4. Extension and lifting problems. As a further illustration of the close relation between small multifunctions and their simplicial approximations we finish with some results on extension and lifting of multifunctions.

THEOREM 4.1. (Extension problem). *Let X, Y be polyhedra, and let A be a subpolyhedron of X. Let φ: $A \to Y$ be a small,* usc *and point-closed multifunction, and f: $A \to Y$ be a simplicial approximation of φ. If f can be extended to a map from X to Y, then φ can be extended to a small,* usc *and point-closed multifunction from X to Y.*

Proof. If g: $X \to Y$ is an extension of f, then its restriction to A equals f. Hence Theorem 2.4 shows the existence of a small homotopy Φ: $A \times I \to Y$ with $\Phi(a, 0) = g(a)$ and $\Phi(a, 1) = \varphi(a)$ for all $a \in A$. Let r: $X \times I \to (X \times 0) \cup (A \times I)$ be a retraction, define Φ': $(X \times 0) \cup (A \times I) \to Y$ by

$$\Phi'(x, t) = \begin{cases} g(x) & \text{if} \quad t = 0, \\ \Phi(x, t) & \text{if} \quad t > 0, \end{cases}$$

and define Ψ: $X \times I \to Y$ by $\Psi = \Phi' \circ r$. Then Ψ is small, usc and point-closed, and $\Psi(x, 1) = \varphi(x)$ if $x \in A$. Therefore $\psi(x) = \Psi(x, 1)$ is an extension of φ.

Remark. It is also easy to prove the converse, i.e. the fact that the existence of an extension of φ implies the existence of an extension of f.

THEOREM 4.2. (Lifting problem). *Let X, E, B be polyhedra and φ: $X \to B$, π: $E \to B$ be small,* usc *and point-closed multifunctions. Let f: $X \to B$ and p: $E \to B$ be simplicial approximations of φ and π. If f can be lifted to a map g: $X \to E$ with $p \circ g \simeq f$, then φ can be lifted to a small,* usc *and point-closed multifunction ψ: $X \to E$ with $\pi \circ \psi \overset{s}{\simeq} \varphi$.*

Proof. Take $\psi = g$, so that $\pi \circ \psi = \pi \circ g$. Let Π: $E \times I \to B$ be a small homotopy from π to p (see Theorem 2.4), and let 1_I: $I \to I$ be the identity map. Then $\pi \circ (g \times 1_I)$: $X \times I \to B$ is a small homotopy from $\pi \circ g$ to $p \circ g$. As $p \circ g \simeq f$ and $f \overset{s}{\simeq} \varphi$ (Theorem 2.4), we have $\pi \circ \psi \simeq \varphi$.

References

[1] T. R. Brahana, M. K. Fort, Jr., and W. G. Horstmam, *Homotopy for cellular set-valued functions*, Proc. Amer. Math. Soc. 16 (1965), pp. 455–459.

[2] C. R. F. Maunder, *Algebraic Topology*, London 1970.

[3] H. Schirmer, *Homotopy for small multifunctions*, to appear in Fund. Math.

[4] — *δ-continuous selections of small multifunctions*, Canad. J. Math. 24 (1972), pp. 631–635.

[5] — *Near fixed points of small* usc *multifunctions*, to appear.

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Stone lattices: a topological approach

by

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Abstract. A $\{0, 1\}$-distributive lattice L can be represented as the lattice of clopen increasing subsets of an appropriate ordered topological space X. It is shown that, when L is a Stone lattice, the dual space X is characterized by subspaces $Y(X)$, $Z(X)$ of X and a continuous increasing map $m(X)$: $Y(X) \to Z(X)$. This enables the structure of Stone lattice dual spaces to be analysed in terms of simpler components and leads to a construction theorem for such dual spaces which is in the same spirit as, but not directly dual to, Chen and Grätzer's triple construction theorem for Stone lattices.

1. Introduction. Chen and Grätzer show in [5] that a Stone lattice L can be studied by investigating an associated triple of simpler components, $(C(L), D(L), \Phi(L))$, where $C(L)$, $D(L)$ are appropriate subsets of L and $\Phi(L)$ a connecting map. In this paper the duality between $\{0, 1\}$-distributive lattices and compact totally order disconnected spaces developed in [19] and [20] is applied to Stone lattices and the dual space X of a Stone lattice L is shown to be characterized by subspaces $Y(X)$, $Z(X)$ of X and a continuous increasing map $m(X)$: $Y(X) \to Z(X)$. The ordered spaces $Y(X)$, $Z(X)$ are the duals of the lattices $D(L)$, $C(L)$; $m(X)$ and $\Phi(L)$ are related, but are not mutually dual maps.

The Construction Theorem in [5] asserts that, given a suitably defined triple (C, D, Φ), there exists a Stone lattice L with $C(L) = C$, $D(L) = D$, $\Phi(L) = \Phi$. Problem 55 of [9] seeks a less computational proof of this theorem than that given in [5]. Motivated by this problem, we show how to construct a space dual to a Stone lattice from a "dual triple" (Y, Z, m) and hence obtain a new method of constructing Stone lattices from simpler components.

Dual triples also provide new information on free Stone algebras and a short proof of Theorem 2 of [2], characterizing injectives.

2. The dual space of a Stone lattice. We refer to [19], [20] for the ordered topological space concepts needed, recalling only two crucial definitions concerning a set X endowed with a partial order $\leqslant$ and a topology $\mathfrak{J}$. A subset E of X is *decreasing* (*increasing*) if $x \leqslant y \in E$ ($x \geqslant y \in E$) implies

$x \in E$. $(X, \mathfrak{J}, \leqslant)$ is *totally order disconnected* (abbreviated to t.o.d.) if, given x, y in X with $x \nleqslant y$, there exists a $\mathfrak{J}$-clopen increasing set U such that $x \in U$, $y \notin U$.

For the lattice theoretic terminology used without explanation, [9], or [4], may be consulted. Let L be a $\{0,1\}$-distributive lattice. The main theorems of [19] allow L to be identified with the lattice of clopen increasing subsets of a compact t.o.d. space $(X, \mathfrak{J}, \leqslant)$ (the *dual space* of L); we shall assume this identification to have been made. Henceforth we adopt a compromise between the incompatible notations of [9] and of [19], [20], and use set theoretic symbols for lattice operations in L and lower case roman letters $a, b, \ldots$ to denote subsets of X, reserving x, y, z for points of X. $\mathrm{cl}(a)$ (or, where desirable, $\mathrm{cl}_X(a)$) denotes the closure of a in the $\mathfrak{J}$-topology of an ordered space $(X, \mathfrak{J}, \leqslant)$. The closure of a in the upper topology $\mathcal{U}$ on X (consisting of the $\mathcal{U}$-open increasing sets) is written $\mathcal{U}\text{-}\mathrm{cl}(a)$ or $\mathcal{U}\text{-}\mathrm{cl}_X(a)$.

A $\{0,1\}$-distributive lattice L is said to be *pseudocomplemented* if, for each element $a \in L$, there exists $a^* \in L$ such that

$$a \cap b = 0 \quad \text{if and only if} \quad b \subseteq a^*.$$

A *Stone lattice* L is a $\{0,1\}$-distributive lattice which is pseudocomplemented and such that, for all $a \in L$, $a^* \cup a^{**} = 1$.

