On the hyperspaces of hereditarily indecomposable continua

by

J. Krasinkiewicz (Warszawa)

Abstract. H. Whitney described a map on the hyperspace $C(X)$ into reals, for a continuum X. If X is hereditarily indecomposable, then the Whitney map can be used to define some new maps on X and $X\times I$. Using these maps, we exhibit close relationships between $C(X)$ and the cone over X. Next we study the structure of arcwise connected subsets of $C(X)$ and we show that some of these results imply several known theorems. Some results on embeddability of $C(X)$ and non-embeddability in $C(X)$ are proved.

Introduction. A continuum X is called *decomposable* if it is the union of two proper subcontinua. Otherwise it is *indecomposable*. A continuum is *hereditarily indecomposable* if its every subcontinuum is indecomposable. For a continuum X the symbol $C(X)$ denotes the *hyperspace* of all non-empty subcontinua of X metricized by the Hausdorff metric: $\operatorname{dist}(A, B) = \inf\{\varepsilon > 0\colon B\subset K(A, \varepsilon) \text{ and } A\subset K(B, \varepsilon)\}$, where $K(A, \varepsilon)$ denotes an open ball with radius ε around A. The hyperspaces possess many interesting properties and the case where X is hereditarily indecomposable is particularly interesting. If X is non-empty, then it is a point of $C(X)$ and we call it the *vertex* of $C(X)$. By $\hat{X}$ we denote the *base* of $C(X)$, i.e. the set of all singletons of $C(X)$. Clearly, $\hat{X}$ is isometric to X and sometimes it is identified with X. A maximal monotone collection of continua between $A, B\in C(X)$, $A\subset B$, is called a *segment* and denoted by AB. It is known (see e.g. [6]) that if $A\neq B$, then AB is an arc in $C(X)$. A segment of the form $\{x\}X$ is called *maximal*. It is obvious that every $A\in C(X)$ belongs to a maximal segment.

1. Maps associated with given hyperspaces. As has been noted by Kelley [6], there exists a continuous map $\mu\colon C(X)\to[0, \infty)$ (originally due to Whitney [16]) with the following properties:

(i) if $A\subset B$ and $A\neq B$, then $\mu(A) < \mu(B)$,

(ii) $\mu(\{x\}) = 0$, for every $x\in X$.

Each map with these properties will be called a *Whitney map*. We denote by I_μ the image of μ. It follows from (i) that $I_\mu = [0, \mu(X)]$ provided X is non-empty, and otherwise $I_\mu = \emptyset$. Clearly, if I_μ contains at least two points, then there exists a Whitney map μ' such that $\mu'(X) = 1$, i.e. $I_\mu = I$. It is known (see [4] and [8]) that $\mu^{-1}(t)$ is a continuum and $\mu\colon C(X)\to I_\mu$ is an open map. It follows that $\boldsymbol{E} = \{\mu^{-1}(t)\colon t \in I_\mu\}$ is a continuous, i.e. upper and lower semicontinuous, decomposition of $C(X)$. A recent result of the author shows that if X is a snake-like (circle-like) continuum, then every non-degenerate element of $\boldsymbol{E}$ is a snake-like (respectively circle-like) continuum [8]. We shall see in the sequel that the Whitney maps possess other important properties.

Let μ be a fixed Whitney map on $C(X)$.

1.1. *If $A \in C(X)$, then $\mu^{-1}(t) \cap C(A)$ is a continuum.*

In fact, $\mu_A = \mu|C(A)$ is a Whitney map on $C(A)$. Moreover,

$$\mu_A^{-1}(t) = \mu^{-1}(t) \cap C(A);$$

hence this set is a continuum because any Whitney map is monotone.

Since $C(X)$ is a continuum, the space $C(C(X))$ is well defined and we denote it by $C^2(X)$. Kelley [6] showed that if $\boldsymbol{A} \in C^2(X)$, then $\bigcup \boldsymbol{A}$ is a continuum, i.e. an element of $C(X)$, and that the correspondence $\boldsymbol{A}\to\bigcup\boldsymbol{A}$ defines a continuous retraction

$$\sigma\colon C^2(X)\to C(X),$$

$C(X)$ being regarded, as usual, as a subspace of $C^2(X)$. Consider for each t the map

$$\sigma_t = \sigma\,|\,C(\mu^{-1}(t)).$$

We have

$$\mu(\sigma_t(\boldsymbol{A})) \geqslant t \quad \text{for} \quad \boldsymbol{A} \in C(\mu^{-1}(t)).$$

