On the hyperspaces of hereditarily indecomposable continua

by

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Abstract. H. Whitney described a map on the hyperspace $C(X)$ into reals, for a continuum $X$. If $X$ is hereditarily indecomposable, then the Whitney map can be used to define some new maps on $X$ and $X 	imes I$. Using these maps, we exhibit close relationships between $C(X)$ and the cones over $X$. Next we study the structure of arcwise connected subsets of $C(X)$ and we show that some of these results imply several known theorems. Some results on embeddability of $C(X)$ and non-embeddability in $C(X)$ are proved.

Introduction. A continuum $X$ is called decomposable if it is the union of two proper subcontinua. Otherwise it is indecomposable. A continuum is hereditarily indecomposable if its every subcontinuum is indecomposable. For a continuum $X$ the symbol $C(X)$ denotes the hyperspace of all non-empty subcontinua of $X$ metricized by the Hausdorff metric: $d(A, B) = \inf \{ \varepsilon > 0 : B \subset \mathcal{K}(A, \varepsilon) \text{ and } A \subset \mathcal{K}(B, \varepsilon) \}$, where $\mathcal{K}(A, \varepsilon)$ denotes an open ball with radius $\varepsilon$ around $A$. The hyperspaces possess many interesting properties and the case where $X$ is hereditarily indecomposable is particularly interesting. If $X$ is non-empty, then it is a point of $C(X)$ and we call it the vertex of $C(X)$. By $\mathring{X}$ we denote the base of $C(X)$, i.e. the set of all singletons of $C(X)$. Clearly, $\mathring{X}$ is isometric to $X$ and sometimes it is identified with $X$. A maximal monotone collection of continua between $A$, $B \in C(X)$, $A \subset B$, is called a segment and denoted by $AB$. It is known (see e.g. [6]) that if $A \neq B$, then $AB$ is an arc in $C(X)$. A segment of the form $(a, x)X$ is called maximal. It is obvious that every $a \notin C(X)$ belongs to a maximal segment.

1. Maps associated with given hyperspaces. As has been noted by Kelley [6], there exists a continuous map $\mu : C(X) \to [0, \infty)$ (originally due to Whitney [16]) with the following properties:

1. if $A \subset B$ and $A \neq B$, then $\mu(A) < \mu(B)$,
2. $\mu(\{x\}) = 0$, for every $x \in X$.

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Each map with these properties will be called a Whitney map. We denote by \( I_0 \) the image of \( \mu \). It follows from (ii) that \( I_0 = [0, \mu(X)] \) provided \( X \) is non-empty, and otherwise \( I_0 = \emptyset \). Clearly, if \( I_0 \) contains at least two points, then there exists a Whitney map \( \mu' \) such that \( \mu'(I) = I \), i.e., \( I_0 \). It is known (see [3] and [20]) that \( \mu^{-1}(t) \) is a continuum and \( \mu : C(X) \to I_0 \) is a continuous, i.e., upper and lower semi-continuous, decomposition of \( C(X) \).

A recent result of the author shows that if \( X \) is a snake-like (circle-like) continuum, then every non-degenerate element of \( E \) is a snake-like (respectively circle-like) continuum [6]. We shall see in the sequel that the Whitney maps possess other important properties.

Let \( \mu \) be a fixed Whitney map on \( C(X) \).

1.1. If \( A \subseteq C(X) \), then \( \mu^{-1}(t) \cap C(A) \) is a continuum.

In fact, \( \mu(A) = \mu(\cup A) \) is a Whitney map on \( C(A) \). Moreover,

\[
\mu^{-1}(t) = \mu^{-1}(t) \cap C(A);
\]

hence this set is a continuum because any Whitney map is monotone.