PROPOSITION 1. *A $\{0,1\}$-distributive lattice L is pseudocomplemented if and only if, for each clopen increasing subset a in the dual space X of L, $d(a) = \{x \in X\colon x \leqslant y$ for some $y \in a\}$ is open in X.*

Proof. Let $a \in L$ and suppose that $d(a)$ is open. Then $\mathfrak{c}(d(a))$ is disjoint from a and is clopen increasing ($d(a)$ is closed since a is closed and X is compact t.o.d.). Also, if b is increasing, $a \cap b = \emptyset$ implies that $d(a) \cap b = \emptyset$. It follows that a has pseudocomplement $\mathfrak{c}(d(a))$ in L.

Conversely suppose that each a in L has a pseudocomplement a^* in L. To show that $d(a)$ is open we show that $d(a) = \mathfrak{c}a^*$. $a \cap a^* = \emptyset$ and a^* is increasing, so $a^* \subseteq \mathfrak{c}(d(a))$. If x does not belong to the closed decreasing set $d(a)$, then total order disconnectedness of X provides a clopen increasing set b such that $x \in b \subseteq \mathfrak{c}(d(a))$. $a \cap b = \emptyset$, so $b \subseteq a^*$. Hence $\mathfrak{c}(d(a)) \subseteq a^*$.

A compact t.o.d. space $(X, \mathfrak{J}, \leqslant)$ will be called an SLD-*space* if $a \in \mathcal{U}$ implies $d(a) \in \mathcal{U}$.

PROPOSITION 2. *X is an* SLD-*space if and only if X is the dual space of a Stone lattice.*

Proof. Suppose that the lattice L of clopen increasing subsets of a compact t.o.d. space X is a Stone lattice. L is then pseudocomplemented and, for all $a \in L$, $a^* \cup a^{**} = X$. The last condition gives $d(a) = a^{**}$, which is open increasing. Any open increasing set is the union of its clopen increasing subsets and the operator d preserves unions. Thus $d(a) \in \mathcal{U}$ whenever $a \in \mathcal{U}$.

For the converse, note that, if $a \in L$ implies that $d(a) \in \mathcal{U}$, then $a^{**} = \mathfrak{c}\Big(d\big(\mathfrak{c}(d(a))\big)\Big) = d(a)$.

PROPOSITION 3. *Let X be the dual space of a pseudocomplemented distributive lattice and let X_m denote the set of points in X which are maximal with respect to the partial order on X. Then the following statements are equivalent*:

(i) *X is an* SLD-*space*;

(ii) *for each $x \in X$ there exists a unique $m(x) \in X_m$ with $x \leqslant m(x)$*;

(iii) *for any increasing sets $p, q \subseteq X$, $d(p \cap q) = d(p) \cap d(q)$.*

Proof. (i) $\Rightarrow$ (ii): A Zorn's lemma argument shows that each $x \in X$ is majorized by at least one element in X_m. Suppose $x \leqslant y, z$, where $y, z \in X_m$, $y \neq z$. There exist clopen increasing sets a, b with $y \in a \cap \mathfrak{c}b$, $z \in b \cap \mathfrak{c}a$, $x \in d(a) \cap d(b)$, which is an increasing set by (i). Hence $y \in d(a) \cap d(b)$ and so, by maximality, $y \in a \cap b$, which is a contradiction.

(ii) $\Rightarrow$ (iii): Let p and q be increasing, and let $x \in d(p) \cap d(q)$ ((iii) holds trivially if $d(p)$, $d(q)$ are disjoint). x is majorized by maximal points $y \in p$, $z \in q$. (ii) forces $y = z = m(x)$, so $x \in d(p \cap q)$. Trivially $d(p \cap q) \subseteq d(p) \cap d(q)$.

(iii) $\Rightarrow$ (i): We show that $d(p)$ is increasing for each increasing set p. Let $x \in d(p)$, $y \geqslant x$. Then $x \leqslant y \leqslant z \in X_m$. Also $x \leqslant z_1 \in p \cap X_m$. $x \in d(\{z\}) \cap d(\{z_1\}) = d(\{z\} \cap \{z_1\})$, by (iii). Hence $z = z_1$, so $y \in d(p)$.

The equivalence (i)$\Leftrightarrow$(ii) is related to the characterization of Stone lattices as those pseudocomplemented distributive lattices in which, for all lattice elements a, b, $a^* \cup b^* = (a \cap b)^*$ (see [8]). (i)$\Leftrightarrow$(iii) is the dual of the theorem of Varlet [22] stating that a pseudocomplemented distributive lattice is a Stone lattice if and only if every prime ideal contains a unique minimal prime ideal. An equivalent form of this theorem was earlier obtained by Grätzer and Schmidt [11]: a pseudocomplemented distributive lattice is a Stone lattice if and only if the join of any two distinct minimal prime ideals in L is L. That the restriction to pseudocomplemented lattices is necessary was shown by an example by Katriňák [13]. Examples of a different type can be constructed as follows. Let X_1 be the dual of any $\{0,1\}$-distributive lattice which is not pseudocomplemented. Then there is a clopen increasing subset a of X_1 such that $d(a)$ is not open. Let $X_2 = \{y, z\}$ be a two element space with discrete topology and order. Form the topological sum X of X_1 and X_2 and order X

by requiring that the given order relations be induced on X_1 and X_2 and that

$$y \geqslant x \in X_1 \quad \text{if and only if} \quad x \in a\,,$$
$$z \geqslant x \in X_1 \quad \text{if and only if} \quad x \in \complement a\,.$$

X is compact and t.o.d. and satisfies condition (ii) of Proposition 3. The set $a \cup \{y\}$ is clopen increasing in X and the smallest decreasing set in X containing it is $d(a) \cup \{y\}$, which is not open in X. Hence the lattice of clopen increasing subsets of X is not pseudocomplemented, but is such that any prime ideal contains a unique minimal prime ideal.

PROPOSITION 4. *In an* SLD*-space X, X_m is $\mathfrak{J}$-closed and the map*

$$m\colon (X, \mathfrak{U}) \to (X_m, \mathfrak{J}|X_m)\,,$$

assigning to each point $x \in X$ the unique maximal point majorizing x, is continuous.

Proof. Let $a = a_1 \cap X_m$, where a_1 is open increasing in X.

$$m^{-1}(a) = d(a_1 \cap X_m) = d(a_1) \cap d(X_m) = d(a_1)\,,$$

which is $\mathfrak{J}$-open. Hence m is continuous when X has the topology $\mathfrak{J}$, X_m the relative $\mathfrak{U}$-topology. It follows that X_m is $\mathfrak{U}$-compact. Take $x \notin X_m$. For each $y \in X_m$, choose a clopen increasing set a_y with $y \in a_y$, $x \notin a_y$. X_m can be covered by a finite number of the sets a_y. The intersection of the complements of these sets is a $\mathfrak{J}$-open neighbourhood of x disjoint from X_m, so X_m is $\mathfrak{J}$-closed.

Finally, if b is closed in X_m (in the relative $\mathfrak{J}$-topology), then b is $\mathfrak{J}$-closed in X. $m^{-1}(b) = d(b)$, which is closed.