Indeed, if $\boldsymbol{A} \in C(\mu^{-1}(t))$, then $\boldsymbol{A} \neq \emptyset$ and $\sigma(\boldsymbol{A}) \in C(X)$. Let $A \in \boldsymbol{A}$. Then $A \subset \sigma(\boldsymbol{A})$; hence by (i) we obtain $t = \mu(A) \leqslant \mu(\sigma(\boldsymbol{A}))$, which proves the above proposition. Thus we may regard σ_t as a map

$$\sigma_t\colon C(\mu^{-1}(t))\to\mu^{-1}([t, \infty)).$$

The map σ_t has the following properties (comp. [11], Lemma 2.1).

1.2. *σ_t is onto.*

In fact, let $A \in \mu^{-1}([t, \infty))$. According to 1.1 the set $\mu_A^{-1}(t)$ is a continuum. Since $\mu(A) \geqslant t$, it is not empty. It follows that $\mu_A^{-1}(t)$ is an element of $C(\mu^{-1}(t))$. Since $\sigma_t(\mu_A^{-1}(t)) \subset A$, we need only to show the reverse inclusion. Let $a \in A$ and let $\{a\}A$ be a segment in $C(X)$. Since $\mu(\{a\}) = 0$, there is a continuum $B \in \{a\}A$ such that $\mu(B) = t$. Hence $B \in \mu_A^{-1}(t)$ and therefore $a \in B \subset \sigma_t(\mu_A^{-1}(t))$, which completes the proof.

1.3. *If X is hereditarily indecomposable, then σ_t is one-to-one.*

Indeed, suppose $\boldsymbol{A}, \boldsymbol{B} \in C(\mu^{-1}(t))$ and $\sigma_t(\boldsymbol{A}) = \sigma_t(\boldsymbol{B})$. By symmetry it suffices to show that $\boldsymbol{A} \subset \boldsymbol{B}$. So let $A \in \boldsymbol{A}$. There is a $B \in \boldsymbol{B}$ such that $A \cap B \neq \emptyset$. Hence either $A \subset B$ or $A \supset B$. But $\mu(A) = \mu(B) = t$; hence by (i) we have in any case $A = B$. Therefore $A \in \boldsymbol{B}$, which completes the proof.

Thus we have shown that *in the case where X is a hereditarily indecomposable continuum the map σ_t is a homeomorphism of $C(\mu^{-1}(t))$ onto* $\mu^{-1}([t, \infty))$ (see [11]).

If A and B are two subsets of X, then $d(A, B) = \inf\{\varrho(a, b)\colon a \in A, b \in B\}$. Now we shall prove a useful simple proposition.

1.4. *If X is a hereditarily indecomposable continuum, then for each $\varepsilon > 0$ there exists a positive real number $\eta > 0$ such that if $d(A, B) < \eta$ and $|\mu(A) - \mu(B)| < \eta$, then $\operatorname{dist}(A, B) < \varepsilon$ for every $A, B \in C(X)$.*

Proof. Suppose that it is not true. Hence there exists a sequence of pairs $(A_n, B_n) \in C(X)\times C(X)$ such that

$$d(A_n, B_n) < \frac{1}{n}, \quad |\mu(A_n) - \mu(B_n)| < \frac{1}{n} \quad \text{and} \quad \operatorname{dist}(A_n, B_n) \geqslant \varepsilon.$$

We may assume that A's converge to $A \in C(X)$ and B's converge to $B \in C(X)$. Then by the continuity of $\varrho(\cdot, \cdot)$, μ and $\operatorname{dist}(\cdot, \cdot)$ we obtain

$$A \cap B \neq \emptyset, \tag{1}$$

$$\mu(A) = \mu(B) \tag{2}$$

and

$$\operatorname{dist}(A, B) \geqslant \varepsilon. \tag{3}$$

By the assumption on X and by (1) we obtain $A \subset B$ or $B \subset A$. By (2) and (i) we have in any case $A = B$. This contradicts (3) and this contradiction completes the proof.

There is another interesting map in the hyperspaces of hereditarily indecomposable continua associated with μ. Namely, for each $t \in I_\mu$ the set $\mu^{-1}(t)$, regarded as a collection of continua, constitutes by 1.4 a continuous decomposition of X. It follows that the quotient topology on $\mu^{-1}(t)$ is the same as the topology inherited from $C(X)$. This means that (see [4], p. 1030)

1.5. *The quotient map*

$$\lambda_t\colon X\to\mu^{-1}(t),$$

i.e. such that $x \in \lambda_t(x)$ for $t \in I_\mu$ and for every $x \in X$, is continuous, monotone, open and onto.