Since \( C(X) \) is a continuum, the space \( C(\mu^{-1}(t)) \) is well defined and we denote it by \( C^0(I) \). Kelley [6] showed that if \( A \subseteq C(X) \), then \( \bigcup A \) is a continuum, i.e., an element of \( C(X) \), and that the correspondence \( A \mapsto \bigcup A \) defines a continuous retraction

\[
\sigma : C(X) \to C^0(I),
\]

\( C(X) \) being regarded, as usual, as a subspace of \( C^0(I) \). Consider for each the map

\[
\sigma_t = \sigma \circ \mu(\mu^{-1}(t)).
\]

We have

\[
\mu(\sigma(A)) \geq t \text{ for } A \subseteq C(\mu^{-1}(t)).
\]

Indeed, if \( A \subseteq C(\mu^{-1}(t)) \), then \( A \neq \emptyset \) and \( \sigma(A) \subseteq C(X) \). Let \( A \subseteq \sigma(A) \); hence by (i) we obtain \( t = \mu(A) \leq \mu(\sigma(A)) \), which proves the above proposition. Thus we may regard \( \sigma_t \) as a map

\[
\alpha_t : C(\mu^{-1}(t)) \to \mu^{-1}(t) \cup \{t, \emptyset\}.
\]

The map \( \alpha_t \) has the following properties (comp. [11], Lemma 2.1):

1.2. \( \alpha_t \) is onto.

In fact, let \( A \subseteq \mu^{-1}(t, \infty) \). According to 1.1 the set \( \mu^{-1}(t) \) is a continuum. Since \( \mu(A) \geq t \), it is not empty. It follows that \( \mu^{-1}(t) \) is an element of \( C(\mu^{-1}(t)) \). Since \( \sigma_t(\mu^{-1}(t)) \subseteq A \), we need only to show the reverse inclusion. Let \( \sigma(A) \subseteq A \) and let \( \alpha_t(\sigma(A)) \) be a segment in \( C(X) \). Since \( \mu(\sigma(A)) = 0 \), there is a continuum \( B \subseteq \sigma(A) \) such that \( \mu(B) = t \). Hence \( B \subseteq \mu^{-1}(t) \) and therefore \( A \subseteq B \cup \alpha_t(\sigma(A)) \), which completes the proof.

1.3. If \( X \) is hereditarily indecomposable, then \( \alpha_t \) is one-to-one.

Indeed, suppose \( A, B \subseteq C(\mu^{-1}(t)) \) and \( \alpha_t(A) = \alpha_t(B) \). By symmetry it suffices to show that \( A \subseteq B \). So let \( A \subseteq B \). There is a \( B \subseteq B \) such that \( A \subseteq B \). Hence either \( A \subseteq B \) or \( A \subseteq B \). But \( \mu(A) = \mu(B) \leq t \); hence by (i) we have in any case \( A = B \). Therefore \( A \subseteq B \), which completes the proof.

Thus we have shown that in the case where \( X \) is a hereditarily indecomposable continuum the map \( \alpha_t \) is a homeomorphism of \( C(\mu^{-1}(t)) \) onto \( \mu^{-1}(t) \cup \{t, \emptyset\} \) (see [11]).

If \( A \) and \( B \) are two subsets of \( X \), then \( d(A, B) = \inf \{g(a, b) : a \in A, b \in B\} \). Now we shall prove a useful simple proposition.

1.4. If \( X \) is a hereditarily indecomposable continuum, then for each \( \varepsilon > 0 \), there exists a positive real number \( \eta > 0 \) such that if \( d(A, B) < \eta \) and \( \mu(A) - \mu(B) < \eta \), then \( \text{dist}(A, B) < \varepsilon \) for every \( A, B \subseteq C(X) \).

