered" implies the axiom of choice in $\mathcal{ZF}$ (see Rubin and Rubin
[15] page 77), but not in FM.

Let $(X, <)$ be an ordered set. Then for some $H \in \mathcal{F}$,

$$H((X, <) \in H).$$

Define $\sim$ on $X$ by $\xi \sim \eta$ if for some $\sigma \in H$, $\sigma \xi = \eta$. Let $X_0$ be a $\sim$-class
and $\xi_0 \epsilon X_0$, and suppose that $H(\xi_0) \in H$. Then there is a $\sigma \in H(\xi_0)$
and as $\sigma\xi_0 \neq \xi_0$, $\sigma\xi_0 < \xi_0$ or $\xi_0 < \sigma\xi_0$. As every member of $G$ has finite
order, $\sigma^* =$ identity, some $a \epsilon \omega$. Also $\sigma \epsilon H$, so $\sigma$ preserves $<$. Therefore

$$\xi_0 > \sigma\xi_0 > \sigma^2\xi_0 > \ldots > \sigma^n\xi_0 = \xi_0,$$

or

$$\xi_0 < \sigma\xi_0 < \sigma^2\xi_0 < \ldots < \sigma^n\xi_0 = \xi_0,$$

each of which is impossible.

Therefore $H(\xi_0) \notin H$. Now by definition of $\sim$, $X_0 = \{\sigma\xi_0 : \sigma \epsilon H\}$
But $\sigma\xi_0 = \xi_0$, all $\sigma \epsilon H$. Therefore $|X_0| = 1$. Thus each $\sim$-class has just
one member. Let $\xi'$ be well-order $X$ in $\mathcal{F}$. Then every member of $H$ preserves
every member of $X$, so also $<$. Thus $X$ can be well-ordered in $\mathbb{R}_I$ (or $\mathbb{R}_I$).

Now we move on to $\mathbb{R}_I$. By Gaunt [3] Lemma 4 we have that for
each $X \epsilon \mathbb{R}_I$ there is a unique minimal finite s$(X) \epsilon U$ such that $X$
is ordinal definable over $\mathbb{R}_I \cup s(X) \cup \{F\}$ (and is contained in any other
such subset of $U$). The function $s$ is definable in $\mathbb{R}_I$, and moreover for
any finite $A \subseteq U$, the class of all $X \epsilon \mathbb{R}_I$ such that $s(X) = A$ has a natural
well-ordering, $w(A)$, say, in $\mathbb{R}_I$, given by the minimal triple $(\theta, a, 0)$,
where $\theta$ is a formula defining $X$ in the 0th member of $\mathbb{R}_I$ and $F$
(if $i = 0$, or not with $F$ (if $i = 1$), and $A$.

We are now able to show that $\mathbb{R}_I$ satisfies $W$: any set of non-empty
well-orderable sets has a choice function. A similar proof shows that it
holds in $\mathbb{R}_I$ and $\mathbb{R}_I$ (see Larche [6] pages 33, 35), but we do not need that
here.

Let $X$ be a set of non-empty well-orderable sets in $\mathbb{R}_I$, and let $Y \epsilon X$.
As $Y$ can be well-ordered in $\mathbb{R}_I$, there is a 1-1 map in $\mathbb{R}_I$ from $Y$ into $\mathcal{O}_s$
(= the class of all ordinals). Hence for some finite $A \subseteq U$, every member
of $Y$ is ordinal definable over $\mathbb{R}_I \cup \{F\}$. Therefore $\bigcup \{s(\xi) : \xi \epsilon Y\}$
is finite. Let $A$ be $A$. Then $F$ defines an ordering of $X$. Let $B$ be the first
member of $P(A)$ in this ordering such that for some $\xi \epsilon Y$, $s(\xi) = B$,
and let $g(Y)$ be the first member of $X \cap (Z : s(Z) = B)$ in $\mathbb{R}(B)$. $g$
is defined in terms of $P(X)$ and so lies in $\mathbb{R}_I$.

$W$ implies as a very special case $\Lambda \subseteq \mathcal{O}_n$ (or indeed $\Lambda \subseteq \mathcal{O}_n$
eto by the corollary to Theorem 3, $w = A_1$ in $\mathbb{R}_I$.