We remark that compactness of X_m (in either $\mathfrak{J}$- or $\mathfrak{U}$-topology) is implicitly obtained by Speed in [21].

A Stone algebra is a Stone lattice in which $0, 1$ are regarded as nullary operations and $*$ as a unary operation. Morphisms in the category of Stone algebras are consequently lattice homomorphisms which preserve these operations. If L, L_1 are $\{0,1\}$-distributive lattices with dual spaces X, X_1, there is ([19], [20]) a one-to-one correspondence between $\{0,1\}$-preserving lattice homomorphisms $\Phi\colon L_1 \to L$ and continuous increasing maps $f\colon X \to X_1$ such that $f^{-1}(a_1) = \Phi(a_1)$ for all $a_1 \in L_1$.

PROPOSITION 5. *If X, X_1 are* SLD*-spaces and $f\colon X \to X_1$ is continuous and increasing, then the dual map $\Phi\colon L_1 \to L$ is a Stone algebra morphism if and only if f maps X_m into $(X_1)_m$.*

Proof. For $a_1 \in L_1$,

$$\Phi(a_1^*) = \Phi\big((a_1)\big)^* \quad \text{if and only if} \quad f^{-1}\big(d(a_1)\big) = d\big(\Phi(a_1)\big)\,.$$

If $x \leqslant y \in \Phi(a_1)$, then $f(x) \leqslant f(y) \in a_1$ and so $d\big(\Phi(a_1)\big) \subseteq f^{-1}\big(d(a_1)\big)$ always. Suppose now that f preserves maximality, so that, for all x in X, $f\big(m(x)\big)$ is the unique maximal point majorizing $f(x)$ in X_1. This implies that, if $x \in f^{-1}\big(d(a_1)\big)$, $m(x) \in f^{-1}(a_1) = \Phi(a_1)$. Thus $x \in d\big(\Phi(a_1)\big)$, so Φ preserves $*$.

Conversely, let $x \in X_m$. Then

$$f(x) \in a_1 \Leftrightarrow x \in \Phi(a_1) \Leftrightarrow x \in d\big(\Phi(a_1)\big) \Leftrightarrow f(x) \in d(a_1)\,.$$

If $f(x)$ were non-maximal and majorized by $y \neq f(x)$, there would exist a clopen increasing set a_1 such that $f(x) \notin a_1$, $y \in a_1$. But then $f(x) \in d(a_1) \cap \complement a_1$, which is impossible.

3. The dual triple associated with a Stone lattice. Two subsets of a Stone lattice L are of special importance: the centre of L,

$$C(L) = \{a^*\colon\ a \in L\}\,,$$

and the dense set,

$$D(L) = \{a\colon\ a^* = 0\}\,.$$

Any element a of L may be represented as

$$a = a^{**} \cap (a \cup a^*)\,.$$

$a^{**} \in C(L)$ and $a \cup a^* \in D(L)$. Thus if $C(L)$ and $D(L)$ are given and the relation between $C(L)$ and $D(L)$ is known, then L is completely determined. The map used by Chen and Grätzer in [5] to link $C(L)$ and $D(L)$ is $\Phi(L)$, which is the $\{0,1\}$-homomorphism mapping $C(L)$ into the lattice of filters of $D(L)$ defined by

$$\big(\Phi(L)\big)(a) = \{b \in D(L)\colon\ b \supseteq a^*\}\,.$$

Chen and Grätzer show in § 3 of [5] that L is determined up to isomorphism by its triple $\big(C(L), D(L), \Phi(L)\big)$ (or see [9], p. 163). One would therefore expect the dual space X of L to be determined by the dual spaces $Y(X)$, $Z(X)$ of $D(L)$, $C(L)$ and an appropriate connecting map $m(X)$.

a^* is given by $a^* = \complement d(a)$ in X. It follows that $C(L)$ is the Boolean algebra of those clopen subsets of the dual space X which are atonic (i.e. simultaneously increasing and decreasing), while $D(L)$ consists of those clopen increasing subsets of X which contain X_m. $C(L)$ and $\bar{D}(L) = D(L) \cup \{0\}$ are $\{0,1\}$-sublattices of L and their dual spaces are, according to [20], § 7, appropriately ordered quotient spaces of $(X, \mathfrak{J})$.

We now identify the dual of $C(L)$. This is (homeomorphic to) the discretely ordered quotient space X/R where $x\,R\,y$ if and only if, for every clopen atonic set a, $x \in a$ if and only if $y \in a$. If $m(x) = m(y)$, then any atonic set containing x contains y, and vice versa, so $x\,R\,y$. On the other hand, if $m(x) \neq m(y)$, then a clopen increasing set a can be found with $m(x) \in a$, $m(y) \in \complement a$. $d(a)$ is clopen atonic and $x \in d(a)$, $y \notin d(a)$.

Thus x, y belong to different R-equivalence classes. We conclude that the elements of X/R are the sets $d(x)$, where $x \in X_m$.

PROPOSITION 6. *The dual space of $C(L)$ is homeomorphic to X_m (with the relative $\mathfrak{J}$-topology).*

Proof. The map $g\colon X_m \to X/R$ defined by $g(x) = d(x)$ is one-to-one and onto. A set of equivalence classes $\{d(x)\colon x \in c \subseteq X_m\}$ is open in X/R if and only if $d(c)$ is open in X. When $d(c)$ is open, $c = d(c) \cap X_m$ is open in X_m and so g is continuous. Since X_m is compact and X/R Hausdorff, g is a homeomorphism.

Next we turn our attention to $\bar{D}(L)$. An equivalence relation R' can be defined on X by letting $x R' y$ if and only if either $x = y$ or x, y both belong to X_m. X/R' is a compact t.o.d. space under the quotient topology and quotient order (cf. [20], Proposition 14 (ii)), and it is clear that the lattice of clopen increasing subsets of X/R' is isomorphic to $\bar{D}(L)$. Hence the dual space of $\bar{D}(L)$ can be identified with X/R'.

X/R' is homeomorphic and order isomorphic to the space obtained by taking the one-point compactification of $X \setminus X_m$ and ordering it by requiring that the given order be induced on $X \setminus X_m$ and that the adjoined point majorize every point of $X \setminus X_m$ (see [20], p. 510 and also [15], [16]). That the dual of $\bar{D}(L)$ takes the form of a one-point order compactification reflects the fact that $\bar{D}(L)$ is obtained from $D(L)$ by the adjunction of $\mathbf{0}$. Since the prime ideals of $\bar{D}(L)$ are the prime ideals of $D(L)$ with $\mathbf{0}$ added to each, together with the zero ideal, $X \setminus X_m$ is the dual space of $D(L)$, according to the definition in [20], § 10. We sum up these results in

PROPOSITION 7. *The dual space of $\bar{D}(L)$ is the ordered quotient space obtained from X by identifying the points of X_m, or equivalently, is a one-point order compactification of $X \setminus X_m$. The dual space of $D(L)$ is $X \setminus X_m$.*

We now define the *dual triple* associated with an SLD-space X to be $(Y(X), Z(X), m(X))$, where $Y(X) = X \setminus X_m$, $Z(X) = X_m$ (both with relative topology and order from X) and $m(X)\colon Y(X) \to Z(X)$ is the continuous increasing map defined by the restriction of m to $X \setminus X_m$. We shall denote by $\bar{Y}(X)$ the one-point order compactification of $Y(X)$ in which X_m is adjoined to $Y(X)$ as a universal maximal point.