Proof. Suppose that it is not true. Hence there exists a sequence of pairs \( (A_n, B_n) \subseteq C(X) \times C(X) \) such that

\[
d(A_n, B_n) < \frac{1}{n}, \quad \mu(A_n) - \mu(B_n) < \frac{1}{n}, \quad \text{and dist}(A_n, B_n) > \varepsilon.
\]

We may assume that \( A_n \) converges to \( A \subseteq C(X) \) and \( B_n \) converges to \( B \subseteq C(X) \). Then by the continuity of \( g(\cdot, \cdot) \), \( \mu \) and \( \text{dist}(\cdot, \cdot) \) we obtain

\[
(1) \quad A \cap B = \emptyset,
\]

\[
(2) \quad \mu(A) = \mu(B),
\]

and

\[
(3) \quad \text{dist}(A, B) > \varepsilon.
\]

By the assumption on \( X \) and by (1) we obtain \( A \subseteq B \) or \( B \subseteq A \). By (2) and (i) we have in any case \( A = B \). This contradicts (3) and this contradiction completes the proof.

There is another interesting map in the hyperspaces of hereditarily indecomposable continua associated with \( \mu \). Namely, for each \( t \subseteq I_0 \) the set \( \mu^{-1}(t) \), regarded as a collection of continua, constitutes by 1.4 a continuous decomposition of \( X \). It follows that the quotient topology on \( \mu^{-1}(t) \) is the same as the topology inherited from \( C(X) \). This means that (see [4], p. 1030)

1.5. The quotient map

\[
\lambda : X \to \mu^{-1}(t),
\]

i.e., such that \( \pi = \lambda(x) \) for \( t \subseteq I_0 \) and for every \( x \subseteq X \), is continuous, monotone, open and onto.
The maps $\lambda_i$ are originally due to Rhein [12]. Setting 
\[ \lambda(x, t) = \lambda(x) \]
for every point $(x, t) \in X \times I_{x}$, we obtain a map 
\[ \lambda : X \times I \rightarrow O(X) \]
Using Lemma 1.4 we shall now prove that

1.6. The map $\lambda$ is a continuous, open and monotone transformation of $X \times I$ onto $O(X)$.

Proof. Let $U$ and $V$ be two open subsets of $X$ and $I$, respectively. It is easy to observe that 
\[ \lambda(U \times V) = \mu^{-1}(V) \cap \{A \in O(X) : A \cap U \neq \emptyset\} \]
Hence, as the intersection of two open subsets of $O(X)$, this set is open in $O(X)$. It follows that $\lambda$ is open. Moreover, by 1.4 we infer that if $\text{diam}(U) \rightarrow 0$ and $\text{diam}(V) \rightarrow 0$, then $\text{diam}(\lambda(U \times V)) \rightarrow 0$, which proves the continuity of $\lambda$. Since for $\alpha \in A \in O(X)$ we have $A = \lambda^{-1}(\alpha) = \lambda(\alpha, \mu(A))$, $\lambda$ is onto. To prove that $\lambda$ is monotone it suffices to observe that for each $A \in O(X)$ we have $\lambda^{-1}(A) = A \times \{\mu(A)\}$. This completes the proof.

Let $S(X) = X \times I \times \{\mu(x)\}$ and let $\pi : X \times I \rightarrow S(X)$ be the corresponding quotient map, where $X$ is a hereditarily indecomposable continuum. Thus a continuous map 
\[ \nu : S(X) \rightarrow O(X) \]
is defined such that $\nu \circ \pi = \lambda$. From 1.6 it follows, and this result was also obtained by J. T. Rogers [15], that

1.7. The map $\nu$ is continuous, monotone, open and onto.

2. Arrows in the hyperspaces of hereditarily indecomposable continua. In this section $X$ denotes a hereditarily indecomposable continuum. In [6] Kelley observed that

2.1. $O(X)$ is 1-arrows connected, i.e., for each pair $A, B \in O(X)$ there is an arrow from $A$ to $B$. In particular, if $A \subseteq B$ and $A \neq B$, then the segment $AB$ is the unique arc in $O(X)$ between $A$ and $B$.

Kelley also observed that $\sigma(A) \in A$, for every arc $A$ in $O(X)$. This result can be strengthened as follows.

