About neighborhoods of surfaces integrally unknotted in S^3

by

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Abstract. A (closed) surface N embedded in S^3 is (integrally) unknotted, if N can be homeomorphically approximated by tame surfaces bounding handlebodies in both complementary domains. It will be shown in this note that every unknotted surface is a deformation retract of a large open/closed neighborhood in S^3.

A subspace Y of a space X will be said to be *mildly embedded* in X, or simply, *mild* in X, if Y is a deformation retract of a neighborhood in X. Otherwise, Y is *wickedly embedded*, or *wicked*, in X [10]. If X is a triangulated space, and Y a tame subset of X, then Y is mild in X. And such is the case where X and Y are AR-spaces or where Y is any topological image of the $(n-1)$-sphere S^{n-1} in $X = S^n$. On the other hand, most simple k-manifolds with $k = 1, 2$ or 3 can be wickedly embedded in very simple 3-manifolds [8], and any closed orientable surface of positive genus can be wickedly embedded in S^3 [9]. We will show here that a closed surface is mild in S^3 if it is integrally unknotted, i.e., if it can be approximated homeomorphically by surfaces bounding handlebodies in their both complementary domains. Of course, this condition is not necessary at all.

On our way we have to discuss a property characterizing unknottedness of a surface in S^3 which may be of some independent interest. Occasionally, one can see that in the space of all embeddings of a surface into S^3 with the metric "sup", the set of all unknotted embeddings is a pathwise connected and locally pathwise connected component. Finally, it turns out that every crumpled handlebody (see the last section for definition) X has an h-spine, i.e., there exists a subset K of X such that K is a deformation retract of X, and the boundary of X is a deformation retract of $X \setminus K$.

1. Preliminaries. Let Y be a subset of a metric space X, a mapping $f\colon Y \to X$ is an ε-mapping if $\operatorname{dist}[f(x), x] < \varepsilon$ for every $x \in Y$. A small mapping is an ε-mapping with ε small. And $f\colon Y \twoheadrightarrow X$ stands for a mapping of Y onto X.

We consider the 3-sphere S^3 as the one-point compactification of the Euclidean 3-space E^3 and all triangulations, polyhedra etc. will be related to the ordinary matric of E^3. Regular neighborhoods will be understood in the sense of Whitehead [15]. By a topological regular neighborhood of a set A in a space X we shall mean a closed neighborhood V of A such that there is an embedding $h\colon V \to E^n$ with $h(V)$ being a regular neighborhood of the polyhedron $h(A)$ in E^n. By a handlebody we shall mean any homeomorphic image of a regular neighborhood of a connected 1-polyhedron in E^3. By a surface we shall understand a closed, connected orientable 2-manifold.

A subset A of a triangulated space X is tame if there exists an autohomeomorphism of X that maps A onto a polyhedron in X. If the surface N is a subset of S^3, and U denotes a component of $S^3 \setminus N$, then if $\overline{U}$ is a handlebody, we shall say that N is *tamely unknotted from the side* of U or unilaterally tamely unknotted. If the closures of both components in $S^3 \setminus N$ are handlebodies, then N is known to be tame and will be said to be *tamely unknotted from both sides* or simply, *tamely unknotted.*

Two compact 3-manifolds M and N with nonempty boundaries and with $N \subset \operatorname{Int} M$ are concentric [7] if the closure of $M \setminus N$ is homeomorphic to $\operatorname{Bd} M \times I$, where $\operatorname{Int} M$ and $\operatorname{Bd} M$ stand for interior and boundary of M, and $I = [0, 1]$. If A and B are disjoint, closed homeomorphic 2-manifolds embedded in a compact 3-manifold, then A and B are concentric if there exists a region G in M such that $\operatorname{Fr} G = A \cup B$ and G is homeomorphic to $A \times I$.

Clearly, two 3-manifolds are concentric if and only if so are their boundaries. Let A and B be concentric surfaces in a 3-manifold bounding regions U and V respectively in M with $\overline{U} \subset V$. Then $\overline{V}$ is a handlebody if so is $\overline{U}$, and conversely, $\overline{U}$ is a handlebody if $\overline{V}$ is a handlebody and if A is tame. For concentric manifolds are homeomorphic.

2. Unknotted surfaces.

LEMMA 1. *Let N be a tame surface in S^3, and U a component of $S^3 \setminus N$. If for every positive number ε there exists an ε-homeomorphism mapping N onto a tame surface that bounds a handlebody in the ε-neighborhood of $\overline{U}$, then $\overline{U}$ is a handlebody, i.e., N is tamely unknotted from the side of U.*

Proof. Since N is a tame, closed orientable surface, there exists a "collaring" homeomorphism $c\colon N \times I^* \to S^3$, where $I^* = [-1, 1]$, such that $c(x, 0) = x$ for every $x \in N$, and every $N_t = c(N \times \{t\})$ is a tame surface. The double collar $C = c(N \times I^*)$ is a tame 3-manifold with $\operatorname{Bd} C = N_{-1} \cup N_1$. Since C is a compact ANR-space, there exists a positive number ε such that any ε-homeomorphism $h\colon N \to C$ is homotopic to the identity mapping on N, id_N, in C; moreover, if $\varepsilon < \operatorname{dist}(N, \operatorname{Bd} C)$, then $h(N)$ separates C between N_{-1} and N_1, because so does N. Therefore, assuming that h is a tame embedding, we infer that $h(N)$ is concentric with N_{-1} and N_1 in C by Theorem 1 of [7]. And N_{-1} and N_1 are concentric with N by construction.