Before describing, in Theorem 9, how X is determined by its dual triple, we explain the relation between $m(X)$ and Chen and Grätzer's structure map $\Phi(L)$. Recall that $\Phi(L)$ is defined by

$$\big(\Phi(L)\big)(a) = \{b \in D(L)\colon\ b \supseteq a^*\}.$$

$\big(\Phi(L)\big)(a)$ is a proper filter in $\bar{D}(L)$ and may be identified with the non-empty closed increasing subset $F(a)$ of $\bar{Y}(X)$ obtained by taking the intersection of the sets in $\big(\Phi(L)\big)(a)$ (regarded as clopen increasing subsets of $\bar{Y}(X)$) (cf. [12], § 19, [19], [20]). Reverting to $D(L)$ and $Y(X)$, we see that $\big(\Phi(L)\big)(a)$ may be identified with the closed increasing subset of $Y(X)$ obtained by deleting the point X_m from $F(a)$. With this interpretation of $\big(\Phi(L)\big)(a)$ we have

PROPOSITION 8.

$$\big(\Phi(L)\big)(a) = \big(m(X)\big)^{-1}(a')$$

for each set a clopen in $Z(X)$.

$$\big(m(X)\big)(x) = \bigcap \{a\colon\ a \text{ is clopen in } Z(X) \text{ and } x \in \big(\Phi(L)\big)(a')\}$$

for all $x \in Y(X)$.

(a' denotes the Boolean complement of a in $C(L)$.)

Proof. The first assertion is an immediate consequence of the definitions; the second can be deduced using compactness and total disconnectedness of $Z(X)$.

THEOREM 9 (Dual structure theorem). *If X is an SLD-space with dual triple $(Y(X), Z(X), m(X))$, then X is homeomorphic and order isomorphic to the increasing subspace*

$$\hat{X} = G_{m(X)} \cup \big(\{Z(X)\} \times Z(X)\big)$$

of $\bar{Y}(X) \times Z(X)$.

($G_{m(X)}$ denotes the graph of $m(X)$.)

Proof. A map ϱ from X onto $\hat{X}$ is defined by

$$\varrho(x) = \begin{cases} (x, m(x)), & x \in Y(X) = X \setminus X_m, \\ (X_m, x), & x \in Z(X) = X_m. \end{cases}$$

$\bar{Y}(X)$ and $Z(X)$ can be regarded as ordered quotients of X; ϱ is then the product of the associated quotient maps and so is continuous.

The proof is completed by showing that ϱ is an order isomorphism. If $x_1 \leqslant x_2$ in X, $m(x_1) = m(x_2)$. It follows that $\varrho(x_1) \leqslant \varrho(x_2)$. Now let $\varrho(x_i) = (y_i, z_i)$, $i = 1, 2$. $z_i = m(x_i)$; $y_i = x_i$ if $x_i \in X \setminus X_m$, $y_i = X_m$ otherwise. Suppose that $\varrho(x_1) \leqslant \varrho(x_2)$. This implies that $y_1 \leqslant y_2$ and $m(x_1) = m(x_2)$. If $x_1, x_2 \notin X_m$, $x_1 \leqslant x_2$ trivially and if $x_1, x_2 \in X_m$, $x_1 = x_2$. $x_1 \in X_m$, $x_2 \notin X_m$ is impossible, while $x_1 \notin X_m$, $x_2 \in X_m$ gives $x_1 \leqslant m(x_1) = m(x_2) = x_2$.

Henceforth, if X and L are dual, with X an SLD-space and L a Stone lattice, we shall refer to $(Y(X), Z(X), m(X))$ and $(C(L), D(L), \Phi(L))$ as being dual to one another and to each as being associated with X and with L. In this situation, it can be inferred from [9], Theorem 12.2, that the dual space of the free $\{0,1\}$-distributive product of $\bar{D}(L)$ and $C(L)$ is the ordered cartesian product of $\bar{Y}(X)$ and $Z(X)$. The formation of a closed subspace of a dual space corresponds to the formation of the quotient by a congruence relation of the corresponding lattice. Theorem 9 may then be interpreted as asserting that

$$L \cong (\overline{D}(L) \times C(L))/\theta\,,$$

where θ is a congruence (cf. [17] and [9], p. 191).

In § 4 of [5], Chen and Grätzer abstract the idea of a triple associated with a Stone lattice. They define a triple (C, D, Φ) to consist of (i) a Boolean algebra C, (ii) a distributive lattice D with 1, (iii) a $\{0, 1\}$-homeomorphism Φ from C into the lattice of filters of D, and they show how a Stone lattice L can be built out of this triple. L is defined by

$$L = \{(b, c):\ c \in C,\ b \in \Phi(c)\}\,.$$

The difficult part of Chen and Grätzer's Construction Theorem is the proof that L can be partially ordered so as to form a distributive lattice. This difficulty can be by-passed if one constructs initially not a Stone lattice but an SLD-space. The collection of clopen increasing subsets of this space then forms automatically a Stone lattice.

4. The dual contruction theorem. An abstract dual triple (Y, Z, m) consists of

(i) an ordered space $(Y, \mathfrak{J}_Y, \leqslant)$ in which the topology of open decreasing sets has a $\mathfrak{J}_Y$-compact base;

(ii) a non-empty compact totally disconnected space $(Z, \mathfrak{J}_Z)$;

(iii) an increasing map $m\colon\ Y \to Z$, continuous with respect to the topologies $\mathfrak{J}_Y, \mathfrak{J}_Z$.

The condition in (i) allows us to form by one-point compactification a compact t.o.d. space $\overline{Y}$, where $\overline{Y} = Y \cup \{Z\}$ and $Z \geqslant y$ for all $y \in Y$. Guided by Theorem 9, we prove

Theorem 10 (Dual construction theorem). *Let (Y, Z, m) be an abstract dual triple. Then there exists an* SLD*-space X such that, up to homeomorphism and order isomorphism, $Y = Y(X)$, $Z = Z(X)$ and $m = m(X)$.*

Proof. Take

$$X = G_m \cup (\{Z\} \times Z) \subseteq \overline{Y} \times Z\,.$$

If $(y, z) \in (\overline{Y} \times Z) \setminus X$, then $y \neq Z$ and $z \neq m(y)$. Take an open subset a in Z containing $m(y)$ but not z. $m^{-1}(a)$ is open in Y and so also in $\overline{Y}$. Consequently $m^{-1}(a) \times a$ is an open neighbourhood of (y, z), disjoint from X. X is thus a closed subset of the compact t.o.d. space $\overline{Y} \times Z$, and so is itself compact and t.o.d.