The maps λ_t are originally due to Rhee [12]. Setting

$$\lambda(x, t) = \lambda_t(x)$$

for every point $(x, t) \in X \times I_\mu$, we obtain a map

$$\lambda\colon X \times I_\mu \to C(X).$$

Using Lemma 1.4 we shall now prove that

1.6. *The map λ is a continuous, open and monotone transformation of $X \times I_\mu$ onto $C(X)$.*

Proof. Let U and V be two open subsets of X and I_μ respectively. It is easy to observe that

$$\lambda(U \times V) = \mu^{-1}(V) \cap \{A \in C(X)\colon A \cap U \neq \emptyset\}.$$

Hence, as the intersection of two open subsets of $C(X)$, this set is open in $C(X)$. It follows that λ is open. Moreover, by 1.4 we infer that if $\operatorname{diam} V \to 0$ and $\operatorname{diam} U \to 0$, then $\operatorname{diam} \lambda(U \times V) \to 0$, which proves the continuity of λ. Since for $x \in A \in C(X)$ we have $A = \lambda_{\mu(A)}(x) = \lambda(x, \mu(A))$, λ is onto. To prove that λ is monotone it suffices to observe that for each $A \in C(X)$ we have $\lambda^{-1}(A) = A \times \{\mu(A)\}$. This completes the proof.

Let $S(X) = X \times I_\mu / X \times \{\mu(X)\}$ and let $p\colon X \times I_\mu \to S(X)$ be the corresponding quotient map, where X is a hereditarily indecomposable continuum. Thus a continuous map

$$\nu\colon S(X) \to C(X)$$

is defined such that $\nu \circ p = \lambda$. From 1.6 it follows, and this result was also obtained by J. T. Rogers [13], that

1.7. *The map ν is continuous, monotone, open and onto.*

2. Arcs in the hyperspaces of hereditarily indecomposable continua. In this section X denotes a hereditarily indecomposable continuum. In [6] Kelley observed that

2.1. *$C(X)$ is 1-arcwise connected, i.e., for each pair $A, B \in C(X)$ there is in $C(X)$ exactly one arc between A and B. In particular, if $A \subset B$ and $A \neq B$, then the segment AB is the unique arc in $C(X)$ between A and B.*

Kelley also observed that $\sigma(\mathcal{A}) \in \mathcal{A}$, for every arc $\mathcal{A}$ in $C(X)$. This result can be strenghtened as follows.

2.2. *If $\mathcal{A}$ is an arcwise connected subcontinuum of $C(X)$, then $\sigma(\mathcal{A}) \in \mathcal{A}$.*

In fact, $M = \sigma(\mathcal{A})$ is a point of $C(X)$. Suppose that $M \notin \mathcal{A}$. Then

$$\inf\{\operatorname{dist}(A, M)\colon A \in \mathcal{A}\} > 0. \tag{1}$$

Let a and b be two points from distinct composants of M. Then there are two continua $A, B \in \mathcal{A}$ such that $a \in A$ and $b \in B$. By (1), A and B are two proper subcontinua of M; hence A and B are in distinct composants of M. Let AM and BM be the segments in $C(X)$. Then

$$AM \cap BM = \{M\}. \tag{2}$$

Indeed, suppose $D \in AM \cap BM$. Then $A \cup B \subset D \subset M$; hence D is a continuum in M meeting two distinct composants of M, and therefore $D = M$, which proves (2). By (2) $AM \cup BM$ is an arc in $C(X)$ between A and B. Since $A, B \in \mathcal{A}$ and $\mathcal{A}$ is arcwise connected, $AM \cup BM \subset \mathcal{A}$, by 2.1. In particular, $M \in \mathcal{A}$, contrary to (1).

2.3. Corollary. *Any arcwise connected subcontinuum of $C(X)$ is contractible in itself. In particular, $C(X)$ is contractible in itself* [6].