Now let $h(N)$ bound a handlebody in the ε-neighborhood of $\overline{U}$. Since either N_{-1} or N_1 is contained in U, the surface N bounds a handlebody in the ε-neighborhood of $\overline{U}$, i.e., $\overline{U}$ is a handlebody.

COROLLARY 2. *If N is a tame surface in S^3, and for every positive number ε there exists an ε-homeomorphism mapping N onto a tame, unilaterally tamely unknotted surface, then N itself is unilaterally tamely unknotted.*

COROLLARY 3. *Let N be a tame surface in S^3, and U_1, U_2 the components of $S^3 \setminus N$. If for $i = 1, 2$ and every positive number ε there exists an ε-homeomorphism h_i mapping N onto a tame surface $h_i(N)$ that bounds a handlebody in the ε-neighborhood of $\overline{U}_i$, then N is a tamely unknotted surface.*

In particular, if a tame surface N can be homeomorphically approximated by tamely unknotted surfaces, then N is tamely unknotted.

Now, let N be an arbitrary surface (not necessarily tame) in S^3, and U a component of $S^3 \setminus N$. N will be said to be *unilaterally unknotted* if for every positive number ε there exists an ε-homeomorphism h mapping N onto a tame surface $h(N)$ which bounds a handlebody in S^3, and if S^3 can be replaced by the ε-neighborhood of $\overline{U}$, N is *unilaterally unknotted from the side* of U. If N can be homeomorphically approximated by tamely unknotted surfaces, then N is said to be *unknotted.* Clearly, these definitions make sense by Corollaries 2 and 3, i.e., they generalize unilateral tame unknottedness and tame unknottedness.

THEOREM 4. *A surface is unknotted in S^3 if and only if it is unknotted from either side.*

Proof. Clearly, one has only to prove the "if part" of this theorem. Let N be a surface in S^3, and U_1, U_2 the components of $S^3 \setminus N$. Assume that for $i = 1, 2$ and for every positive number ε there exists an ε-homeomorphism h_i mapping N onto a tame surface $h_i(N)$ which bounds a handlebody H_i in the ε-neighborhood of $\overline{U}_i$. By Theorem 8.2 of [6], there exists a positive number δ such that if $\operatorname{dist}[x, h_i(x)] < \delta$ for every $x \in N$, then there exists an ambient isotopy G with $G_0 = \mathrm{id}_{S^3}$ and $G_1 h_1 = h_2$. Therefore, if $\varepsilon < \delta$, then both $h_i(N)$'s are tamely unknotted. Thus, N is unknotted.

3. The base of a surface. Let N be a surface of genus n in S^3. A system of simple closed curves $\mathfrak{L} = \{L_{11}, \ldots, L_{1n}; L_{21}, \ldots, L_{2n}\}$ will be called the *base* of N if

(i) $L_{ij} \cap L_{i'j'} = x_0 \in N$ for $(i, j) \neq (i', j')$,

(ii) arbitrary orientations of curves L_{ij} constitute the base of 1-dimensional homology of N.

It can easily be seen that for any base $\mathfrak{L}$ with properly ordered elements there exist two systems $\{L'_{i1}, \ldots, L'_{in}\}$, $i = 1, 2$, of pairwise disjoint simple closed curves in N such that $L'_{1j} \cap L'_{2k} = \emptyset$ for $j \neq k$, or a point for $j = k$, and for every pair (i, j) there exists a homotopy deforming L_{ij} onto L'_{ij} in N, $L_{ij} \simeq L'_{ij}$ in N.

Let U denote a component of $S^3 \setminus N$. If $L_{ij} \simeq 0$ in $\overline{U}$ with $j = 1, \ldots, n$ for a fixed value of i, then the base $\mathfrak{L}$ will be called *regular relatively to* U. A base of N will be called *regular* if it is regular relatively to both components of $S^3 \setminus N$.

THEOREM 5. *Let N be a tame surface in S^3, and U a component of $S^3 \setminus N$. Then N is tamely unknotted from the side of U if and only if there exists a base of N regular relatively to U.*

Proof (compare [12]). We can assume that $\overline{U}$ is a polyhedral 3-manifold with the boundary N. Clearly, we have only to prove the "if part". Let L be a base of N such that $L_{1j} \simeq 0$ in $\overline{U}$ for $j = 1, \ldots, n$. Let $\{L_1, \ldots, L_n\}$ be a system of pairwise disjoint, polyhedral, simple closed curves in N such that $L_j \simeq L_{1j}$ in N for $j = 1, \ldots, n$. By Dehn's lemma [13], L_1 bounds a polyhedral disk D_1 in $\overline{U}$ with $D_1 \cap N = \operatorname{Bd} D_1 \cap N = L_1$. Obviously, choosing a small regular neighborhood B of D_1 in $\overline{U}$, we can split $\overline{U}$ so that $\overline{U} = M \cup B$, M is a polyhedral 3-manifold bounded by a surface N' of genus $n-1$, and B is a polyhedral 3-cell which intersects M in two disjoint disks D' and D'' with $\operatorname{Bd} M \cap \operatorname{Bd} B = D' \cup D''$. Moreover, we can assume that $L_j \subset N' \setminus (D' \cup D'')$ for $j = 2, \ldots, n$. And it is easily seen that arbitrary orientations of $L_2, \ldots, L_n$ constitute a base of 1-dimensional homology of N'. Finally, since M is a retract of $\overline{U}$, $L_j \simeq 0$ in M for $j = 2, \ldots, n$. Thus, we can apply induction on the genus of N and infer that $\overline{U}$ can be obtained by attaching 1-handles to a 3-manifold Q bounded by a polyhedral 2-sphere in S^3. Therefore, by Alexander's theorem [1], Q is a 3-cell which proves the theorem.