Let a be a non-empty increasing subset of X:

$$a = \{(y, m(y)):\ y \in b \subseteq Y\} \cup \{(Z, z):\ z \in c \subseteq Z\}\,,$$

where $m(b) \subseteq c$ since $y \leqslant Z$ for all $y \in Y$. Then a non-empty implies c non-empty, and we conclude that, in X, $d(a) = (\overline{Y} \times c) \cap X$, which is atonic. Suppose in addition that a is open in X and that $z \in c$. It is possible to find p open in $\overline{Y}$, q open in Z with

$$(Z, z) \in (p \times q) \cap X \subseteq a\,,$$

$z \in q \subseteq c$, which shows that c is open in Z. It follows that $d(a)$ is open increasing in X whenever a is open increasing in X, so that X is an SLD-space.

We now identify the triple $(Y(X), Z(X), m(X))$. $(y_1, z_1) \leqslant (y_2, z_2)$ in $\overline{Y} \times Z$ if and only if $y_1 \leqslant y_2$ and $z_1 = z_2$. Thus $X_m = \{(Z, z):\ z \in Z\}$. Clearly Z is homeomorphic to $X_m = Z(X)$ under the map $\sigma\colon\ z \mapsto (Z, z)$. Further, the $\tau\colon\ y \mapsto (y, m(y))$ is a homeomorphism from Y onto $X \setminus X_m = Y(X)$; τ is also an order isomorphism by condition (iii) above. Finally,

$$(m(X))(y, m(y)) = (Z, m(y))\,,$$

i.e.

$$(m(X))(\tau(y)) = \sigma(m(y))\,.$$

A morphism from a triple (Y, Z, m) to a triple (Y_1, Z_1, m_1) is a pair (f, g) where f is a continuous increasing map from $\overline{Y}$ into $\overline{Y}_1$, g is a continuous map from Z into Z_1 and the diagram

$$\begin{tikzcd} Y \arrow[r, "m"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y_1 \arrow[r, "m_1"'] & Z_1 \end{tikzcd}$$

commutes. Analogous to Theorem 1 of [5] we have Proposition 11. We omit the routine proof.

Proposition 11. *Let X, X_1 be* SLD*-spaces with dual triples (Y, Z, m), (Y_1, Z_1, m_1). Then if (f, g) is a triple morphism, the map*

$$f \times g\colon\ \overline{Y} \times Z \to \overline{Y}_1 \times Z_1$$

induces by restriction an SLD*-space morphism $h\colon\ X \to X_1$. Conversely, an* SLD*-space morphism $h\colon\ X \to X_1$ induces maps $f\colon\ \overline{Y} \to \overline{Y}_1$, $g\colon\ Z \to Z_1$ (regarding $\overline{Y}$, Z as quotient spaces of X, $\overline{Y}_1$, Z_1 as quotient spaces of X_1). The resulting pair (f, g) is a triple morphism. The correspondence so established between* SLD*-space morphisms and triple morphisms is one-to-one and onto.*

h is onto if and only if each of f, g is onto. If $f(y) = Z_1$ implies $y = Z$, then h is an order isomorphism if and only if f is an order isomorphism and g is one-to-one.

We remark that dual versions of the "fill-in" theorems in [6], § 6, can be obtained by straightforward diagram-chasing.

Given a triple (C, D, Φ) associated with a Stone lattice L, the reduced triple associated with L is (C_1, D, Φ_1), where C_1 is the quotient of C by θ, the congruence determined by Φ, and $\Phi_1(\theta[a]) = \Phi(a)$ for all $a \in C$ ([6], § 5; see also [7]). Dually, given a dual triple (Y, Z, m) with associated SLD-space X, we define the *reduced dual triple* to be (Y, Z_1, m) where $Z_1 = \mathrm{cl}_Z(m(Y))$.

Proposition 12. *If (C, D, Φ) and (Y, Z, m) are dual to one another, so too are the reduced triples (C_1, D, Φ_1) and (Y, Z_1, m).*

Proof. It is enough to show that C_1 and Z_1 are mutually dual. For a, b clopen in Z,

$$\begin{aligned} a \sim b(\theta) &\Leftrightarrow \Phi(a) = \Phi(b) \\ &\Leftrightarrow m^{-1}(a') = m^{-1}(b') \\ &\Leftrightarrow a' \cap m(Y) = b' \cap m(Y) \\ &\Leftrightarrow a \cap m(Y) = b \cap m(Y) \end{aligned}$$

(dashes denote complements in Z). Since a is open

$$a \cap \mathrm{cl}(m(Y)) \subseteq \mathrm{cl}(a \cap m(Y)),$$

and because a is closed,

$$\mathrm{cl}(a \cap m(Y)) \subseteq a \cap \mathrm{cl}(m(Y)).$$

It follows that $a \sim b(\theta)$ if and only if $a \cap \mathrm{cl}(m(Y)) = b \cap \mathrm{cl}(m(Y))$. We deduce that $C_1 = C/\theta$ is isomorphic to the Boolean algebra of sets $a \cap Z_1$, where a is clopen in Z and $Z_1 = \mathrm{cl}_Z(m(Y))$. An application of Lemma 17.1 of [12] completes the proof.

5. A description of L in terms of (Y, Z, m). Proposition 2 and Theorem 10 together show that Stone lattices may be built from dual triples: the lattice L of clopen increasing subsets of $X = G_m \cup (\{Z\} \times Z)$ is a Stone lattice for any abstract dual triple (Y, Z, m).

Theorem 14 characterizes order and topological properties of subsets of X. It provides an explicit description of L in terms of its dual triple; this is used in Theorem 16, which concerns completeness of L. In addition Theorem 14 yields an explicit, though unwieldy, description of the ideals and filters of L (respectively open increasing, closed increasing, subsets of X) and of the minimal Boolean extension of L (clopen subsets of X).

A typical subset of X is

$$a = \{(y, m(y)) \colon y \in b \subseteq Y\} \cup \{(Z, z) \colon z \in c \subseteq Z\};$$

more concisely, $a = G_{m|b} \cup (\{Z\} \times c)$. For each $b \subseteq Y$, we define

$$\hat{b} = \{z \in Z \colon z = \lim m(y_\mu) \text{ for some net } \{y_\mu\} \text{ in } b,\ y_\mu \to Z \text{ in } \overline{Y}\}.$$

$\hat{b} = \emptyset$ whenever $Z \notin \mathrm{cl}_{\overline{Y}}(b)$; this happens for all $b \subseteq Y$ if Z is an isolated point of $\overline{Y}$ (equivalently if $D(L)$ has a smallest element). We note one property of $\hat{b}$:

Proposition 13. *For each $b \subseteq Y$, $\hat{b}$ is closed in Z.*

Proof. Let $z \in \mathrm{cl}_Z(\hat{b})$ and let p be any open neighbourhood of Z in $\overline{Y}$, q any open neigbourhood of z in Z. There exists a point $y_{(p,q)} \in b \cap p$ such that $m(y_{(p,q)}) \in q$. $\{y_{(p,q)}\}$ forms a net in $\overline{Y}$ when the indices are directed by pairwise set inclusion. By compactness of $\overline{Y}$, we can find a convergent subnet $\{y_{(p',q')}\}$ of $\{y_{(p,q)}\}$. Necessarily $y_{(p',q')} \to Z$ and $m(y_{(p',q')}) \to z$.