Proof. Let $\mathcal{A}$ be an arcwise connected subcontinuum of $C(X)$. By 2.2, $E = \sigma(\mathcal{A}) \in \mathcal{A}$. For every $A \in \mathcal{A}$, the segment AE is contained in $\mathcal{A}$. Let μ be a Whitney map on $C(X)$. Since the map μ restricted to any segment is a homeomorphism, the map

$$\varphi\colon \mathcal{A} \times I \to \mathcal{A}$$

given by the formula

$$\varphi(A, t) = (\mu|AE)^{-1}\big((1-t)\mu(A) + t\mu(E)\big),$$

is well defined. The continuity of φ follows from 1.4. Since

$$\varphi(A, 0) = A \quad \text{and} \quad \varphi(A, 1) = E, \quad \text{for every } A \in \mathcal{A},$$

φ is a contraction of $\mathcal{A}$ in itself to E, which completes the proof.

2.4. Corollary. *Any arcwise connected curve in $C(X)$ is a contractible dendroid.*

In fact, any curve contractible in itself is hereditarily unicoherent.

If $\mathcal{A}$ is an arc in $C(X)$, then the continuum $\sigma(\mathcal{A}) \in \mathcal{A}$ is called the *top* of $\mathcal{A}$. Let A and B be the end-points of $\mathcal{A}$. Since $A, B \subset \sigma(\mathcal{A})$, we have, by 2.1, $\mathcal{A} = A\sigma(\mathcal{A}) \cup B\sigma(\mathcal{A})$. Hence we have

2.5. *If $\mathcal{A}$ is an arc in $C(X)$, then it is either a segment (in this case the top of $\mathcal{A}$ is an end-point of $\mathcal{A}$) or is the union of two segments with a common end-point being the top of $\mathcal{A}$.*

2.6. Corollary. *If $\mathcal{A}'$ is a subarc of an arc $\mathcal{A}$ and $\mathcal{A}'$ is not a segment, then the top of $\mathcal{A}'$ coincides with the top of $\mathcal{A}$.*

One can easily prove that

2.7. *If $\mathbf{L}_1 \subset \mathbf{L}_2 \subset \ldots$ is an increasing sequence of segments in $C(X)$ with a common top T, then $\mathbf{L} = \overline{\bigcup_n \mathbf{L}_n}$ is a segment with the end-points T and $P = \bigcap_n P_n$, where P_n is the other end-point of $\mathbf{L}_n$, for every n.*

Similarly,

2.8. *If $L_1 \subset L_2 \subset \ldots$ is an increasing sequence of segments in $C(X)$ with a common end-point P and T_n is the top of L_n, for every n, then $L = \overline{\bigcup_n L_n}$ is the segment PT, where $T = \overline{\bigcup_n T_n}$ is the top of L.*

Using 2.6, 2.7 and 2.8 it is easy to see that

2.9. *The union of an increasing sequence of arcs in $C(X)$ is contained in an arc.*

According to a theorem of Young [17] we obtain, by 2.9, the following known result.

2.10. *The space $C(X)$ has the fixed point property* [13].

2.11. *If L is an arc in $C(X)$ with the end-points A and B and if $D \in C(X)$ is such that*

$$A \subset D \quad \text{and} \quad \mu(D) \leqslant \mu(B),$$

then $D \in L$.

In fact, $B \subset \sigma(L) = P$; hence $\mu(D) \leqslant \mu(P)$ and finally $D \in AP \subset L$, by 2.1.

Lemma 2.11 implies the following theorem,

2.12. *Let L be an arc in $C(X)$ with the end-points A and B such that $\mu(A) < \mu(B)$, where μ is a Whitney map on $C(X)$. Then there exists a positive real number $\varepsilon > 0$ with the following property: if M is an arc in $C(X)$ such that $d(B, M) < \varepsilon$ and if there is a continuum $E \subset C(X)$ with the diameter $\operatorname{diam} E < \varepsilon$ joining A and M, then M intersects L.*

Proof. Let

$$r = \mu(B) - \mu(A)$$

and let

$$\alpha = \mu \circ \sigma \colon \; C^2(X) \to I_\mu .$$

Since μ and α are continuous and $\alpha(\{A\}) = \mu(A)$, there exists an $\varepsilon > 0$ such that

(1) if $\operatorname{dist}(B, C) < \varepsilon$, then $\mu(C) \geqslant \mu(B) - \frac{1}{2}r$,

(2) if $E \in C^2(X)$ and $\operatorname{dist}(\{A\}, E) < \varepsilon$, then $\alpha(E) \leqslant \mu(A) + \frac{1}{2}r$.