COROLLARY 6. *A surface N tamely embedded in S^3 is tamely unknotted if and only if there is a regular base on N.*

Proof. The "if part" follows by Theorems 5 and 4. The "only if part" follows by a theorem of [14] to the effect that, up to isotopy of S^3, there exists only one Heegaard splitting of S^3 into the union of two handlebodies with common boundary surface of a fixed genus. Thus, we have only to consider standard splittings with standard bases, which ends the proof.

Now, we are going to show that the existence of a regular base, relative or not, is invariant under small homeomorphic displacements of the surface.

A simple calculation shows the following.

LEMMA 7. *If $X = A_1 \cup A_2 \cup A_3 = B_1 \cup B_2 \cup B_3$ with $A_i \cap A_j = \emptyset = B_i \cap B_j$ for $i \neq j$, then $X \setminus (A_1 \cap B_1 \cup A_2 \cap B_2) = A_3 \cup B_3 \cup (A_k \div B_k)$ where $k = 1, 2$, and $A \div B = (A \setminus B) \cup (B \setminus A)$.*

LEMMA 8. *Let A be a compact subset of E^n, $n \geqslant 1$. Assume that $E^n \setminus A$ has exactly two components A_1 and A_2, a bounded and unbounded one. Let $d\colon A \times I \to E^n$ be a map such that $d_0 = \mathrm{id}_A$ and d_1 maps A onto B homeomorphically, and thus $E^n \setminus B$ has likewise two components B_1 and B_2, a bounded and unbounded one. Then*

$$E^n \setminus (A_1 \cap B_1 \cup A_2 \cap B_2) \subset d(A \times I)\,.$$

Proof. By Lemma 7, it suffices to show $(A_1 \div B_1) \setminus (A \cup B) \subset d(A \times I)$. Assume that $p \in (A_1 \div B_1) \setminus (A \cup B)$ and pick out a point q in $A_2 \cap B_2 \setminus d(A \times I)$. If $p \in (A_1 \setminus B_1) \setminus (A \cup B)$, then $p \in A_1$ and $q \in A_2$. But $p, q \in B_2$, and therefore, $p \in d(A \times I)$ by a standard theorem on deformation. If $p \in (B_1 \setminus A_1) \setminus (A \cup B)$, then $p \in B_1$ and $q \in B_2$. But $p, q \in A_2$. Now, define the map $d^*\colon B \times I \to E^n$ by the condition, $d^*(x, t) = d[d_1^{-1}(x), 1-t]$ for all $x \in B$ and $t \in I$. Thus, $d_0^* = \mathrm{id}_B$, $d_1^* = d_1^{-1}$ maps B homeomorphically onto A, and $d^*(B \times I) = d(A \times I)$. Hence, $q \notin d^*(B \times I)$. Therefore, $p \in d^*(B \times I)$.

Though the base of a surface is an intrinsic topological invariant by definition, the regular base, relative or not, is not invariant even under the autohomeomorphisms of the surface. However, the following holds true.

THEOREM 9. *For any surface N embedded in S^3, there exists a positive number ε such that every ε-homeomorphism $h\colon N \to S^3$ satisfies the following conditions,*

(i) *the base $\mathfrak{L} = \{L_{ij}\}$, $i = 1, 2$ and $j = 1, \ldots, n$, of N is regular relatively to the component U of $S^3 \setminus N$ if and only if $\mathfrak{L}' = \{h(L_{ij})\}$, $i = 1, 2$ and $j = 1, \ldots, n$, is a base of $h(N)$ regular relatively to that component U' of $S^3 \setminus h(N)$ which is contained in the ε-neighborhood of $\overline{U}$. And therefore,*

(ii) *$\mathfrak{L}$ is a regular base of N if and only if $\mathfrak{L}'$ is a regular base of $h(N)$.*

Proof. Let U_1, U_2 denote the components of $S^3 \setminus N$ with $U_1 = U$ and $\infty \in U_2$, where $\{\infty\} = S^3 \setminus E^3$. Since N is an ANR-space, there exists a polyhedron P such that $N \subset \operatorname{Int} P \subset P \subset S^3 \setminus \{\infty\}$, and N is a retract of P. Let $r\colon P \to N$ denote a retraction. Also P is an ANR-space, so there exists a positive number δ such that for every pair of maps $f, g\colon N \to P$ the condition $\operatorname{dist}[f(x), g(x)] < \delta$ for every $x \in N$, implies that $f \simeq g$ in P. On the other hand, as N is a retract of P, there exists a positive number η such that for any r-mapping $f\colon N \to P$ which moves the points by as much as η, the set $f(N)$ is a retract of P [4]. Now, choose a positive number ε so that $\varepsilon < \min\{\delta, \eta, \operatorname{dist}(N, S^3 \setminus \operatorname{Int} P)\}$. We are going to show that ε is the required number.

