be referred to as a *crumpled handlebody*. Lemma 19 implies immediately the following

COROLLARY 22. *Any crumpled handlebody X has an h-spine, which is the wedge of simple closed curves or a point and possesses a topological regular neighborhood in $X \setminus \mathrm{Bd}\, X$.*

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Reçu par la Rédaction le 7. 2. 1973

A characterization of Hurewicz space

by

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Abstract. A characterization of Hurewicz spaces is given. In particular, in a regular Lindelöf space the Hurewicz property is equivalent to each normal sequence having a point-finite (or locally-finite) subcollection covering the space.

A topological space X is a *Hurewicz space* [2] if each sequence $G_1, G_2, \ldots$ of open covers of X has a subcollection H that covers X such that $H = H_1 \cup H_2 \cup \ldots$ where each H_n is a finite subcollection of G_n. A topological space X is *totally paracompact* (*metacompact*) provided each open basis of X has a locally-finite (point-finite) subcollection covering X. The sequence $G_1, G_2 \ldots$ is a normal sequence provided G_{n+1} star-refines G_n for each positive integer n. Let $\mathrm{st}(x, G) = \bigcup \{g\colon x \in g \text{ and } g \in G\}$. All spaces are assumed to be Hausdorff.

The author would like to thank Professor A. Lelek for his encouragement while working on this problem.

THEOREM. *A regular Lindelöf space X is Hurewicz if and only if each normal sequence of open covers has a point-finite (or locally-finite) subcollection covering X.*

Suppose X is such a space and let $G_1, G_2, \ldots$ be a sequence of open covers of X. Since X is paracompact and hence fully normal, let $U_1, U_2, \ldots$ be a sequence of open covers of X such that U_1 star-defines G_1 and for each positive integer greater than 1, U_n star-refines both U_{n-1} and G_n. Define $\mathrm{int}(\mathrm{st}(x, U_n))$ to be $\{y\colon \mathrm{st}(y, U_k)$ is contained in $\mathrm{st}(x, U_n)$ for some $k\}$. Let (X, U) be the topological space having basis the set $\{\mathrm{int}(x, U_n)\colon n \in N \text{ and } x \in X\}$. The set $\mathrm{int}(\mathrm{st}(x, U_n))$ is open in X since if P is a point of $\mathrm{int}(\mathrm{st}(x, U_n))$ then there is an integer k so that $\mathrm{st}(P, U_k)$ is contained in $\mathrm{st}(x, U_n)$ and $\mathrm{st}^2(P, U_{k+1})$ is contained in $\mathrm{st}(P, U_k)$ hence $\mathrm{st}(P, U_{k+1})$ is contained in $\mathrm{int}(\mathrm{st}(x, U_n))$. To show we have a basis suppose the point P is common to both $\mathrm{int}(\mathrm{st}(x, U_n))$ and $\mathrm{int}(\mathrm{st}(y, U_m))$ then there is an integer k so that $\mathrm{st}(P, U_k)$ is contained in $\mathrm{st}(x, U_n)$ and there is an integer j so that $\mathrm{st}(P, U_j)$ is contained in $\mathrm{st}(y, U_m)$. Suppose k is greater than or equal to j, then $\mathrm{int}(\mathrm{st}(P, U_k))$ is common to both $\mathrm{int}(\mathrm{st}(x, U_n))$ and $\mathrm{int}(\mathrm{st}(y, U_m))$. The set $\{\bigcap_n \mathrm{st}(x, U_n)\colon n \in N\}$ is a partition of X into pairwise disjoint closed sets. The set $\bigcap_n \mathrm{st}(x, U_n)$ is closed since $\mathrm{st}^2(x, U_{n+1})$ is contained in $\mathrm{st}(x, U_n)$ so the closure of $\mathrm{st}(x, U_{n+1})$ is contained in $\mathrm{st}(x, U_n)$ hence $\bigcap_n \mathrm{st}(x, U_n)$ equals $\bigcap_n$ (closure of $\mathrm{st}(x, U_n)$). To show the set $\{\bigcap_n \mathrm{st}(x, U_n)\colon n \in N$ and $x \in X\}$ is pairwise disjoint suppose there is a point x common to $\bigcap_n \mathrm{st}(p, U_n)$ and $\bigcap_n \mathrm{st}(q, U_n)$ and a point y in $\bigcap_n \mathrm{st}(q, U_n)$ but not in $\bigcap_n \mathrm{st}(p, U_n)$. Therefore there is a positive integer m so that y is not in $\mathrm{st}(q, U_m)$. Since q is in $\mathrm{st}(x, U_{n+1})$ and y is in $\mathrm{st}(q, U_{n+1})$, y is in $\mathrm{st}^2(x, U_{n+1})$ which is a subset of $\mathrm{st}(x, U_n)$ for each positive integer n putting y in $\mathrm{st}(x, U_{m+1})$ and since x is in $\mathrm{st}(p, U_{m+1})$, y is in $\mathrm{st}^2(p, U_{m+1})$ giving the contradiction y is in $\mathrm{st}^2(p, U_{m+1})$ which is a subset of $\mathrm{st}(p, U_m)$. Let X/R be the quotient space obtained by identifying the points of $\bigcap_n \mathrm{st}(x, U_n)$ for each x in X. Define a function φ from (X, U) onto X/R by $\varphi(x) = \bigcap_n \mathrm{st}(x, U_n)$. The function φ is continuous and $\varphi\big(\mathrm{int}(\mathrm{st}(x, U_n))\big)$ is open since $\varphi^{-1}\varphi\big(\mathrm{int}(\mathrm{st}(x, U_n))\big) = \mathrm{int}\big(\mathrm{st}(x, U_n)\big)$. To see this suppose y is a point not in $\mathrm{int}\big(\mathrm{st}(x, U_n)\big)$ this means $\bigcap_n \mathrm{st}(y, U_n)$ does not intersect $\mathrm{int}\big(\mathrm{st}(x, U_n)\big)$ hence y cannot be in $\varphi^{-1}\varphi\big(\mathrm{int}(\mathrm{st}(x, U_n))\big)$. Define $H_n = \{\varphi\big(\mathrm{int}(\mathrm{st}(x, U_n))\big)\colon x \in X\}$. Then H_n covers X/R and $\{H_n\colon n \in N\}$ satisfies Moore's metrization theorem, that is, $\{\mathrm{st}^2(y, H_n)\colon n \in N$ and $y \in X/R\}$ is a basis for the T_1 space X/R. This follows because in $(X, U)\,\mathrm{st}^2\big(p, \{\mathrm{int}(\mathrm{st}(x, U_{n+3}))\colon x \in X\}\big)$ is contained in