Theorem 14. *Let $a = G_{m|b} \cup (\{Z\} \times c)$.*

(i) *a is increasing if and only if b is increasing in Y (equivalently $b \cup \{Z\}$ increasing in $\overline{Y}$) and $m(b) \subseteq c$;*

(ii) *a is decreasing if and only if b is decreasing in Y (equivalently b decreasing in $\overline{Y}$) and $m^{-1}(c) \subseteq b$;*

(iii) $\mathrm{cl}_X(a) = \{(y, m(y)) \colon y \in \mathrm{cl}_Y(b)\} \cup \{(Z, z) \colon z \in \hat{b} \cup \mathrm{cl}_Z(c)\}$;

(iv) *a is open if and only if b is open in Y, c is open in Z and $\widehat{(Y \setminus b)} \subseteq c'$;*

(v) *$\hat{b} \subseteq c$ whenever c is closed and $m(b) \subseteq c$; $\widehat{(Y \setminus b)} \subseteq c'$ whenever c is open and $m^{-1}(c) \subseteq b$.*

Proof. (i) and (ii) are elementary. To prove (iii) let

$$\mathrm{cl}_X(a) = G_{m|b_1} \cup (\{Z\} \times c_1).$$

If $y \in \mathrm{cl}_Y(b)$, $(y, m(y)) \in \mathrm{cl}_X(a)$, by continuity of m. Hence $\mathrm{cl}_Y(b) \subseteq b_1$. On the other hand, if $x = (y, m(y)) \in \mathrm{cl}_X(a)$ then it is possible to choose a net $\{(y_\mu, m(y_\mu))\}$ of points in a converging to x. Hence $y \in \mathrm{cl}_Y(b)$.

It is clear that each point (Z, z) with $z \in \mathrm{cl}_Z(c)$ is in $\mathrm{cl}_X(a)$ and that, by definition of $\hat{b}$, so too is each point (Z, z) for which $z \in \hat{b}$. Now take $x = (Z, z) \in \mathrm{cl}_X(a)$ and $\{x_\lambda\}_{\lambda \in \Lambda}$ a net of points in a with $x_\lambda \to x$. We may write $\Lambda = \Lambda_1 \cup \Lambda_2$, where $x_\lambda \in G_m$ for $\lambda \in \Lambda_1$, $x_\lambda \in \{Z\} \times Z$ for $\lambda \in \Lambda_2$. At least one of Λ_1, Λ_2 is cofinal. If Λ_1 is cofinal, $z \in \hat{b}$; if Λ_2 is cofinal, $z \in \mathrm{cl}_Z(c)$. We conclude that $c_1 = \hat{b} \cup \mathrm{cl}_Z(c)$.

(iv) is proved by applying (iii) to $\complement a$. Finally, (v) is obvious from (i), (ii) and the definitions.

Corollary 15. *$a = G_{m|b} \cup (\{Z\} \times c) \in L$ if and only if*

(i) *b is clopen increasing in Y;*

(ii) *c is clopen in Z;*

(iii) $\widehat{(Y \setminus b)} \subseteq c'$;

(iv) $m(b) \subseteq c$.

To illustrate Corollary 15 we consider the lattice $B^{(2)}$. By definition ([2], [9]), for any Boolean algebra B, $B^{(2)}$ is the sublattice

$$\{(b, c)\colon\ b, c \in B,\ b \subseteq c\}$$

of $B \times B$. $B^{(2)}$ is a Stone lattice with pseudocomplement given by $(b, c)^* = (c', c')$. Clearly $C(B^{(2)}) \cong B$, $D(B^{(2)}) \cong B$. If X is the dual space of B, then the dual triple for $B^{(2)}$ is (X, X, j), j being the identity map on X. The dual space of $B^{(2)}$ is

$$X^{(2)} = \{(x, x)\colon\ x \in X\} \cup \{(X, x)\colon\ x \in X\}\,.$$

A subset a of $X^{(2)}$ is described by a pair b, c of subsets of X. a is clopen increasing if and only if b and c are clopen in X and $b = j(b) \subseteq c$ (condition (iii) of Corollary 15 is vacuous here). Thus we recover

$$B^{(2)} \cong \{(b, c)\colon\ b, c \in B,\ b \subseteq c\}\,.$$

We remark finally that $X^{(2)}$ can be identified with the ordered topological sum $X_1 \oplus X_2$, where $X_1 = X_2 = X$ and $x \leqslant y$ if and only if $x = y$ in $X_1 \oplus X_2$ or $x \in X_1$, $y \in X_2$ and $x = y$ in X. This interpretation of the dual of $B^{(2)}$ will be used in § 6.

Completeness of a Stone lattice L is discussed using the triple for L in § 4 of [6]. We conclude this section with an analogous theorem using L's dual triple. We recall that a $\{0, 1\}$-distributive lattice is complete if and only if its dual space X is extremally order disconnected (i.e., for each set a open and decreasing in X, $\mathfrak{U}\text{-cl}(a)$ is open ([20], Proposition 16).

THEOREM 16. *Let L be a Stone lattice with dual triple (Y, Z, m). Then L is complete if and only if*

(i) *Z is extremally disconnected;*

(ii) *$\overline{Y}$ is extremally order disconnected;*

(iii) *for each b open decreasing in Y, $\hat{b}$ is open in Z.*

Proof. The first step is to show that (ii) is equivalent to (ii′) $d(\mathrm{cl}_Y(b))$ is open in Y whenever b is open in Y and $Z \notin \mathrm{cl}_{\overline{Y}}(b)$. (ii) holds if and only if, for all b open in $\overline{Y}$, $d(\mathrm{cl}_{\overline{Y}}(b)) = \mathfrak{U}\text{-cl}_{\overline{Y}}(b)$ is open in $\overline{Y}$. Suppose b is open in $\overline{Y}$. If $Z \in \mathrm{cl}_{\overline{Y}}(b)$, $d(\mathrm{cl}_{\overline{Y}}(b)) = \overline{Y}$. So assume $Z \notin \mathrm{cl}_{\overline{Y}}(b)$. Then

$$d(\mathrm{cl}_Y(b)) = d(\mathrm{cl}_{\overline{Y}}(b)) \cap Y = d(\mathrm{cl}_{\overline{Y}}(b))$$

which is open in Y if and only if it is open in $\overline{Y}$. Hence (ii) is equivalent to (ii′).

Let a be open decreasing in X. By [20], Lemma 2,

$$\begin{aligned}\mathfrak{U}\text{-cl}(a) &= d(\mathrm{cl}(a))\\ &= \{(y, m(y))\colon\ y \in d(\mathrm{cl}_Y(b)) \cup m^{-1}(\hat{b} \cup \mathrm{cl}_Z(c))\} \cup \{(Z, z)\colon\ z \in \hat{b} \cup \mathrm{cl}_Z(c)\}\,,\end{aligned}$$

by Theorem 14. $\mathfrak{U}\text{-cl}(a)$ is open if and only if

(a) $d(\mathrm{cl}_Y(b)) \cup m^{-1}(\hat{b} \cup \mathrm{cl}_Z(c))$ is open in Y and

(b) $\hat{b} \cup \mathrm{cl}_Z(c)$ is open in Z

(Theorem 14, (iv), (v)).

Since, when a is open decreasing, b is open decreasing in Y and c is open in Z, the sufficiency of conditions (i)-(iii) is now clear.