We shall now show that the ε is the required real. Let M and E satisfy the hypothesis of 2.12. We have to show that M intersects L. There exist continua $C, F \in M$ such that $\operatorname{dist}(B, C) < \varepsilon$ and $F \in E$. Thus, by (1), we obtain

$$\mu(C) \geqslant \mu(B) - \tfrac{1}{2}r . \tag{3}$$

Furthermore, since

$$\operatorname{dist}(\{A\}, E) \leqslant \delta(E) < \varepsilon ,$$

we have by (2)

$$\alpha(E) = \mu(\sigma(E)) \leqslant \mu(A) + \tfrac{1}{2}r . \tag{4}$$

Let $D = \sigma(E)$. It suffices to prove that $D \in L \cap M$. We have $A \in E$; hence $A \subset D$ and by (4), $\mu(D) \leqslant \mu(A) + \frac{1}{2}r < \mu(B)$. Therefore, by 2.11, $D \ni L$. Likewise, let N be the subarc of M between F and C. Since $F \in E$, $F \subset D$. By (3) and (4), we obtain

$$\mu(D) \leqslant \mu(A) + \tfrac{1}{2}r = \mu(B) - \tfrac{1}{2}r \leqslant \mu(C);$$

hence, again by 2.11, $D \in N \subset M$, which completes the proof.

2.13. *Let L be an arc in $C(X)$ with the end-points A and B. Let $D \ni A$ and $E \ni B$ be two subcontinua of $C(X)$. Then there exists an $\varepsilon > 0$ with the following property: if M is an arc in $C(X)$ joining D and E and $\operatorname{diam} D$, $\operatorname{diam} E < \varepsilon$, then M intersects L.*

Proof. Let P be the top of L. We may assume that $A \neq P$. Then $A \subset P$ and $A \neq P$, hence A lies in a composant K of P. Since $\sigma \colon C^2(X) \to C(X)$ is continuous and $\sigma(\{A\}) = A$, then there exists an $\varepsilon_1 > 0$ such that for every continuum $D \subset C(X)$ containing A with the diameter less than ε_1, $\sigma(D)$ lies in K. Clearly $B \subset P$. Consider two cases.

Case (a). $B \neq P$. Then there exists an $\varepsilon_2 > 0$ such that, for every continuum $E \subset C(X)$ containing B, with the diameter less then ε_2, $\sigma(E)$ lies in the same composant of P as does B. Note that B lies in a composant of P different from K, for otherwise L would not contain P. So, if M is an arc in $C(X)$ joining $A' \in D$ and $B' \in E$ and $\operatorname{diam} D$, $\operatorname{diam} E < \min(\varepsilon_1, \varepsilon_2)$, then M contains P because $A' \subset \sigma(D)$ and $B' \subset \sigma(E)$ and $\sigma(D)$ and $\sigma(E)$ lie in distinct composants of P. Hence in case (a) the proof is finished because $P \in L \cap M$.

Case (b). $B = P$. Then $\mu(A) < \mu(P) = \mu(B)$ and applying 2.12 we obtain the conclusion of 2.13 in this case. This completes the proof.

3. On the non-embeddability of certain spaces in the hyperspaces. An immediate consequence of 2.3 is the following result.

3.1. *No space which contains an arcwise connected subcontinuum not contractible in itself can be embedded in the hyperspace of a hereditarily indecomposable continuum.*

A sequence $A_1, A_2, \ldots$ of subsets of a metric space is said to be a 0-*sequence* provided the diameters of A_n's converge to zero when n tends to infinity. The following notion will play the main role in the main theorem of this section.

A connected space Y is called a *ladder* provided that it can be represented as the union

$$Y = L \cup \bigcup_{n=1}^{\infty} (A_n \cup B_n \cup L_n)$$

where L and L_n's are arcs and A_n's and B_n's are continua satisfying the following conditions:

1° $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ are 0-sequences such that

$$\bigcap_n A_n = \{a\} \quad \text{and} \quad \bigcap_n B_n = \{b\}.$$

2° L is an arc joining a and b, $a \neq b$.

3° L_n is an arc joining A_n and B_n, for every integer n.

4° L_n is disjoint with L, for every n.

It is evident that for every point p of a non-degenerate continuum X there exists a 0-sequence of non-degenerate sub-continua of X each of which contains p. It follows that

3.2. *If X is a non-degenerate continuum, then $X \times I$ contains a ladder. In particular, the cone over X contains a ladder.*

Theorem below is an immediate consequence of 2.13 and is the main result of this section.

3.3. THEOREM. *If X is a hereditarily indecomposable continuum, then no ladder can be embedded in $C(X)$.*

3.4. COROLLARY. *If X is a non-degenerate continuum, then $X \times I$ cannot be embedded in the hyperspace of a hereditarily indecomposable continuum. In particular, the cone over X cannot be embedded in the hyperspace* (see [10]).