Now to show that for $\omega \neq A_1$ in $\mathbb{R}_I$ we show that $|U| < \omega_1$. Let $(V, <)$
be an infinite ordered partition of $U$ in $\mathbb{R}_I$. Then for some condition $p$ in $\mathbb{R}_I$, $p$
forces "$\omega$ is a linear ordering." Let $A$ be the set of all urelements
involved in $p$ and in $s(V, <)$. Define $\sim$ on $U \setminus A$ by $u \sim v$ if whenever
$\mathcal{G}(B \subseteq A) : \mathcal{F}(B \cup \{u\}) = \mathcal{F}(B \cup \{v\}) \in p$ and $\mathcal{F}(B \cup \{u\}) = \mathcal{F}(B \cup \{v\})$.

$\sim$ is an equivalence relation and there are only finitely many equiva-
rence classes. Since $V$ is infinite, there are $u_1, u_2 \epsilon U \setminus A$, such that the
$\sim$-class of $u_1$ is infinite, $u_1 \sim u_2$, and if $u_1 \epsilon \sigma \epsilon \mathcal{F}$, $u_1 \epsilon \mathcal{F} \epsilon V, \bar{V}$, $\sim u_2$.
(\text{In fact every $\sim$-class is infinite but we do not need that here.})

Let $g \epsilon \mathcal{F}$, $g \epsilon \mathcal{F}$ forces $u_1 \epsilon u_2$. Let $B$ be finite, $B \setminus A = B_1$, and be
such that every member of $U$ occurring in $g$ or $s(u_1), s(u_2), s(u_3)$ is in
$A \cup B_1$, and $u_1, u_2, u_3, B_3$.

Let $R = \{u_1, u_2, u_3, u_4, u_5, \ldots, u_n\}$. Since the $\sim$-class of $u_1$ is infinite,
there is a $u_4 \epsilon U \setminus (A \cup B_1)$ such that $u_4 \sim u_1$. Let $a, b, c, d, e, f, g, h$
be distinct members of $U \setminus (A \cup B_1)$.

Let $B_3 = \{u_1, u_2, a, b, c, d, e, f, g, h\}$

and $B_1 = \{u_1, u_2, a, b, c, d, e, f, g, h\}$.

and $\sigma, \tau$ permutations of $U$ are defined thus:

$$\sigma u_1 = u_2, \sigma u_2 = u_3, \sigma u_3 = u_4, \sigma u_4 = u_1, \sigma u_5 = u_6, \sigma u_6 = u_7, \sigma u_7 = u_8, \sigma u_8 = u_9, \sigma u_9 = u_10,$$

$$\tau u_1 = u_3, \tau u_2 = u_4, \tau u_3 = u_5, \tau u_4 = u_6, \tau u_5 = u_7, \tau u_6 = u_8, \tau u_7 = u_9, \tau u_8 = u_10, \tau u_9 = u_1,$$

and $\sigma$ and $\tau$ are the identity on points not mentioned.

We observe that $\sigma B = B_1$, $\tau B = B_2$, and $\sigma, \tau$ both fix $A$ pointwise.

We wish to show that $q \epsilon \mathcal{F}$ and $q \epsilon \mathcal{F}$ is a condition. For this we show
that if $X \epsilon \mathcal{F}$ domain $q \epsilon \mathcal{F}$, $q(X) = (\sigma q)(X)$, and similarly for
$q \epsilon \mathcal{F}$ and $q \epsilon \mathcal{F}$.

By the choice of $A$ and $B$, domain $q \subseteq P(A \cup B_1)$. Hence domain
$q \subseteq P(A \cup B_1)$ domain $q \subseteq P(A \cup B_1)$.

Thus if $X \epsilon \mathcal{F}$ domain $q \epsilon \mathcal{F}$ domain $q(X) = (\sigma q)(X)$, and similarly for
$q \epsilon \mathcal{F}$ and $q \epsilon \mathcal{F}$.

By the choice of $A$ and $B$, domain $q \subseteq P(A \cup B_1)$. Hence domain
$q \subseteq P(A \cup B_1)$ domain $q \subseteq P(A \cup B_1)$.

Thus if $X \epsilon \mathcal{F}$ domain $q \epsilon \mathcal{F}$ domain $q(X) = (\sigma q)(X)$, and similarly for
$q \epsilon \mathcal{F}$ and $q \epsilon \mathcal{F}$.


This just says that $(X, u_0) \in aP$. Since $eq \subseteq aP$, $X$ is domain of $eq$, and $aP$ is a function,

$$(eq)(X) = (aP)(X) = u_0 = q(X),$$
as desired.