2.2. If $A$ is an arrow connected continuum of $O(X)$, then $\sigma(A) \in A$.

In fact, $M = \sigma(A)$ is a point of $O(X)$. Suppose that $M \notin A$. Then 
\[ \inf\{\text{dist}(A, M) : A \in \alpha\} > 0 \]
Let $\alpha$ and $\beta$ be two points from distinct components of $M$. Then there are two continua $A, B \in \alpha$ such that $\alpha \in A$ and $\beta \in B$. By (1), $A$ and $B$ are two proper subcontinua of $M_1$; hence $A$ and $B$ are in distinct components of $M$. Let $AM$ and $BM$ be the segments in $O(X)$. Then
\[ (2) \quad AM \cap BM = \{M\} \]
Indeed, suppose $D \subseteq AM \cap BM$. Then $A \cup B \subseteq D \subseteq M_1$; hence $D$ is a continuum in $M$ meeting two distinct components of $M$, and therefore $D = M$, which proves (2). By (2) $AM \cap BM$ an arc in $O(X)$ between $A$ and $B$. Since $A, B \in \alpha$ and $A$ is arcwise connected, $AM \cap BM \subseteq A$, by 2.1. In particular, $M \in A$, contrary to (1).

2.3. Corollary. Any arcwise connected continuum of $O(X)$ is contractible in itself. In particular, $O(X)$ is contractible in itself [6].

Proof. Let $A$ be an arcwise connected continuum of $O(X)$. By 2.2, $E = \sigma(A) \in A$. For every $A \in A$, the segment $AE$ is contained in $A$. Let $\mu$ be a Whitney map on $O(X)$. Since the map $\mu$ restricted to any segment is a homeomorphism, the map 
\[ \varphi : A \times I \rightarrow A \]
given by the formula 
\[ \varphi(A, 0) = (\mu(AE))^{-1}(1 - 1)\mu(A) + 1\mu(B) \]
is well defined. The continuity of $\varphi$ follows from 1.4. Since 
\[ \varphi(A, 0) = A \quad \text{and} \quad \varphi(A, 1) = E, \quad \text{for every} \quad A \in A, \]
$\varphi$ is a contraction of $A$ in itself to $E$, which completes the proof.

2.4. Corollary. Any arcwise connected curve in $O(X)$ is a contractible dendroid.

In fact, any curve contractible in itself is hereditarily unicoherent. If $A$ is an arc in $O(X)$, then the continuum $\sigma(A) \in A$ is called the top of $A$. Let $A$ and $B$ be the end-points of $A$. Since $A, B \subseteq \sigma(A)$, we have, by 2.1, $A = \sigma(A) \cup B \sigma(A)$. Hence we have

2.5. If $A$ is an arc in $O(X)$, then it is either a segment (in this case the top of $A$ is an end-point of $A$) or is the union of two segments with a common end-point being the top of $A$.

2.6. Corollary. If $A_1$ is a subarc of an arc $A$ and $A_2$ is not a segment, then the top of $A_2$ coincides with the top of $A$.

One can easily prove that

2.7. If $L_n \subseteq \mathbb{L}_n \subseteq \mathbb{L}_1$ is an increasing sequence of segments in $O(X)$ with a common top $T$, then $L_n = \bigcup L_n$ is a segment with the end-points $T$ and $P = \cap P_n$, where $P_n$ is the other end-point of $L_n$, for every $n$.

Similarly,
2.8. If $L_1 \subseteq L_2 \subseteq \ldots$ is an increasing sequence of segments in $C(X)$ with a common end-point $P$ and $T_n$ is the top of $L_n$, for every $n$, then $L = \bigcup_n T_n$ is the segment $PT$, where $T = \bigcup_n T_n$ is the top of $L$.

Using 2.6, 2.7 and 2.8 it is easy to see that

2.9. The union of an increasing sequence of arcs in $C(X)$ is contained in an arc.

According to a theorem of Young [17] we obtain, by 2.9, the following known result.