This corollary answers a question raised in [10].

PROBLEM 1. *Can the hyperspace of a hereditarily indecomposable continuum ever contain a topological copy of the Cartesian product of two non-degenerate continua* (see [10])?

4. Dimension of hyperspaces of hereditarily indecomposable tree-like continua. It is known that for every non-degenerate continuum X the dimension of $C(X)$ is at least two (see [4] and [7]). In the case where X is hereditarily indecomposable, $\dim C(X)$ is either 2 or ∞ [4]. This interesting result solves almost entirely the problem of dimension of the hyperspaces for hereditarily indecomposable continua. However, the following problem is still open.

PROBLEM 2. *Is it or is it not true that the dimension of the hyperspace of a hereditarily indecomposable curve equals two* (see [4])?

A partial solution of this problem is given by the following result:

4.1. *If X is a hereditarily indecomposable tree-like curve, then* $\dim C(X) = 2$.

A well-known result of Hurewicz [9, p. 114] asserts that if $f\colon X \to Y$ is a continuous transformation of a compact space X and

$$\dim f^{-1}(y) \leqslant k, \quad \text{for every } y \in Y,$$

then $\dim X \leqslant \dim Y + k$. Hence 4.1 is implied by the following general result.

4.2. *If X is a hereditarily indecomposable tree-like curve and μ is a Whitney map on $C(X)$, then $E = \{\mu^{-1}(t)\colon t \in I_\mu\}$ is a continuous decomposition of $C(X)$ such that each non-degenerate element of E is a hereditarily indecomposable tree-like curve.*

Let S denote the unit circle, i.e. $S = \{z \in E^2\colon |z| = 1\}$.

LEMMA [9, p. 433]. *Let $g\colon D \to F$ be a continuous monotone map of a compact space D onto F. Then, if $h\colon F \to S$ is a continuous map such that $h \circ g \sim 1$, then $h \sim 1$ (where $f \sim 1$ means: f is homotopic to a constant map).*

4.3. *If X is a tree-like curve and $f\colon X \to Y$ is a monotone map onto Y, then Y is a tree-like curve.*

This theorem is probably well known to many people but the author does not known any reference and therefore a proof is presented here.

Proof of 4.3. First we show that

$$\dim Y = 1. \tag{1}$$

Let F be a closed subset of Y and let $h\colon F \to S$ be a continuous map. By [9], p. 354, we need only to prove that h can be extended onto Y. Let $D = f^{-1}(F)$ and let $g = f|D$. Then $h \circ g\colon D \to S$ is continuous. Since $\dim X = 1$, $h \circ g$ can be extended to a map $k\colon X \to S$ (see [9], p. 354). Then $k \sim 1$ (see [1]), and therefore $h \circ g \sim 1$. Since g is monotone we infer by the lemma above that $h \sim 1$. By the Borsuk extension theorem, h can be extended onto Y, which completes the proof of (1).

By (1), to prove that Y is tree-like we need only to show that if $g\colon Y \to P$ is a continuous map of Y into a 1-dimensional polyhedron P, then (see [1])

$$g \sim 1. \tag{2}$$

Let Z be the universal covering space for P and let $p\colon Z \to P$ be the projection map. Since $g \circ f\colon X \to P$ and X is tree-like, we have $g \circ f \sim 1$ (see [1]); hence we can lift f to Z, i.e., there is a continuous map $k\colon X \to Z$ such that

$$p \circ k = g \circ f. \tag{3}$$

Since the fibre $p^{-1}(u)$ is a discrete space for every $u \in P$ and $k(f^{-1}(y))$ is a continuum contained, by (3), in $p^{-1}(g(y))$ for every $y \in Y$, we infer that

$k(f^{-1}(y))$ is a one-point set. Hence the map $g'\colon Y \to Z$ given by the formula

$$\{g'(y)\} = k(f^{-1}(y)) \tag{4}$$

is well defined. Note that, by (3),

$$p \circ g'(y) = g(y) \tag{5}$$

for every $y \in Y$. We shall show that

(6) g is a continuous map.