Similarly if $g(X) \subseteq A$.

Hence $q \cup eq$ is a condition. In a similar way we get that $q \cup gq \subseteq qg$ are conditions, and hence that $q \cup eq \cup gq$ is a condition.

Since $g$ forces $q \subseteq qg$, we have $eq \subseteq gq \subseteq qg$ for $qg \subseteq qg$, where $qg \subseteq gq \subseteq qg$, because $e, f$ fix $A$ and hence $q$ and $f$. Therefore $q \cup eq \cup gq$ forces $u \subseteq v$ and $vy$ forces $u \subseteq v$ and $vy$ forces $u \subseteq v$, which contradicts $g \neq p$ and $p$ forces $u \subseteq v$ to be a linear ordering.

Hence $U$ has no infinite ordered partition in $\mathcal{R}$, and so $|U| \notin A\omega$. We have now incidentally proved the following, which is a strengthening of Läuchli [6].

**Theorem 6.** If ZF is consistent, then so is ZF+W: any set of non-empty well-orderable sets has a choice function, there is an infinite set of generic reals, (See Gauntt [3].)

We are now left with the following to prove from Theorem 5.

$\mathcal{R}_0 \vdash A_0 = A_1$, $\mathcal{R}_0 \vdash A_0 \neq A_1 = A_2$, $\mathcal{R}_0 \vdash \omega \neq A_1$, $\mathcal{R}_0 \vdash A_1$. Initially we may suppose that $A_0 = A_1$ in $\mathcal{R}_0$. Let $(X, <)$ be an infinite ordered set in $\mathcal{R}_0$. If for some finite subset $A$ of $U$, $\xi (\xi : \xi : A = A)$ is infinite, this provides a countable subset of $X$ by $\omega (\omega)$. If for each finite $A \subseteq U$, $\xi (\xi : \xi : A = A)$ is finite, then by $\mathcal{R}_0$, $\omega$ we may suppose that each $(\xi : \xi : \xi : A = A)$ is $< 1$ (replacing $X$ by an infinite subset). This amounts to supposing $X \subseteq \omega (\omega)$. But by Theorem 4 and $\mathcal{R}_0 \vdash A_1$, $\omega$ cannot be ordered in $\mathcal{R}_0$, a contradiction.

Now suppose that $V$ is an infinite subset of $U$ in $\mathcal{R}_0$. Then for some finite $A \subseteq U$, $H(V) \subseteq H(A)$. As $V$ is infinite, there is some $u \in V \subseteq A$. Let $u_0$ be any other member of $U - A$. Then $\sigma$, the permutation of $U$ which interchanges $u_0$ and $u_1$, lies in $H(A)$, so $o \in V$. Therefore $o \in V$. Thus $U \subseteq V$ is finite. This shows that $U$ is not the disjoint union of two infinite sets in $\mathcal{R}_0$, i.e. that $(U) \notin A_0 \omega$. Therefore $o \neq A_0 \omega$ in $\mathcal{R}_0$.

We next show that $(U) \notin A_0 A_1 R_0$. Otherwise $o \neq A_0$. Then by Lemma 8 there is a map $f$ from $o (U)$ onto $e (U) \cup \{1 \in e (U)\}$ and an increasing function $u : o \to o$ such that for each $i, f_i (e (U) \cup \{1 \in e (U)\})$ and $f_i (e (U) \cup \{1 \in e (U)\}) = C (U)^{o^b}$. As $f \in R_0$, for some finite $A \subseteq U$, $H(f) \subseteq H(A)$. For some $i, f_i (e (U) \cup \{1 \in e (U)\}) = C (U)^{o^b}$. Therefore there is an $X \subseteq e (U)$ satisfying $X \subseteq A$ and $X \subseteq f_i (e (U) \cup \{1 \in e (U)\})$, and so with respect to this, $|X \cap A|$ is maximal. (We may assume maximality because $A$ is finite.)

Let $f(X) = A$. Then as $o$ is an increasing function,

$$|X| = u (i + 1) > u (i) = |X|.$$  

Case I. $X - (Y \cup A) \neq 0$. Let $u \in X - (Y \cup A)$. Let $a \in U - (X \cup Y)$. $(X \cup Y \cup A)$ is finite. Then the permutation $u a$ is in $K(A)$, and hence also in $H(f)$. Since $(X, Y) \epsilon f_1$, $(X, Y) \epsilon f_1$, where $e$ is this permutation. But $o \epsilon Y$ as neither $u$ nor $u'$ is in $Y$, and $o \epsilon X$ as $u \in X$ but $u' \notin X$. This contradicts $f$ a function.