2.10. The space $C(X)$ has the fixed point property [13].

2.11. If $L$ is an arc in $C(X)$ with the end-points $A$ and $B$ and if $D \subseteq C(X)$ is such that

$$A \subset D \quad \text{and} \quad \mu(D) \leq \mu(B),$$

then $D \subseteq L$.

In fact, $B \subset \sigma(L) = P$; hence $\mu(D) \leq \mu(P)$ and finally $D \subseteq AP \subseteq L$, by 2.1.

Lemma 2.11 implies the following theorem.

2.12. Let $L$ be an arc in $C(X)$ with the end-points $A$ and $B$ such that $\mu(A) < \mu(B)$, where $\mu$ is a Whitney map on $C(X)$. Then there exists a positive real number $\epsilon > 0$ with the following property: if $M$ is an arc in $C(X)$ such that $d(B, M) < \epsilon$ and if there is a continuum $E \subseteq C(X)$ with the diameter $\text{diam} E < \epsilon$ joining $A$ and $M$, then $M$ intersects $L$.

Proof. Let

$$r = \mu(B) - \mu(A)$$

and let

$$a = \mu \circ \sigma: C(X) \to \mathbb{R}.$$ 

Since $\mu$ and $a$ are continuous and $a(\sigma(A)) = \mu(\sigma(A))$, there exists an $\epsilon > 0$ such that

(1) if $\text{dist}(B, C) < \epsilon$, then $\mu(C) > \mu(B) - \frac{1}{2} r$;

(2) if $E \subset C(X)$ and $\text{dist}((A), E) < \epsilon$, then $\sigma(E) \leq \mu(A) + \frac{1}{2} r$.

We shall now show that the $\epsilon$ is the required real. Let $M$ and $E$ satisfy the hypothesis of 2.12. We have to show that $M$ intersects $L$. There exist continua $C, F \subseteq M$ such that $\text{dist}(B, C) < \epsilon$ and $F \subseteq E$. Thus, by (1), we obtain

$$\mu(C) > \mu(B) - \frac{1}{2} r.$$ 

Furthermore, since

$$\text{dist}((A), E) \leq \delta(E) < \epsilon,$$

we have by (2)

$$\sigma(E) \leq \mu(\sigma(E)) \leq \mu(A) + \frac{1}{2} r.$$

Let $D = \sigma(E)$. It suffices to prove that $D \not\subseteq L \land M$. We have $A \subseteq E$; hence $A \subseteq D$ and by (4), $\mu(D) \leq \mu(A) + \frac{1}{2} r < \mu(B)$. Therefore, by 2.11, $D \not\subseteq L$.

Likewise, let $N$ be the subarc of $M$ between $F$ and $C$. Since $F \subseteq E, E \subseteq D$, by (3) and (4), we obtain

$$\mu(D) \leq \mu(A) + \frac{1}{2} r = \mu(B) - \frac{1}{2} r = \mu(C);$$

hence, again by 2.11, $D \not\subseteq N \subseteq M$, which completes the proof.

2.13. Let $L$ be an arc in $C(X)$ with the end-points $A$ and $B$. Let $D \supseteq A$ and $E \supseteq B$ be two subcontinua of $C(X)$. Then there exists an $\epsilon > 0$ with the following property: if $M$ is an arc in $C(X)$ joining $D$ and $E$, then $M$ intersects $L$.

Proof. Let $P$ be the top of $L$. We may assume that $A \not\subseteq P$. Then $A \cap P$ and $A \not\subseteq P$, hence $A$ lies in a component $X$ of $P$. Since $X$ is $C(X)$ is continuous and $\sigma(A) = A$, then there exists an $\epsilon > 0$ such that for every continuum $C \subseteq C(X)$ containing $A$ with the diameter less than $\epsilon$, $C \subseteq X$ lies in $X$. Clearly $B \subseteq P$. Consider two cases.