Indeed, let $y \in Y$ and let V be an open set in Z containing $g'(y)$. By (4), $k(f^{-1}(y)) \subset V$; hence by the continuity of k there is an open subset G of X such that $f^{-1}(y) \subset G$ and

$$k(G) \subset V. \tag{7}$$

Since f is a closed map, there exists an open subset U of Y such that $y \in U$ and

$$f^{-1}(U) \subset G. \tag{8}$$

By (7) and (8) we have, $g'(U) = k(f^{-1}(U)) \subset V$. Hence U is a neighbourhood of y such that $g'(U) \subset V$, which proves (6).

By (6) and (5), g' is a lifting of g. Since Z is an infinite acyclic 1-dimensional polyhedron, $g'(Y)$ is a compact finite tree. Thus $g'(Y)$ is contractible in itself and therefore $g' \sim 1$. Hence by (5) we obtain (2), which completes the proof.

Proof of 4.2. By [4 or 8], $\mu\colon C(X) \to I_\mu$ is monotone and open; hence E is a continuous decomposition of $C(X)$. As we remarked in § 1, the map $\lambda_t\colon X \to \mu^{-1}(t)$ is monotone and onto. By 4.3, $\mu^{-1}(t)$, $t < \mu(X)$, is a tree-like curve. Since any monotone image of a hereditarily indecomposable continuum is hereditarily indecomposable, $\mu^{-1}(t)$ is hereditarily indecomposable, which completes the proof of 4.2.

5. On the embeddability of hyperspaces in Euclidean spaces. H. Cook described in [2] a class of plane hereditarily indecomposable continua. Eberhart and Nadler [4] asked: Can the hyperspaces of Cook's continua be embedded in E^3?

W. R. R. Transue [15] proved that hyperspaces of hereditarily indecomposable tree-like plane continua can be embedded in E^3. None of the above-mentioned Cook continua is tree-like, hence the Transue theorem cannot be applied in this case. However, if X is a Cook continuum, then each proper subcontinuum of X is a snake-like continuum (a pseudoarc), and hence it does not separate the plane (see [2], th. 1,2). We shall now prove a generalization of the Transue theorem from which it follows that the hyperspaces of the Cook continua can be embedded in E^3. Namely:

5.1. *If X is a hereditarily indecomposable plane continuum such that no proper subcontinuum of X separates the plane, then $C(X)$ can be embedded in E^3.*

Proof. We may assume that X is non-degenerate and $\mu(X) = 1$. Let $\nu\colon S(X) \to C(X)$ be the map described in § 1. We regard X as a continuum lying in $E^2 \subset E^3$, where $E^2 = \{z \in E^3\colon z = (x, y, 0)\}$, and $S(X)$ as the geometrical cone, i.e.,

$$S(X) = \{(1-t)x + tw\colon x \in X,\ t \in I\},$$

where $w = (0, 0, 1)$ is the vertex of $S(X)$. Then, by 1.5,

$$D = \{\nu^{-1}(A)\colon A \in C(X)\} \cup \{\text{single points of } E^3 \setminus S(X)\}$$

is a monotone and upper-semicontinuous decomposition of E^3 such that each element of D lies in a plane parallel to E^2 and does not separate this plane because of

$$\nu^{-1}(A) = \{(1-\mu(A))x + \mu(A)\}w\colon x \in A\} \subset \{(x, y, z)\colon z = \mu(A)\}$$

is a continuum homeomorphic to A provided that $A \neq X$ and $\nu^{-1}(X) = \{w\}$. According to [3], the quotient space E^3/D is homeomorphic to E^3. Let $q\colon E^3 \to E^3/D$ be the projection. Then, by 1.7, $q\nu^{-1}\colon C(X) \to E^3/D$ is the desired embedding. This completes the proof.

5.2. *If X is a hereditarily indecomposable plane continuum such that no proper subcontinuum of X separates the plane, then* $\dim C(X) = 2$.

In fact, by 5.1, $\dim C(X) \leqslant 3$ and therefore $\dim C(X) = 2$ because X is hereditarily indecomposable (see [4]).

Finally, we recal that the hyperspaces of snake-like and circle-like plane continua are embeddable in E^3 (see [5] and [14]).

PROBLEM 3. *Can the hyperspace of a hereditarily indecomposable plane continuum be embedded in E^3?*

PROBLEM 4. *Suppose that X is a continuum such that $C(X)$ can be embedded in E^3. Can X be embedded in the plane?*

A positive answer to Problem 4 in the case where X is a locally connected continuum follows from a formula of Kelley [6, p. 30].

The author acknowledges his gratitude to the referee, who made several valuable suggestions.

References

[1] J. H. Case and R. E. Chamberlin, *Characterization of tree-like continua*, Pacific J. Math. 10 (1960), pp. 73–84.