For the converse, assume that a open decreasing in X implies $\mathfrak{U}\text{-cl}_X(a)$ is open. Let c be open in Z.

$$a = G_{m|(m^{-1}(c))} \cup (\{Z\} \times c)$$

is open decreasing in X and

$$\widehat{(m^{-1}(c))} \subseteq \mathrm{cl}_Z(c)\,.$$

Hence, by (b) above, $\mathrm{cl}_Z(c)$ is open, so (i) holds.

Now take b open decreasing in Y. $a = G_{m|b}$ is open decreasing in X. By (b), $\hat{b}$ is open in Z, whence (iii) holds. Now suppose also $Z \notin \mathrm{cl}_{\overline{Y}}(b)$. Then $\hat{b} = \emptyset$. By (a), $d(\mathrm{cl}_Y(b))$ is open in Y. Therefore (ii′) holds and the proof is complete.

Comparing Theorem 16 with Theorem 5 of [6], we see that our conditions (i), (ii) are the duals of conditions (1), (2) in Chen and Grätzer's completeness theorem. The relation of their condition (3) to our condition (iii) is however not clear. Finally we note that the corollaries to Chen and Grätzer's theorem are equally easy consequences of Theorem 16.

6. Free and injective Stone algebras. The Stone space of a free Stone algebra L has been used by Grätzer and Lakser [10] to examine the structure of L (see also [1], [3], [9], [18]). Our approach is a variant on that adopted by Grätzer and Lakser, emphasizing the order structure. The dual space of $\mathrm{FSL}(a)$, the free Stone algebra on a generators, is the ordered cartesian product of a copies of a discretely topologized 3-element space $S = \{x_0, x_1, x_2\}$ in which $x_i \geqslant x_j$ if and only if $i = j$, or $i = 1$, $j = 2$ (cf. [9], Theorem 17.5). S is the disjoint union of the partially ordered sets $S_0 = \{x_0\}$ and $S_1 = \{x_1, x_2\}$.

If n is finite, the $\mathfrak{I}$-topology on $X(n)$, the dual of $\mathrm{FSL}(n)$, is discrete and only the order concerns us. We see (cf. [9], Theorem 17.6) that

$$X(n) \text{ is order isomorphic to } \sum_{r=0}^{n} n\,Cr\,(S_1)^r,$$

where summation denotes the disjoint union of partially ordered sets, $(S_1)^0 = S_0$, and, for $r \geqslant 1$, $(S_1)^r$ is the cartesian product of r copies of S_1, and the binomial coefficient nCr indicates the number of copies of $(S_1)^r$ occurring in the sum. The dual triple for $\mathrm{FSL}(n)$, $(Y(n), Z(n), m(n))$, is easily recognized. For $r \geqslant 1$, $(S_1)^r$ has a unique maximal point $(x_1, \dots, x_1)$. Hence $Z(n)$ has 2^n elements. Let T_r be the partially ordered set obtained by deleting the maximal element from $(S_1)^r$. It is clear that, to within order isomorphism,

$$Y(n) = \sum_{r=1}^{n} nCrT_r .$$

We may write

$$Y(n) = \sum_{s=1}^{2^n-1} W_s ,$$

where, for $r \geqslant 1$, $W_s = T_r$ for $\sum_{k=0}^{r-1} nCk \leqslant s < \sum_{k=0}^{r} nCk$;

$$Z(n) = \{s:\ s = 0, \dots, 2^n-1\} .$$

Then $m(n)$ is given by

$$\big(m(n)\big)(x) = s \quad \text{if and only if} \quad x \in W_s .$$

$C(n)$ is the lattice of subsets of $Z(n)$; $D(n)$ is the lattice of all increasing subsets of $Y(n)$, as is the lattice of filters of $D(n)$. We sum up our results in Proposition 17 (cf. [6], Problem 1), using the notation of [9].

PROPOSITION 17. *Let* FSL(n) *have triple* $\big(C(n), D(n), \Phi(n)\big)$. *Then* $C(n) = F_B(n)$ *and* $D(n) = \prod\big((F_{D(0,1)}(|A|)\setminus\{0\}): A \subseteq \{1, 2, \dots, n\}\big)$. *If* $a \in C(n)$ *is a* p*-element set,* $\big(\Phi(n)\big)(a)$ *is an increasing subset of* $Y(n)$, *isomorphic as a partially ordered set to the dual of a product either of* 2^n-p *or of* 2^n-p-1 *lattices, each of which is a free* $\{0,1\}$*-distributive lattice with zero deleted.*

Proof. The statement concerning $\Phi(n)$ is obtained from Proposition 8.

In answer to a question posed by Chen and Grätzer ([6], Problem 2) we state

PROPOSITION 18. *Let* $\big(C_1(n), D(n), \Phi_1(n)\big)$ *be the reduced triple associated with* FSL(n). *Then* $C_1(n) = F_B(n-1)$. *If* $a \in C_1(n)$, $\big(\Phi_1(n)\big)(a)$ *can be represented by a product of* $2^{n-1}-p$ *lattices, each of which is a free* $\{0,1\}$*-distributive product with zero deleted.*

We have been unable to find any worthwhile description of the dual triple for FSL(α) when α is a general cardinal (although it is clear that $Z(\alpha)$ is the Cantor space 2^{α}).

Finally we present a new proof of Theorem 2 of [2], which characterizes injective Stone algebras as Stone lattices expressible in the form $B_0 \times B_1^{(2)}$, where B_0, B_1 are complete Boolean algebras (see also [14]).

THEOREM 19. *If X is a projective* SLD*-space then the dual triple for X takes the form* $(Y_1, Y_0 \oplus Y_1, j)$, *where* Y_0, Y_1 *are compact, discretely ordered and extremally disconnected and* $j\colon Y_1 \to Y_0 \oplus Y_1$ *the embedding map.*

Proof. Let $(X, \mathfrak{J}, \leqslant)$ be projective, with dual triple (Y, Z, m). The first step is to show that the order on Y is discrete (cf. the proof of Proposition 21 of [20]). Suppose not, and take $y_1, y_2 \in Y$, $y_1 \neq y_2$, $y_1 \geqslant y_2$, $y_1 \not\leqslant y_2$, so we can find a clopen increasing set X_1 such that $x \in X_1$, $y \in X_2 = X \setminus X_1$. We may assume that $X_1 \supseteq Z$. Define an order relation $\leqslant'$ on X by

$$\begin{aligned} x \leqslant' y \text{ if and only if } &\text{(i) } x = y ,\\ \text{or } &\text{(ii) } x, y \in X_1,\ x \leqslant y ,\\ \text{or } &\text{(iii) } x \in X_2,\ y \in Z,\ x \leqslant y; \end{aligned}$$

this gives a genuine partial order. We claim that $(X, \mathfrak{J}, \leqslant')$ is t.o.d. Take $x \not\leqslant' y$ and note that a set increasing $(\leqslant)$ is increasing $(\leqslant')$. In either of the cases (i) $x, y \in X_1$, (ii) $x \in X_2$, $y \in Z$, $x \not\leqslant y$ and so there is a clopen increasing $(\leqslant')$ set containing x but not y. If $x \in X_1$, $y \in X_2$, X_1 is clopen increasing $(\leqslant')$ and contains x but not y. Finally let $x \in X_2$, $y \in X_1 \cap Y$. There is a clopen decreasing $(\leqslant)$ set p disjoint from Z and containing y, $p \cap X_1$ is decreasing $(\leqslant')$ and contains y but not x.