Case (a). $B \not\subseteq P$. Then there exists an $\epsilon > 0$ such that for every continuum $E \subseteq C(X)$ containing $B$, with the diameter less than $\epsilon$, $\sigma(\sigma(E))$ lies in the same component of $P$ as $E$. Note that $E$ lies in a component of $P$ different from $X$, for otherwise $L$ would not contain $P$. So, if $M$ is an arc in $C(X)$ joining $A' \subseteq D$ and $B' \subseteq E$ and $\text{diam} \sigma(D) < \min(\epsilon, \epsilon)$, then $M$ contains $P$ because $A' \subseteq \sigma(D)$ and $B' \subseteq \sigma(E)$ and $\sigma(D)$ and $\sigma(E)$ lie in distinct components of $P$. Hence in case (a) the proof is finished because $P \subseteq L \subseteq M$.

Case (b). $B \subseteq P$. Then $\mu(A) < \mu(P) = \mu(B)$ and applying 2.13 we obtain the conclusion of 2.13 in this case. This completes the proof.

3. On the non-embeddability of certain spaces in the hyperspaces. An immediate consequence of 2.3 is the following result.

3.1. No space which contains an arcwise-connected subcontinuum not contractible in itself can be embedded in the hyperspace of a hereditarily indecomposable continuum.

A sequence $A_1, A_2, \ldots$ of subsets of a metric space is said to be a $0$-sequence provided the diameters of $A_n$'s converge to zero when $n$ tends to infinity. The following notion will play the major role in the main theorem of this section.

A connected space $Y$ is called a ladder provided that it can be represented as the union

$$Y = L \cup \bigcup_{n=1}^{\infty} (A_n \cup B_n \cup L_n),$$
where \( L \) and \( L_n \)'s are arcs and \( A_n \)'s and \( B_n \)'s are continua satisfying the following conditions:

1. \( A_1, A_2, \ldots \) and \( B_1, B_2, \ldots \) are \( n \)-sequences such that
   \[
   \bigcap_n A_n = \{a\} \quad \text{and} \quad \bigcap_n B_n = \{b\}.
   \]

2. \( L \) is an arc joining \( a \) and \( b \), \( a \neq b \).

3. \( L_n \) is an arc joining \( A_n \) and \( B_n \), for every integer \( n \).

4. \( L_n \) is disjoint with \( L \), for every \( n \).

It is evident that for every point \( p \) of a non-degenerate continuum \( X \) there exists a \( 0 \)-sequence of non-degenerate sub-continua of \( X \) each of which contains \( p \). It follows that

3.2. If \( X \) is a non-degenerate continuum, then \( X \times I \) contains a ladder.

In particular, the cone over \( X \) contains a ladder.

Theorem below is an immediate consequence of 2.13 and is the main result of this section.

3.3. \textbf{THEOREM.} If \( X \) is a hereditarily indecomposable continuum, then no ladder can be embedded in \( C(X) \).

3.4. \textbf{COROLLARY.} If \( X \) is a non-degenerate continuum, then \( X \times I \) cannot be embedded in the hyperspace of a hereditarily indecomposable continuum. In particular, the cone over \( X \) cannot be embedded in the hyperspace (see [10]).

This corollary answers a question raised in [10].

\textbf{PROBLEM 1.} Can the hyperspace of a hereditarily indecomposable continuum ever contain a topological copy of the Cantor product of two non-degenerate continua? (see [10]).


It is known that for every non-degenerate continuum \( X \) the dimension of \( C(X) \) is at least two (see [4] and [7]). In the case where \( X \) is hereditarily indecomposable, \( \dim C(X) \) is either 2 or \( \infty \) [4]. This interesting result solves almost entirely the problem of dimension of the hyperspaces for hereditarily indecomposable continua. However, the following problem is still open.