Let $h\colon N\to S^3$ be an ε-homeomorphism. Clearly, $h(N)\subset \operatorname{Int}P$, $h\simeq \mathrm{id}_N$ in P, and $h(N)$ is a retract of P. Let $d\colon N\times I\to P$ be the homotopy binding id_N with h, and $r'\colon P\twoheadrightarrow h(N)$ a retraction. Let U_1' denote that component of $S^3\setminus h(N)$ which lies in the ε-neighborhood of $\overline{U}_1$, and U_2' the other component. Thus, $\infty\in U_2'$. Since $\infty\in U_2\cap U_2'$, on applying Lemma 8 we infer that $S^3\setminus(\overline{U}_1\cap \overline{U}_1'\cup \overline{U}_2\cap \overline{U}_2')\subset d(N\times I)\subset P$. And hence, it follows that, by Lemma 7, $\overline{U}_i\dot{-}\overline{U}_i'\subset P$ for $i=1,2$. Thus, $\overline{U}_i\cup P=\overline{U}_i'\cup P$ and let denote this set by V_i, $i=1,2$.

Next, we show that both $\overline{U}_i$ and $\overline{U}_i'$ are retracts of V_i, $i=1,2$. In fact, the conditions $r_i=r|P\cap\overline{U}_i$ and $r_i'=r'|P\cap\overline{U}_i'$ define the retractions $r_i\colon P\cap\overline{U}_i\twoheadrightarrow N$ and $r_i'\colon P\cap\overline{U}_i'\twoheadrightarrow h(N)$, respectively. Assuming

$$\bar{r}_i(x)=\begin{cases} x & \text{for} \quad x\in\overline{U}_i, \\ r_{i+1}(x) & \text{for} \quad x\in P\cap\overline{U}_{i+1}\end{cases}$$

with subscripts reduced mod 2, we obtain the retractions $\bar{r}_i\colon V_i\twoheadrightarrow\overline{U}_i$. And the retractions $\bar{r}_i'\colon V_i\twoheadrightarrow\overline{U}_i'$ we obtain in the same way, replacing r_i by r_i', and $\overline{U}_i$ by $\overline{U}_i'$.

Now, we can finish our argument. (a) Let $L_{1j}\in\mathfrak{L}$. By hypothesis, $L_{1j}\simeq 0$ in $\overline{U}_1\subset V_1$ for each j. Thus, $h(L_{1j})\simeq 0$ in V_1 because $L_{1j}\simeq h(L_{1j})$ in $P\subset V_1$. On applying the retraction $\bar{r}_1'$ to V_1, $\bar{r}_1'h(L_{1j})\simeq 0$ in $\bar{r}_1'(V_1)=\overline{U}_1'$. But, $\bar{r}_1'h(L_{1j})=h(L_{1j})$, and therefore, $h(L_{1j})\simeq 0$ in $\overline{U}_1'$. Which shows that $\mathfrak{L}'$ is a base of $h(N)$ regular relatively to $\overline{U}_1'$. (b) Conversely, assume that $h(L_{1j})\simeq 0$ in $\overline{U}_1'\subset V_1$. But, $h(L_{1j})\simeq L_{1j}$ in $P\subset V_1$, thus $L_{1j}\simeq 0$ in V_1. And on applying the retraction $\bar{r}_1$, $\bar{r}_1(L_{1j})=L_{1j}\simeq 0$ in $\bar{r}_1(V_1)=\overline{U}_1$.

4. Characterization of unknotted surfaces.

THEOREM 10. *Let N be a surface embedded in S^3, and U a component of $S^3\setminus N$. Then N is unknotted from the side of U if and only if there exists a base of N regular relatively to U.*

Proof. Let ε be a positive number granted by Theorem 9.

First, assume that N is uknotted from the side of U. Then there exists an ε-homeomorphism $h\colon N\to S^3$ such that $h(N)$ bounds a handlebody H in the ε-neighborhood of $\overline{U}$. Therefore, $h(N)$ has a regular base relatively to $\operatorname{Int}H$, and thus, by (i) of Theorem 9, N has a regular base relatively to U.

Second, assume that N has a regular base relatively to U. By Bing's approximation theorem [2], there exists an ε-homeomorphism $h\colon N\to S^3$ such that $h(N)$ is a tame surface, which bounds a 3-manifold M in the ε-neighborhood of $\overline{U}$. By Theorem 9 (i), $h(N)$ has a regular base relatively to $\operatorname{Int}M$, and therefore, by Theorem 5, $h(N)$ is tamely unknotted from the side of $\operatorname{Int}M$, i.e., M is a handlebody. Thus, N is unknotted from the side of U.

Immediately we get the following.

COROLLARY 11. *A surface is unknotted if and only if it has a regular base.*

COROLLARY 12. *Let N be an unknotted (unilaterally unknotted) surface in S^3. There exists a positive number ε such that for every ε-homeomorphism $h\colon N\to S^3$, the surface $h(N)$ is unknotted (unilaterally unknotted).*

5. Digression. Let $\mathcal{E}$ denote the space of embeddings of a fixed surface N into S^3 with the "sup" metric, and let $\mathcal{E}_0$ denote the subset of all unknotted embeddings.