[2] H. Cook, *Concerning three questions of Burgess about homogeneous continua*, Colloq. Math. 19 (1968), pp. 241–244.

[3] E. Dyer and M. E. Hamstrom, *Completely regular mappings*, Fund. Math. 45 (1958), pp. 113–118.
[4] C. E. Eberhart and S. B. Nadler, *The dimension of certain hyperspaces*, Bull. Polon. Acad. Sci. Sér. Sci. Math. Astronom. Phys. 19 (1971), pp. 1027–1034.
[5] G. W. Henderson, *On the hyperspaces of subcontinua of an arc-like continuum*, Proc. Amer. Math. Soc. 27 (1971), pp. 416–417.
[6] J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. 52 (1942), pp. 22–36.
[7] J. Krasinkiewicz, *No 0-dimensional set disconnects the hyperspace of a continuum*, Bull. Polon. Acad. Sci. Sér. Sci. Math. Astronom. Phys. 19 (1971), pp. 755–758.
[8] — *On the hyperspaces of snake-like and circle-like continua* (to appear).
[9] K. Kuratowski, *Topology*, vol. 2, New York–London–Warszawa 1968.
[10] S. B. Nadler, Jr., *Locating cones and Hilbert cubes in hyperspaces* (to appear).
[11] T. Nishiura and C. J. Rhee, *The hyperspace of a pseudoarc is a Cantor manifold*, Proc. Amer. Math. Soc. 31 (1972), pp. 550–556.
[12] C. J. Rhee, *On dimension of hyperspace of a metric continuum*, Bull. de la Soc. Royale des Sciences de Liege, 38 Année, 11–12 (1969), pp. 602–604.
[13] J. T. Rogers, *The cone = hyperspace property*, Can. J. Math. 24 (1972), pp. 279–285.
[14] — *Embedding the hyperspaces of circle-like plane continua*, Proc. Amer. Math. Soc. 29 (1971), pp. 165–168.
[15] W. R. R. Transue, *On the hyperspace of subcontinua of the pseudoarc*, Proc. Amer. Math. Soc. 18 (1967), pp. 1074–1075.
[16] H. Whitney, *Regular families of curves*, Proc. Nat. Acad. Sci., U.S.A. 18 (1932), pp. 275–288.
[17] G. S. Young, *The introduction of local connectivity by change of topology*, Amer. J. Math. 68 (1946), pp. 479–494.

Reçu par la Rédaction 10. 11. 1972

Classes of Dedekind finite cardinals (1)

by

John Truss (Oxford)

Abstract. We discuss seven possible definitions of "finiteness" of cardinal numbers, and associate with each definition the class of cardinals "finite" in that sense. The results are extensions of those in Levy's paper "The independence of various definitions of finiteness" (Fund. Math. 46 (1958), pp. 1–13). We investigate the closure of the classes under addition, multiplication, unions, and disjoint unions. In the final section we give a wide variety of possible combinations of inclusions and equalities between the classes. Also we give an affirmative answer to Tarski's question "Can there be exactly $2^{\aleph_0}$ Dedekind finite cardinals?"

§ 1. The object of this paper is to investigate various properties of seven classes of Dedekind finite cardinals, and to discover what possible combinations of inclusions and equalities can hold between them. The starting point is Levy's paper "The independence of various definitions of finiteness" [8]. All but two of the seven classes correspond to definitions of Levy (which were in turn taken from Tarski [18]), and the other two arise quite naturally.

The classes are defined in § 2, and some of their elementary properties given. In § 3 we discuss the closure of the classes under $+$, $\times$, unions, disjoint unions, and $\leqslant$, and a model is given in which Δ_5 is not closed under $\times$. In § 4 we show that if any two Dedekind finite cardinals are comparable, then any infinite set is the disjoint union of two infinite sets. § 5 contains five models which establish various possibilities of strict inclusion and equality between the classes. Various combinations of these models actually yield thirteen models in which the combination of strict inclusion and equality is different. It is shown that there can be no more than twenty-three possible combinations, using the results of § 2 and § 3. All of the ten unsolved cases involve the following situation:

There is an infinite set with no infinite orderable subset, and if a set X has a countable partition, there is a map from X onto X which is not 1-1.

(1) This paper is part of the author's Ph. D. thesis at the University of Leeds. He would like to thank Dr. F. R. Drake and Prof. A. Levy for their supervision, and the Science Research Council for their financial support.