Case II. $X \cap Y = A$. Then $X \subseteq A \subseteq Y \cap A$. By the maximality of $|X \cap A| = |X \cap A| \geq |X| - |X \cap A| 

|X| - |X \cap A| < |X| - |X \cap A| = |X| - A.$

Since $X \subseteq A$, we get

$$|X| - A = |X| - |X \cap A| < |X| - |X \cap A| = |X| - A.$$

As in Case I this gives a contradiction. Therefore $A_0 \neq A_1$ in $\mathcal{R}_0$.

There remains now to prove, $A_0 = A_1$ in $\mathcal{R}_0$. Let $|X| \notin A_1$, $|X| \notin A_0$. Then $X = \bigcup \{X_t : t \epsilon o\}$ disjoint, and the sequence $(\{X_t : t \epsilon o\})$ is in $\mathcal{R}_0$. This means that for some finite $A \subseteq U$, $H(X) \subseteq H(A)$, each $X_t$, $t \epsilon o$.

~ is defined on $X_t$ by $\xi = \eta$ if for some $\sigma : K(A), \eta \epsilon = \eta$. Let $\theta \notin X_t \subseteq X_t$, $\eta \epsilon = \eta$. Then $H(X_t) \subseteq H(A)$, each $i$, so $\bigcup X_t : t \epsilon o \subseteq \mathcal{R}_0$. This amounts to supposing, without loss of generality, that each $X_t$ is a $\eta$-class. Let $\xi \epsilon X_t$. Then by Mostowski [11] page 339 we know that there is a unique $B_1, B_1$ finite and contained in $U', B_1 \cap A = \emptyset$ such that $H(X_t) \subseteq H(A) = H(B_1 \cup A)$, each $i$, $\epsilon o$. Let $I_1, I_2, \ldots, I_n$ be the open intervals of $U$ under $A$. Then if $\eta \subseteq U - A$ and $|\eta| \subseteq I_1$, $\eta \subseteq B_1 \cap I_1$, each $j < n$, there is a $\sigma \subseteq K(A)$ mapping $B_1$ 1-1 onto $\eta_1$.

Therefore if we map $X_t = \{\xi : \eta : K(A)\}$ to $(\{\eta : \sigma \subseteq K(A)\}$ by $\sigma \epsilon \epsilon \eta$, the map is 1-1 and onto, because $H(X_t) \subseteq H(A) = H(B_1 \cup A)$, it lies in $R_0$ (easily checked), and its image is

$$\eta \subseteq U - A : \eta \subseteq B_1 \cap I_1, \eta \subseteq B_1 \cap I_1, \eta \subseteq B_1 \cap I_1, \eta \subseteq B_1 \cap I_1.$$
Thus we may suppose that $X_i$ is of the form

$$\{n \in U \cap A : [n \cap I_j = m_a(j)] \}$$

for some $(a+1)$-tuple $(m_a(\bar{d}), m_a(\bar{e}), \ldots, m_a(\bar{r}))$.

By discarding some of the $X_i$'s, (but retaining infinitely many) we may suppose that $i < j \Rightarrow m_a(i) < m_a(j)$. (Consider the two cases $m_a(i)$ bounded and unbounded, and similarly for the other co-ordinates one by one. Thus we may suppose that for each $k < n_i$, $i < j \Rightarrow m_a(k) < m_a(j)$.

We are now able to map $X$ onto $\{ \xi \in I : (1 \not\in \xi) \}$. If $\eta \in X_i$, let $f(\eta) = 1$. If $\eta \notin X_i$, let $f(\eta) = i$. For each $j \leq n_i$, $\zeta_j$ and $\eta_j$ are the first $m_a(i+1)$ members of $\eta_j \cap I_j$ in the ordering of $U$.

It is then easily checked that $f \in \mathcal{R}_i$ and $f$ is "onto", as desired. Hence $\mathcal{A}_i = \mathcal{A}_i \in \mathcal{R}_i$.

In fact by pushing this proof a little further one can show in $\mathcal{R}_i$ any cardinal $\lambda$ can be written as $y + z$, where $y \in \mathcal{A}_i$, and $2z \leq \lambda$.