THEOREM 13. *$\mathcal{E}_0$ is a pathwise connected and locally pathwise connected component of $\mathcal{E}$.*

Proof. By definition of the unknotted embedding, $\mathcal{E}_0$ is closed, and by Corollary 12, $\mathcal{E}_0$ is open. To complete the proof we have to show that $\mathcal{E}_0$ is pathwise connected and locally pathwise connected. By Theorem 8.2 of [6], for any embedding $h\colon N\to S^3$ and for every positive number ε there exists a positive number δ such that for any two tame embeddings $h_i\colon N\to S^3$, $i=1,2$, with $\operatorname{dist}[h_i(x),h(x)]<\delta$ for every $x\in N$, there exists an isotopy $\boldsymbol{h}\colon S^3\times I\twoheadrightarrow S^3$ such that $\boldsymbol{h}_1h_1=h_2$, and $\operatorname{diam}\boldsymbol{h}(\{x\}\times I)<\varepsilon$ for any $x\in N$. Thus, any two sufficiently close, tamely unknotted embeddings can be joined by a small path in $\mathcal{E}_0$. Similarly, any unknotted embedding can be joined to a sufficiently close, tamely unknotted embedding by a small path in $\mathcal{E}_0$. In fact, if h is an unknotted embedding of N, using the definition and picking eventually a subsequence, we can choose a uniformely converging sequence of tamely unknotted embeddings $h_k\colon N\to S^3$ such that h_k converges to h as k goes to ∞, and $\operatorname{dist}[h_k(x),h(x)]<\delta_k$ for any $x\in N$ where $\{\delta_k\}$ is a given decreasing sequence of reals converging to 0. Now, by Theorem 8.2 of [6], for any decreasing sequence of reals $\{\varepsilon_k\}$ converging to 0 there exists a (decreasing) sequence of reals $\{\delta_k\}$ (converging to 0) such that for every positive integer k there exists an ambient isotopy $\boldsymbol{h}^k\colon S^3\times I\twoheadrightarrow S^3$ such that $\boldsymbol{h}_1^kh_k=h_{k+1}$ and $\operatorname{diam}\boldsymbol{h}^k(\{x\}\times I)<\varepsilon_k$ for every $x\in N$. Thus, properly choosing the sequence $\{\varepsilon_k\}$ and piecing isotopies $\boldsymbol{h}^k$ with embeddings h_k and h together, we can describe an isotopy $\boldsymbol{g}\colon N\times I\to S^3$ such that $\boldsymbol{g}_{(1-1/k)}=h_k$, $\boldsymbol{g}_1=h$ and $\operatorname{diam}\boldsymbol{g}(\{x\}\times I)$ is less than any given positive number for every $x\in N$. Finally, $\mathcal{E}_0$ is also integrally pathwise connected for, by the mentioned theorem of Waldhausen [14], any two tamely unknotted embeddings are ambient isotopic.

6. The spine and the concentricity of 3-manifolds. In order to make some progress, we have to discuss a few well known concepts in certain details. A mapping cylinder M_f of a map $f\colon X\to Y$ is the disjoint union $X\times I\cup Y$ with every pair $(x,1)$ identified to $f(x)$. By the identification of x with $(x,0)$ for every $x\in X$ and $(x,0)$ in M_f, we consider X and Y as closed subsets of M_f. Let $A=\overline{A}\subset X$. An open subset U of X containing A is an open cylinder neighborhood (MCN) of A if there exist a map $f\colon \operatorname{Fr} U \to\!\!\!\to \operatorname{Fr} A$ and a homeomorphism $h\colon \overline{U}\setminus \operatorname{Int} A \to\!\!\!\to M_f$ such that $h|\operatorname{Fr} U \cup \operatorname{Fr} A$ is an identity mapping.

Let M be an n-manifold with boundary. A subset K of M is a spine of M if there exists a map $f\colon \operatorname{Bd} M \times I \to\!\!\!\to M$ such that $f|\operatorname{Bd} M \times (0,1]$ is a homeomorphism onto $M\setminus K$, $f(\operatorname{Bd} M \times \{0\}) = K$, and $\dim K < n$. Clearly, M is an MCN of K. It is well-known that every n-manifold with non-empty boundary has a spine [5]. Moreover, if $\dim M = 3$ and its spine K is a topological polyhedron, then K is tame in M [11].

If X is a n-manifold with boundary or a homeomorphic image of a closed domain in E^n bounded by an $(n-1)$-manifold, then it is clear what should be meant by the boundary of X. A subset K of X will be said to be an *h-spine* of X if K is a deformation retract of X, and $\operatorname{Bd} X$ is a deformation retract of $X\setminus K$. Obviously, any spine of a manifold is its h-spine.