\textbf{PROBLEM 2.} Is it or is it not true that the dimension of the hyperspace of a hereditarily indecomposable curve equals two (see [4])?

A partial solution of this problem is given by the following result:

4.1. If \( X \) is a hereditarily indecomposable tree-like curve, then \( \dim C(X) = 2 \).

A well-known result of Hurewicz [9, p. 114] asserts that if \( f: X \to Y \) is a continuous transformation of a compact space \( X \) and

\[
\dim f^{-1}(y) = k, \quad \text{for every } y \in Y,
\]

then \( \dim X \leq \dim Y + k \). Hence 4.1 is implied by the following general result.

4.2. \textbf{If} \( X \) \textbf{is a hereditarily indecomposable tree-like curve and} \( \mu \) \textbf{is a Whitney map on} \( C(X) \), then \( E = \{\mu^{-1}(t) : t \in I\} \) is a continuous decomposition of \( C(X) \) such that each non-degenerate element of \( E \) is a hereditarily indecomposable tree-like curve.

Let \( S \) denote the unit circle, i.e., \( S = \{t \in \mathbb{R}^2 : |t| = 1\} \).

\textbf{LEMMA [9, p. 435].} Let \( g: D \to \mathbb{R}^2 \) be a continuous monotone map of a compact space \( D \) onto \( \mathbb{R}^2 \). Then, if \( h: \mathbb{R}^2 \to \mathbb{R}^2 \) is a continuous map such that \( h \circ g \sim 1 \), then \( h \sim 1 \) (where \( f \sim 1 \) means \( f \) is homotopic to a constant map).

4.3. If \( X \) is a tree-like curve and \( f: X \to Y \) is a monotone map onto \( Y \), then \( Y \) is a tree-like curve.

This theorem is probably well known to many people but the author does not know any reference and therefore a proof is presented here.

\textbf{Proof of 4.3.} First we show that

\[
(1) \quad \dim Y = 1.
\]

Let \( P \) be a closed subset of \( Y \) and let \( h: P \to \mathbb{R}^2 \) be a continuous map. By [9], p. 354, we need only to prove that \( h \) can be extended onto \( Y \). Let \( D = f^{-1}(P) \) and let \( g = f|D \). Then \( h \circ g: D \to \mathbb{R}^2 \) is continuous. Since \( \dim X = 1 \), \( h \circ g \) can be extended to a map \( h: X \to \mathbb{R}^2 \) (see [9], p. 354).

Then \( h \sim 1 \) (see [1]), and therefore \( h \circ g \sim 1 \) since \( g \) is monotone we infer by the lemma above that \( h \sim 1 \). By the Borsuk extension theorem, \( h \) can be extended onto \( Y \), which completes the proof of (1).

By (1), to prove that \( Y \) is tree-like we need only to show that if \( g: Y \to P \) is a continuous map of \( Y \) into a 1-dimensional polyhedron \( P \), then (see [13])

\[
(2) \quad g \sim 1.
\]

Let \( Z \) be the universal covering space for \( P \) and let \( p: Z \to P \) be the projection map. Since \( g \circ f: X \to P \) and \( X \) is tree-like, we have \( g \circ f \sim 1 \) (see [1]); hence we can lift \( f \) to \( Z \), i.e., there is a continuous map \( h: X \to Z \) such that

\[
(3) \quad p \circ h = g \circ f.
\]

Since the fibre \( p^{-1}(u) \) is a discrete space for every \( u \in P \) and \( k(p^{-1}(y)) \) is a continuum contained, by (3), in \( p^{-1}(g(y)) \) for every \( y \in Y \), we infer that
5.1. If $X$ is a hereditarily indecomposable plane continuum such that no proper subcontinuum of $X$ separates the plane, then $C(X)$ can be embedded in $E^3$.