Finally we show that in $\mathcal{R}_i$, $\mathcal{A}_i \neq \mathcal{A}_i$. Let $\mathcal{W} \in \mathcal{U}$, and for each $n \in \omega$,

$$X_n = \{ \xi \in \mathcal{W} : \xi \not\in \eta \cap \mathcal{A}_i \}$$

Each $X_n$ is non-empty because no condition can force it to be empty. Thus if $\mathcal{X} = \bigcup\{X_n : n \in \omega\}$, $\mathcal{Y} \subseteq \mathcal{X}$.

We show that $|\mathcal{X}| \not\in \mathcal{A}_i \in \mathcal{R}_i$. If not there is a function $f : \mathcal{X} \to \mathcal{X} \cup \{1, 2\}$, "onto", as given by Lemma 5. (Each $X_n \in \mathcal{A}_i \subseteq \mathcal{R}_i$ by Theorem 1 (vi)).

Then for some $p \in \mathcal{A}_i$, $p$ forces "$\exists f$. Let $A$ be a finite subset of $U$ containing $p$, $\sigma(f)$, and any members of $U$ involved in $p$. By choice of $f$ there are $\xi, \eta \subseteq U$ such that $0 < |\xi| < |\eta|, (\xi, \eta) \not\subseteq \mathcal{A}_i$, some $n > 0, \eta \cap A = \xi \cap A$.

Let $q \in \mathcal{A}_i, q \supset p$, and $g$ force $(\xi, \eta) \not\subseteq \mathcal{A}_i$. If $\mathcal{Y} \subseteq \mathcal{A}_i$, by definition of $X_n$, and since $\mathcal{Y} \neq \mathcal{A}_i$, $P(\mathcal{W} \cap \mathcal{Y}) = \mathcal{U}$. However this is contrary to $\mathcal{X} \not\subseteq \mathcal{Y}$.

Therefore $\mathcal{X} \subseteq \mathcal{Y}$.

Let $\sigma, \eta \subseteq \mathcal{Y}$, and let $B$ be the set of all members of $U$ involved in $\lambda$. $\eta, \sigma, q, l \subseteq U \supseteq B$. Since $\eta \cap A = \xi \cap A \subseteq \mathcal{Y}$. If $\sigma$ is the permutation of $U$ interchanging $\xi$ and $\eta$, then $\sigma \not\in \mathcal{A}_i$.

But $\sigma$ fixes $(\xi, \eta) \not\subseteq \mathcal{A}_i$.

Then, as $f$ is $1-1$, $\mathcal{A}_i$ can be mapped $1-1$ into $\mathcal{A}_i$. Since $\mathcal{A}_i \not\subseteq \mathcal{A}_i$, $\mathcal{A}_i \not\subseteq \mathcal{A}_i$. The same argument that was used to show that $\mathcal{A}_i \not\subseteq \mathcal{A}_i$ shows that $\mathcal{A}_i \neq \mathcal{A}_i$.

Hence $\mathcal{A}_i \neq \mathcal{A}_i$.

Therefore $\mathcal{A}_i \not\subseteq \mathcal{A}_i$.

This means that $|\mathcal{A}_i| \not\subseteq \mathcal{A}_i$. (Because $\mathcal{A}_i \not\subseteq \mathcal{A}_i$ holds in $\mathcal{R}_i$).

Now let $\mathcal{B}$ be a set of $\mathcal{A}_i$ disjoint $\mathcal{A}_i$ infinite subsets of $\mathcal{A}_i$.

It is easily seen that $\mathcal{A}_i$ and $\mathcal{A}_i$ are compatible.
Thus if we map \( a \times a^{\aleph_0} \) by \( g \), thus:

\[ g([u, v], f) = [\bigcup \{ f(t) \times X_{\alpha} : t \in a \}] \]

we certainly have \( A \subset g^{\aleph_0}(a \times a^{\aleph_0}) \), (and in fact =).

We now show that \( g \in R_{\aleph_0} \) (or \( R_{\aleph_0} \)). In fact \( H(g) = G \). To show this it is enough that \( \sigma[X] = [X] \), any \( \sigma \in G \times X = \in R_{\aleph_0} \). (This actually shows that any set of cardinals can be well-ordered.)

For \( R_{\aleph_0} \) we show that \( H(\sigma[X] \supset H(X) \times X) \hspace{1em} \cap \hspace{1em} \cap H(\sigma[X]) \)，where \( n \) is the least integer such that \( \sigma \) fixes all \( u_m \) with \( m \geq n \).