LEMMA 14. *Let M be a triangulated 3-manifold with boundary, K a polyhedral spine of M, and V a regular neighborhood of K in* Int M. *Then the manifolds M and V are concentric.*

Proof. V is a polyhedral manifold with boundary N, say, and let $C(N)$ be a double collar of N in $\operatorname{Int} M\setminus K$. Since K is a spine of M, there exists a map $f\colon \operatorname{Bd} M \times I \to\!\!\!\to M$ such that $f|\operatorname{Bd} M \times (0,1]$ is a homeomorphism onto $M\setminus K$, and $f(\operatorname{Bd} M\times\{0\}) = K$. By the uniform continuity of f, there exists a positive number δ such that $f(\operatorname{Bd} M\times\{\delta\}) \subset V\setminus[C(N)\cup K]$. Therefore, $C(N) \subset W = f(\operatorname{Bd} M \times [\delta, 1])$ and N is a tame subset of W separating it between $f(\operatorname{Bd} M\times\{\delta\})$ and $f(\operatorname{Bd} M\times\{1\})$. Thus, by Theorem 1 of [7], N is concentric with $\operatorname{Bd} M$, which proves our lemma.

LEMMA 15. *Let M_1, M_2 be 3-manifolds with boundary such that* (1) $M_1 \cap M_2 = \operatorname{Bd} M_1 \cap \operatorname{Bd} M_2 = F$ *is a surface, and* (2) *there exists a homeomorphism $h_i\colon F\times I \to\!\!\!\to M_i$ for $i = 1, 2$. Then there exists a homeomorphism $h\colon F\times I\to\!\!\!\to M_1\cup M_2$.*

Proof. We can assume that $h_i|F\times\{0\}$ is a mapping onto F and we only need to identify the bases of two disjoint copies of the product $F\times I$ by the homeomorphism $h_2^{-1}h_1|F\times\{0\}$.

Theorem 1 cf [4] can apparently be strengthened at no extra expence in the following way.

LEMMA 16. *Let B be a compact* ANR-*space. There exists a positive number ε such that for every retract A of B and every r-map $f\colon A\to B$ moving points by as much as ε, $f(A)$ is a retract of B.*

Proof. We can assume that B is embedded in the Hilbert cube Q^ω. There exists an open neighborhood U of B in Q^ω such that B is a retract of U. Choose $\varepsilon = \operatorname{dist}(Q^\omega\setminus U, B)$. The rest of argument follows word for word the one of [4].

COROLLARY 17. *Let X be a compact* ANR-*space. There exists a positive number ε such that for every deformation retract A of X and every r-map $f\colon A\to X$ moving points by as much as ε, $f(A)$ is a deformation retraction of X.*

Proof. Let α be a positive number granted for the space X by Lemma 16. And by hypothesis on X there exists also a positive number β such that for every pair of maps $g_1, g_2\colon Y\to X$ with $\operatorname{dist}[g_1(x), g_2(x)] < \beta$ for every $x\in Y$, $g_1\simeq g_2$ in X. Let $\varepsilon = \min\{\alpha,\beta\}$. Thus, (1) $f(A)$ is a retract of X, and (2) X can be deformed over itself to $f(A)$; because, $f\simeq \mathrm{id}_A$ in X, and by hypothesis, A is a deformation retract of X. Hence, the conclusion follows by a well-known theorem of Fox.

7. Unknotted surfaces. To save words let us agree that henceforth N denotes a surface in S^3, U_i's with $i = \pm 1$ denote the components of $S^3\setminus N$, and $I^* = [-1, 1]$.

LEMMA 18. *If N is an unknotted surface, there exists a positive number ε such that for every number δ with $0<\delta\leqslant\varepsilon$ there is a δ-homeomorphism $h\colon N\to S^3$ satisfying the following condition, if V_i denotes the component of $S^3\setminus h(N)$ contained in the δ-neighborhood of $\overline{U}_i$ for $i=\pm 1$, then $\overline{V}_i$ is a handlebody with a spine $K_i\subset V_i\cap U_i$. Moreover, it can be required that K_i be a wedge of polyhedral, simple closed curves or a point.*

Proof. Let ε be a positive number granted by Corollary 12 for N. Apply Bing's Side Approximation Theorem for 2-Manifolds [3] with $M^3 = S^3$, $M^2 = N$, $U_1 = U_{-1}$, $U_2 = U_1$, and $f(x) = \delta$ for every $x\in N$ and $0<\delta\leqslant\varepsilon$. Then there exists a homeomorphism $h\colon N\times I^*\to S^3$ such that (1) every $h_t(N)$ is tame, (2) every h_t is a δ-homeomorphism, (3) for $0<t\leqslant 1$, $U_{-1}\cap h_t(N)$ is covered by the interiors of a finite collection of mutually exclusive disks in $h_t(N)$ each of diameter no greater than δ, and (4) for $0<t\leqslant 1$, $U_1\cap h_{-t}(N)$ is covered by a finite collection of mutually exclusive disks in $h_{-t}(N)$ each of diameter no greater than δ.