$$\mathrm{st}^2\big(p, \{\mathrm{st}(x, U_{n+3})\colon x \in X\}\big) = \mathrm{st}(p, U_{n+1}) \subset \mathrm{int}\big(\mathrm{st}(p, U_n)\big).$$

Since X is Lindelöf, (X, U) is Lindelöf and therefore X/R is a separable metric space. Let β be an open basis for X/R. Since X/R is metric and hence paracompact β can be written in a normal sequence by letting β_1 be β, and for each integer n greater than 1 let β_n be a refinement, using members of β, of a star-refinement of β_{n-1}. Therefore $\{\varphi^{-1}(b)\colon b \in \beta_n$, $n = 1, 2, \ldots\}$ is a normal sequence of open covers of (X, U) and hence also for X and therefore has a point-finite subcollection α covering X. The set $\{\varphi(a)\colon a \in \alpha\}$ is a point-finite subcollection of β that covers X/R. Hence by Theorem 2 of [2] X/R is Hurewicz. For each integer n greater than 1 let α_n be a finite subcollection of H_n such that $\alpha_1 \cup \alpha_2 \cup \ldots$ covers X/R. Therefore $\{\varphi^{-1}(a)\colon a \in \alpha_m,\ m > 1\}$ is an open cover of X such that the finite set $\{\varphi^{-1}(a)\colon a \in \alpha_n,\ n$ fixed$\}$ refines U_{n-1}. Hence there is a sequence $\lambda_1, \lambda_2, \ldots$ such that λ_n is a finite subcollection of G_n and $\lambda_1 \cup \lambda_2 \cup \ldots$ covers X. Therefore X is Hurewicz.

To prove each normal sequence of open covers in a Hurewicz space X has a locally finite subcollection covering X let $G_1, G_2, \ldots$ be a normal

sequence of open covers of X and let $G = G_1 \cup G_2 \cup \ldots$ Using the Hurewicz property let $H = H_1 \cup H_2 \cup \ldots$ where each H_n is a finite subcollection of G_{2n+1} and H covers X. For each member h of H_n select one and only one member g of G_{2n-1} so that $\mathrm{st}(h, G_{2n+1})$ is contained in g. Let this collection be denoted by K. Let L be the collection to which g belongs only in case g belongs to K and if g is a member of G_n then g is not a subset of a member of G_i with i less than n. L is a locally-finite subcollection of G covering X because each member of H can intersect at most finitely many members of L.

Corollary 1 (Theorem 3.1 of [1]). *A regular Hurewicz space is totally paracompact.*

In a paracompact Hausdorff space each basis can be written as a normal sequence.

Corollary 2. *If each continuous one-to-one image of a regular Lindelöf space X is totally metacompact, then X is Hurewicz.*

Question 1. In a Lindelöf space does total paracompactness imply that each normal sequence of open covers have a point-finite subcollection covering the space?

Question 2. In a paracompact or a regular Lindelöf space are total paracompactness and total metacompactness equivalent?

An answer to Question 1 would answer problem 3.2 of [1].

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Reçu par la Rédaction le 3. 3. 1972