Let $d'(a)$ be the smallest decreasing $(\leqslant')$ set containing the clopen increasing $(\leqslant')$ set a. Then $a \cap X_1$ is increasing $(\leqslant)$. Take $x \leqslant' y \in a$ and suppose first that $x, y \in X_1$, $x \leqslant y$. Since X_1 is increasing $(\leqslant)$,

$$x \in d(a) \cap X_1 \subseteq d'(a) .$$

Secondly, if $x \in X_2$, $y \in Z$, $x \leqslant y$,

$$x \in d(a) \cap X_2 \subseteq d'(a)$$

(to obtain this inclusion, note that if $z \in d(a) \cap X_2$, $z \leqslant z_1 \in a \cap Z$, because a is increasing). Thus $d'(a)$ is open, since $d(a)$ is open. It follows from Proposition 3 that $(X, \mathfrak{J}, \leqslant')$ is an SLD-space, since each point in X is majorized in the order $\leqslant'$ by a unique maximal point.

The identity map from $(X, \mathfrak{J}, \leqslant')$ onto $(X, \mathfrak{J}, \leqslant)$ is an SLD-space morphism. Projectivity of X implies that $\leqslant'$ and $\leqslant$ coincide. But this is false since $y_1 \geqslant y_2$, $y_1 \not\geqslant' y_2$. Therefore Y has the discrete order.

Let $\hat{\mathfrak{J}}$ denote the discrete topology on X. The identity map from $(X, \hat{\mathfrak{J}})$ to $(X, \mathfrak{J})$ extends to a continuous map α from βX (the Stone-Czech compactification of $(X, \hat{\mathfrak{J}})$) to $(X, \mathfrak{J})$. βY and $\beta\big(Z \setminus m(Y)\big)$ may be regarded as totally disconnected subspaces of βX. Let $\hat{X} = X_1 \oplus X_2 \oplus X_3$, where $X_1 = X_2 = \beta X$ and $X_3 = \beta\big(Z \setminus m(Y)\big)$. We can construct a continuous map $\hat{\alpha}$ from $\hat{X}$ onto $(X, \mathfrak{J})$ by stipulating

$$\hat{\alpha}|X_1 = \alpha|\beta Y, \qquad \hat{\alpha}|X_3 = \alpha|\beta\big(Z \setminus m(Y)\big), \qquad \hat{\alpha}|X_2 = \pi(\alpha|\beta Y),$$

where $\pi\colon X \to Z$ is the projection map. $\hat{X}$ may be ordered by defining $x \leqslant y$ if and only if $x = y$ in $\hat{X}$ or $x \in X_1$, $y \in X_2$ and $x = y$ in βY. $\hat{X}$ is then an SLD-space with dual triple $(X_1, X_2 \oplus X_3, i)$, with $i(x) = x$ for all $x = X_1$. Further, $\hat{\alpha}$ is easily seen to be an SLD-space morphism. By projectivity of X, the identity map from X into $\hat{X}$ is an SLD-space morphism. We conclude that $Y, Z \setminus m(Y)$ are compact and that each point in $m(Y)$ majorizes precisely one point in Y. Thus the dual triple of X is of the form $(Y_1, Y_0 \oplus Y_1, j)$, with Y_0, Y_1 Boolean spaces and j the natural embedding. That Y_0, Y_1 are extremally disconnected is clear since X is, a priori, projective in the category of Boolean spaces ([17]; cf. also [20]).

COROLLARY 20 ([2], Theorem 2). *A Stone algebra is injective if and only if it can be represented in the form* $B_0 \times B_1^{(2)}$, *where* B_0, B_1 *are complete Boolean algebras.*

Proof. Necessity follows from Theorem 19 since the dual triple for $B_0 \times B_1^{(2)}$ is of the form $(Y_1, Y_0 \oplus Y_1, j)$ (see the discussion of $B^{(2)}$ in § 5). For sufficiency we refer to Lakser's paper [14], remarking that a dual version of Lakser's proof can easily be constructed to prove directly the converse of Theorem 19.

References

[1] G. Ja. Areškin, *Free distributive lattices and free bicompact T_0-spaces* (Russian), Dokl. Akad. Nauk SSSR. N. S. 90 (1953), pp. 485–486.

[2] R. Balbes and G. Grätzer, *Injective and projective Stone algebras*, Duke Math. J. 38 (1971), pp. 339–347.

[3] — and A. Horn, *Stone lattices*, Duke Math. J. 37 (1970), pp. 537–546.

[4] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Publ. 25, 3rd edn, Providence, R. I. 1967.

[5] C. C. Chen and G. Grätzer, *Stone lattices I. Construction theorems*, Canad. J. Math. 21 (1969), pp. 884–894.

[6] — *Stone lattices II. Structure theorems*, Canad. J. Math. 21 (1969), pp. 895–903.

[7] P. D. Finch, *On the structure of Stone lattices*, Bull. Austral. Math. Soc. 2 (1970), pp. 401–413.

[8] O. Frink, *Pseudo-complements in semi-lattices*, Duke Math. J. 29 (1962), pp. 505–514.

[9] G. Grätzer, *Lattice Theory*, Vol. I, San Francisco, California 1971.

[10] — and H. Lakser, *Some applications of free distributive products*, Notices Amer. Math. Soc. 16 (1969), p. 405.

[11] — and E. T. Schmidt, *On a problem of M. H. Stone*, Acta Math. Acad. Sci. Hungar. 8 (1957), pp. 455–460.

[12] P. Halmos, *Boolean algebras*, Mathematical Studies 1, Princeton, N. J. 1963.

[13] T. Katriňák, *Notes on Stone lattices I* (Russian), Mat. Casopis Sloven. Akad. Vied. 16 (1966), pp. 128–142.

[14] H. Lakser, *Injective hulls of Stone algebras*, Proc. Amer. Math. Soc. 24 (1970), pp. 524–529.

[15] T. McCallion, *Compactifications of ordered topological spaces*, Proc. Camb. Phil. Soc. 71 (1972), pp. 463–473.

[16] S. D. McCartan, *Separation axioms for topological ordered spaces*, Proc. Camb. Phil. Soc. 64 (1968), pp. 965–973.

[17] W. C. Nemitz, *Implicative semi-lattices*, Trans. Amer. Math. Soc. 114 (1965), pp. 128–142.

[18] A. Nerode, *Some Stone spaces and recursion theory*, Duke Math. J. 26 (1959), pp. 397–406.

[19] H. A. Priestley, *Representation of distributive lattices by means of ordered Stone spaces*, Bull. London Math. Soc. 2 (1970), pp. 186–190.

[20] — *Ordered topological spaces and the representation of distributive lattices*, Proc. London Math. Soc. (3) 24 (1972), pp. 507–530.

[21] T. P. Speed, *Some remarks on a class of distributive lattices*, J. Austral. Math. Soc. 9 (1969), pp. 289–296.

[22] J. Varlet, *On the characterization of Stone lattices*, Acta Sci. Math. (Szeged) 27 (1966), pp. 81–84.

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