By the choice of δ and by (1), the closure of any component in $S^3\setminus h_t(N)$ is a handlebody. Let $h = h_0$. Because of complete symmetry with respect to U_{-1}, U_1 we have to consider only the case of one of them, U_1 say. To this purpose, next to V_1, already defined, consider the set V_1', which is the component of $S^3\setminus h_1(N)$ contained in the δ-neighborhood of $\overline{U}_1$. Clearly, $\overline{V}_1$ and $\overline{V}_1'$ are concentric handlebodies with $\overline{V}_1'\subset V_1$. The handlebody $\overline{V}_1$ admits a spine K_1', which is a (tame) wedge of simple closed curves or a point and lies in $\operatorname{Bd}\overline{V}_1' = h_1(N)$. By (3), there is a finite collection of mutually disjoint disks $D_1,\ldots,D_m$ in $h_1(N)$ such that $U_{-1}\cap h_1(N)\subset \operatorname{Int}\bigcup_{j=1}^{m} D_j$. Therefore, it is easy to construct an isotopy

$g: h_1(N)\times I \to h_1(N)$ such that every g_t is an identity map off an arbitrarily small neighborhood of $\bigcup_{j=1}^{m} D_j$ in $h_1(N)$ and that $g_1(K_1')\subset h_1(N)\setminus\bigcup_{j=1}^{m} D_j \subset U_1$. Since the tame surface $h_1(N)$ has a double collar in V_1, we can easily extend g to an isotopy $\boldsymbol{g}: \overline{V}_1\times I\to \overline{V}_1$ such that $\boldsymbol{g}_t$ is an identity map off an arbitrarily small neighborhood of $\bigcup_{j=1}^{m} D_j$ in V_1. Thus, $K_1 = \boldsymbol{g}_1(K_1') = g_1(K_1')$ is contained in $V_1\cap U_1$. And since K_1 is clearly tame in V_1, it can be required to be polyhedral.

LEMMA 19. *If N is a unknotted surface, then there exist sets K_i in U_i, $i=\pm 1$, being wedges of polyhedral, simple closed curves or one-point sets, such that there exist a regular neighborhood V_i of K_i in U_i and a homeomorphism $h: N\times I^* \to W = \mathrm{Cl}[S^3\setminus(V_{-1}\cup V_1)]$ with the following properties.*

(i) *Every set K_i is in standard position, i.e., there exists an autohomeomorphism of E^3 mapping K_i into a plane;*

(ii) *every $N_t = h(N\times\{t\})$ is a tamely unknotted surface;*

(iii) *N is a deformation retract of W;*

(iv) *N is a deformation retract of $S^3\setminus(K_{-1}\cup K_1)$;*

(v) *every K_i is a deformation retract both of $S^3\setminus K_{-1}$ and of $\overline{U}_i$, and*

(vi) *N is a deformation retract of $\overline{U}_i\setminus K_i$ for $i=\pm 1$.*

Proof. By Lemma 18, there exists an embedding $f: N\to S^3$ such that $N' = f(N)$ is a tamely unknotted surface bounding two handlebodies X_i with spines $K_i\subset \mathrm{Int}\, X_i\cap U_i$ for $i=\pm 1$, each K_i being a wedge of polyhedral, simple closed curves or a point. Then condition (i) follows by Waldhausen's theorem [14].

Choose V_i to be a regular neighborhood of K_i in $\mathrm{Int}\, X_i\cap U_i$. $N_i = \mathrm{Bd}\, V_i$. By Lemma 14, the manifolds V_i and X_i are concentric, and hence, so are their boundaries. Therefore, by Lemma 15, the surfaces N_{-1} and N_1 are concentric, and so there exists a homeomorphism $h: N\times I^*\to W$ satisfying condition (ii).

In order to prove (iii), let us remind that by Bing's approximation theorem for surfaces [2], there exists an arbitrarily small homeomorphism $g: N\to \mathrm{Int}\, W$ mapping N onto a tame surface $g(N)$. Thus, $g(N)$ separates W between N_{-1} and N_1 and hence, by Theorem 1 of [7], $g(N)$ is concentric both with N_{-1} and N_1, which implies that $g(N)$ is a deformation retract of W. As g is an arbitrarily small homeomorphism, we have only to apply Corollary 17.

Now, consider condition (iv). First, observe that N_i is a strong deformation retract of $V_i\setminus K_i$. Thus, extending given retracting deformations $(V_i\setminus K_i)\times I\to V_i\setminus K_i$ by "identity" deformation $\delta: W\times I\to W$ with $\delta(x,t) = x$ for every (x,t), we infer that W is a deformation retract of $S^3\setminus(K_{-1}\cup K_1)$, which proves (iv) by (iii).

Since K_i is a deformation retract of V_i, and the latter set can be shown to be a deformation retract of $V_i\cup W$ along similar lines as in (iv), and next, $V_i\cup W$ is a deformation retract of $S^3\setminus K_{-i} = V_i\cup W\cup (V_{-i}\setminus K_{-i})$, we infer that K_i is a deformation retract of $S^3\setminus K_{-i}$. To complete the proof of (v), observe that, by (iv) there exists a retraction $r_i: S^3\setminus K_{-i}\to \overline{U}_i$, and by preceding paragraph, there is a deformation $d^i: \overline{U}_i\times I\to S^3\setminus K_{-i}$ with $d^i(\overline{U}_i\times\{1\}) = K_i$, which is the restriction to the set $\overline{U}_i\times I$ of the deformation given there. Then $\delta^i = r_i d^i$ deforms $\overline{U}_i$ over itself to K_i. By a similar reason, there is a retraction $\varrho_i: \overline{U}_i\to K_i$. Thus, K_i is a deformation retract of $\overline{U}_i$.

Finally, to show (vi) we have only to combine the proper restriction of the retracting deformation granted by (iv) with the proper retraction r_i.

As an immediate corollary, we get the following.

THEOREM 20. *If N is an unknotted surface, then N is a deformation retract of an open and a closed neighborhood. Thus, N is mild in S^3.*

The converse of this is obviously false.