Proof. We may assume that $X$ is non-degenerate and $\mu(X) = 1$. Let $v: S(X) \rightarrow \bar{C}(X)$ be the map described in §1. We regard $X$ as a continuum lying in $E^3 \cap E^2$, where $E^2 = \{ x \in E^3 : z = (x, y, z) \}$, and $S(X)$ as the geometrical cone, i.e.,

$$S(X) = \{(1-t)x + tw: x \in X, \; t \in I\},$$

where $w = (0, 0, 1)$ is the vertex of $S(X)$. Then, by 1,5,

$$D = \{v^{-1}(A): A \in \bar{C}(X) \} \cup \{\text{single points of } E^3 \setminus S(X)\}$$

is a monotone and upper-semicontinuous decomposition of $E^3$ such that each element of $D$ lies in a plane parallel to $E^2$ and does not separate this plane because of

$$v^{-1}(A) = \{(1 - \mu(A))x + \mu(A)w: x \in A \} \subset \{(x, y, z): z = \mu(A)\}$$

is a continuum homeomorphic to $A$ provided that $A \neq X$ and $v^{-1}(X) = \{w\}$. According to [3], the quotient space $E^3/D$ is homeomorphic to $E^2$. Let $g: E^3 \rightarrow E^2/D$ be the projection. Then, by 1,7, $q^{-1}: C(X) \rightarrow E^2/D$ is the desired embedding. This completes the proof.

5.2. If $X$ is a hereditarily indecomposable plane continuum such that no proper subcontinuum of $X$ separates the plane, then $\dim C(X) = 2$.

In fact, by 5.1, $\dim C(X) \leq 3$ and therefore $\dim C(X) = 2$ because $X$ is hereditarily indecomposable (see [4]).

Finally, we recall that the hyperspaces of snake-like and circle-like plane continua are embeddable in $E^2$ (see [5] and [14]).

Problem 3. Can the hyperspace of a hereditarily indecomposable plane continuum be embedded in $E^2$?

Problem 4. Suppose that $X$ is a continuum such that $C(X)$ can be embedded in $E^2$. Can $X$ be embedded in the plane?

A positive answer to Problem 4 in the case where $X$ is a locally connected continuum follows from a formula of Kelley [6, p. 30]. The author acknowledges his gratitude to the referees, who made several valuable suggestions.

References

Classes of Dedekind finite cardinals (1)

by

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Abstract. We discuss seven possible definitions of “finiteness” of cardinal numbers, and associate with each definition the class of cardinals “finite” in that sense. The results are extensions of those in Levy’s paper “The independence of various definitions of finiteness” (Fund. Math. 46 (1958), pp. 1–13). We investigate the closure of the classes under addition, multiplication, unions, and disjoint unions. In the final section we give a wide variety of possible combinations of inclusions and equalities between the classes. Also we give an affirmative answer to Tarski’s question “Can there be exactly 2n Dedekind finite cardinals?”

§ 1. The object of this paper is to investigate various properties of seven classes of Dedekind finite cardinals, and to discover what possible combinations of inclusions and equalities can hold between them. The starting point is Levy’s paper “The independence of various definitions of finiteness” [8]. All but two of the seven classes correspond to definitions of Levy (which were in turn taken from Tarski [18]), and the other two arise quite naturally.

The classes are defined in § 2, and some of their elementary properties given. In § 3 we discuss the closure of the classes under +, ×, unions, disjoint unions, and ⊆, and a model is given in which A1 is not closed under ×. In § 4 we show that if any two Dedekind finite cardinals are comparable, then any infinite set is the disjoint union of two infinite sets. § 5 contains five models which establish various possibilities of strict inclusion and equality between the classes. Various combinations of these models actually yield thirteen models in which the combination of the classes actually yield thirteen models in which the combination is different. It is shown that there can be no more than twenty-three possible combinations, using the results of § 2 and § 3. All of the ten unsolved cases involve the following situation:

There is an infinite set with no infinite orderable subset, and if a set X has a countable partition, there is a map from X onto X which is not 1-1.

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