For \( \tau \in H(X) \times (H(X) \cap H(\sigma[X])) \), respectively, \( \tau \) commutes with \( \sigma \), so

\[ \tau(\sigma[X]) = \tau(\xi_\sigma, \sigma) : \xi \in \tau[X] = \{ (\tau, \tau_\sigma) : \xi \in \tau[X] \}
\]

\[ = (\xi \tau, \sigma) : \xi \in \tau[X] \hspace{1em} \text{as} \hspace{1em} \tau \in H(X) \times (H(X) \cap H(\sigma[X])) \]

\[ = (\xi, \sigma) : \xi \in \tau[X] = \sigma[X] \]

Therefore \( |\sigma| \leq 2^{\aleph_0} \). Since \( 2^{\aleph_0} \) is a well-ordered cardinal, we have \( |\sigma| \leq 2^{\aleph_0} \).

Finally to show that in \( R_{\aleph_0} \) and \( R_{\aleph_0} \) there is a set of \( 2^{\aleph_0} \) mutually incomparable members of \( A \), let \( A \) again be a set of \( 2^{\aleph_0} \) almost disjoint infinite subsets of \( a \).

Then \( \{ X_{\alpha} : A \in A \} \) is a set of \( 2^\aleph_0 \) mutually incomparable members of \( A \), where \( X_A = \bigcup \{ X_{\alpha} : A \in A \} \).

References

О теореме Викториса в категории гомотопий и одной проблеме Ворсука

С. Богатый (Москва)

Абстракт. Рассматриваются отображения (*) метрических компактов, у которых все прямые прообразы точек аппроксимированы связным в некоторой размерности в [7]. В частности, показывается аналог теоремы Викториса [23] и Смейла [22]: если для отображения \( f: X \to Y \) компакта \( X \in \text{LC}^m \) на компакт \( Y \) прообразы \( f^{-1}(y) \in \text{LC}^m \) для всех \( y \in Y \), то \( Y \in \text{LC}^m \) и индуцированное отображение \( f_*: \pi_r(X) \to \pi_r(Y) \) (как отображение \( X \) гомотопичных отображений) означает, как только \( \dim X \leq m \). Это позволяет доказать итеративный метод на одну проблему К. Ворсука [1]: если для отображения \( f: X \to Y \) компактного компакта \( X \) на компактный компакт \( Y \) прообразы \( f^{-1}(y) \) ГАЕ для всех \( y \in Y \), то \( \text{Sh}(X) = \text{Sh}(X) \). Кроме того, в ответ на вопрос К. Ворсука [7] доказывается, что для компактных абстрактных гомотопий это в точности подложные компакты, которые аппроксимированы связным в размерности.

Пусть \( X \) и \( Y \) компактные метрические пространства и пусть отображение \( f: X \to Y \) является \( \text{LC}^m \). Тогда для всех \( 0 < m \leq r \leq m-1 \), индуцированная гомотопия \( f_*: \pi_r(X) \to \pi_r(Y) \) является изоморфизмом на \( r < m-1 \). Существуют примеры, показывающие, что аналогичная теорема для гомотопий неверна. Тем не менее, аналог непротиворечивых условий, Смейл [22] доказал аналогочную теорему для гомотопий (иногда применима для требования двойной компактности).

Пусть \( f: X \to Y \) отображение связных пространств \( X \) и \( Y \) в \( \text{LC}^m \) и для всех \( y \in Y \), \( f^{-1}(y) \) является локально связным связным в размерности \( m-1 \) пространством (т. е. \( f^{-1}(y) \in \text{LC}^{m-1} \)). Тогда \( Y \in \text{LC}^m \) и индуцированный гомотопизм \( f_*: \pi_r(X) \to \pi_r(Y) \) является изотопию на всех \( 0 \leq r \leq m-1 \) и на всех \( r = m \).

Основная цель работы состоит в демонстрации теоремы Викториса в категории гомотопий, в которой на пространства \( X \) и \( f^{-1}(Y) \) не накладываются никаких локальных условий, но, как уже говорилось, в токе Смейла просто зачеркнуты \( \text{LC}^m \) и \( \text{LC}^{m-1} \) нельзя, потому что множеств

(*) После представления статьи автор узнал, что некоторые аналогичные результаты получены для топологических отображений в [25], для компактно-дискретных отображений в [27], [28] и для "U-r"-отображений в [35].