THEOREM 21. *The following conditions are equivalent.*

(1) *The surface N is unknotted.*

(2) *There exists an embedding $h: N\times I^*\to S^3$ such that* (i) *every $N_t = h(N\times\{t\})$ is tamely unknotted, and* (ii) *N separates $W = h(N\times I^*)$ between N_{-1} and N_1.*

(3) *For $i=\pm 1$, in U_i there exists a set K_i, which is the wedge of n polyhedral, simple closed curves, n being the genus of N, or a point, if $n=0$, such that for arbitrarily small, regular neighborhood V_i of K_i, the surfaces $\mathrm{Bd}\, V_{-1}$ and $\mathrm{Bd}\, V_1$ are concentric in S^3.*

Proof. By Lemma 19, (1) implies both (2) and (3). On the other hand, (3) implies (2). For, the surfaces $\mathrm{Bd}\, V_i$ and N are homeomorphic having the same genus. Next, if V_i is so small that $V_i\subset U_i$ and since $\mathrm{Bd}\, V_i$'s can be assumed concentric, there exists a homeomorphism $h: N\times I^*\to S^3$ such that $N_i = \mathrm{Bd}\, V_i$ and N separates W between N_{-1} and N_1. Moreover, since N_{-1} and N_1 are concentric and bound the handlebodies V_{-1} and V_1 respectively, the surfaces N_i's are tamely unknotted, and so are all surfaces N_t by their mutual concentricity.

Finally, (2) implies (1). We have only to apply Bing's approximation theorem for surfaces [2] to construct an arbitrarily small homeomorphism $g: N\to \mathrm{Int}\, W$ such that $g(N)$ is a tame surface in W separating it between N_{-1} and N_1. Therefore, by Theorem 1 of [7], $g(N)$ is concentric with both N_i's, and thus, $g(N)$ is tamely unknotted, which proves (1).

8. Crumpled handlebodies. Let N be an unknotted surface. A homeomorphic image of the closure of a complementary domain U_i of N will be referred to as a *crumpled handlebody*. Lemma 19 implies immediately the following

COROLLARY 22. *Any crumpled handlebody X has an h-spine, which is the wedge of simple closed curves or a point and possesses a topological regular neighborhood in $X \setminus \mathrm{Bd}\, X$.*

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A characterization of Hurewicz space

by

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Abstract. A characterization of Hurewicz spaces is given. In particular, in a regular Lindelöf space the Hurewicz property is equivalent to each normal sequence having a point-finite (or locally-finite) subcollection covering the space.

A topological space X is a *Hurewicz space* [2] if each sequence $G_1, G_2, \ldots$ of open covers of X has a subcollection H that covers X such that $H = H_1 \cup H_2 \cup \ldots$ where each H_n is a finite subcollection of G_n. A topological space X is *totally paracompact* (*metacompact*) provided each open basis of X has a locally-finite (point-finite) subcollection covering X. The sequence $G_1, G_2 \ldots$ is a normal sequence provided G_{n+1} star-refines G_n for each positive integer n. Let $\mathrm{st}(x, G) = \bigcup \{g\colon x \in g \text{ and } g \in G\}$. All spaces are assumed to be Hausdorff.

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THEOREM. *A regular Lindelöf space X is Hurewicz if and only if each normal sequence of open covers has a point-finite* (*or locally-finite*) *subcollection covering X.*

Suppose X is such a space and let $G_1, G_2, \ldots$ be a sequence of open covers of X. Since X is paracompact and hence fully normal, let $U_1, U_2, \ldots$ be a sequence of open covers of X such that U_1 star-defines G_1 and for each positive integer greater than 1, U_n star-refines both U_{n-1} and G_n. Define $\mathrm{int}(\mathrm{st}(x, U_n))$ to be $\{y\colon \mathrm{st}(y, U_k)$ is contained in $\mathrm{st}(x, U_n)$ for some $k\}$. Let (X, U) be the topological space having basis the set $\{\mathrm{int}(x, U_n)\colon n \in N \text{ and } x \in X\}$. The set $\mathrm{int}(\mathrm{st}(x, U_n))$ is open in X since if P is a point of $\mathrm{int}(\mathrm{st}(x, U_n))$ then there is an integer k so that $\mathrm{st}(P, U_k)$ is contained in $\mathrm{st}(x, U_n)$ and $\mathrm{st}^2(P, U_{k+1})$ is contained in $\mathrm{st}(P, U_k)$ hence $\mathrm{st}(P, U_{k+1})$ is contained in $\mathrm{int}(\mathrm{st}(x, U_n))$. To show we have a basis suppose the point P is common to both $\mathrm{int}(\mathrm{st}(x, U_n))$ and $\mathrm{int}(\mathrm{st}(y, U_m))$ then there is an integer k so that $\mathrm{st}(P, U_k)$ is contained in $\mathrm{st}(x, U_n)$ and there is an integer j so that $\mathrm{st}(P, U_j)$ is contained in $\mathrm{st}(y, U_m)$. Suppose k is greater than or equal to j, then $\mathrm{int}(\mathrm{st}(P, U_k))$ is common to both $\mathrm{int}(\mathrm{st}(x, U_n))$ and $\mathrm{int}(\mathrm{st}(y, U_m))$. The set $\{\bigcap_n \mathrm{st}(x, U_n)\colon n \in N\}$ is