Stone lattices: a topological approach

by

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Abstract. A (0,1)-distributive lattice L can be represented as the lattice of clopen increasing subsets of an appropriate ordered topological space X. It is shown that when L is a Stone lattice, the dual space X is characterized by subspaces Y(X), Z(X) of X and a continuous increasing map m(X): Y(X) -> Z(X). This enables the structure of Stone lattice dual spaces to be analyzed in terms of simpler components and leads to a construction theorem for dual spaces which is in the same spirit as, but not directly dual to, Chen and Grätzer’s triple construction theorem for Stone lattices.

1. Introduction. Chen and Grätzer show in [2] that a Stone lattice L can be studied by investigating an associated triple of simpler components, ([L], D(L), Φ(L)), where [L], D(L) are appropriate subsets of L and Φ(L) is a connecting map. In this paper the duality between (0,1)-distributive lattices and compact totally order disconnected spaces developed in [19] and [20] is applied to Stone lattices and the dual space X of a Stone lattice L is shown to be characterized by subspaces Y(X), Z(X) of X and a continuous increasing map m(X): Y(X) -> Z(X). The ordered spaces Y(X), Z(X) are the duals of the lattices D(L), C(L); m(X) and Φ(L) are related, but are not mutually dual maps.

The Construction Theorem in [5] asserts that, given a suitably defined triple (Ω, D, Φ), there exists a Stone lattice L with C(L) = Ω, D(L) = D, Φ(L) = Φ. Problem 55 of [9] seeks a less computational proof of this theorem than that given in [5]. Motivated by this problem, we show how to construct a space dual to a Stone lattice from a "dual triple" (Y, Z, m) and hence obtain a new method of constructing Stone lattices from simpler components.

Dual triples also provide new information on free Stone algebras and a short proof of Theorem 2 of [2], characterizing injectives.

2. The dual space of a Stone lattice. We refer to [19], [20] for the ordered topological space concepts needed, recalling only two crucial definitions concerning a set X endowed with a partial order ≤ and a topology τ. A subset E of X is decreasing (increasing) if x ≤ y ⇒ E(x) → E(y) implies...
For the lattice theoretic terminology used without explanation, [9], or [4], may be consulted. Let \( L \) be a \((0,1)\)-distributive lattice. The main theorems of [19] allow \( L \) to be identified with the lattice of clopen increasing subsets of a compact t.o.d. space \((X, \mathcal{J}, \ll)\) (the dual space of \(L\)); we shall assume this identification to have been made. Henceforth we adopt a compromise between the incompatible notations of [9] and of [19], [20], and use set theoretic symbols for lattice operations in \( L \) and lower case roman letters \( a, b, \ldots \) to denote subsets of \( X \), reserving \( x, y, z \) for points of \( X \). \( cl(a) \) (or, where desirable, \( cl_{U}(a) \)) denotes the closure of \( a \) in the \( 3 \)-topology of an ordered space \((X, \mathcal{J}, \ll)\). The closure of \( a \) in the upper topology \( \mathcal{U} \), on \( X \) (consisting of the \( \mathcal{U} \)-open increasing sets) is written \( \mathcal{U}cl(a) \) or \( \mathcal{U}cl_{U}(a) \).

A \((0,1)\)-distributive lattice \( L \) is said to be pseudocomplemented if, for each element \( a \in L \), there exists \( a^* \in L \) such that

\[
a \wedge b = 0 \quad \text{if and only if} \quad b \subseteq a^*.
\]

A Stone lattice \( L \) is a \((0,1)\)-distributive lattice which is pseudocomplemented and such that, for all \( a \in L \), \( a^* \wedge a^* = 1 \).

**Proposition 1.** A \((0,1)\)-distributive lattice \( L \) is pseudocomplemented if and only if, for each clopen increasing subset \( A \) in the dual space \( X \) of \( L \),

\[
d(\mathcal{U}cl(a), x \ll y) = 0 \quad \text{for some} \quad y \in a.
\]

**Proof.** Let \( a \in L \) and suppose that \( d(a) \) is open. Then \( d(a) \) is disjoint from \( a \) and is clopen increasing \( d(a) \) is closed since \( a \) is closed and \( X \) is compact t.o.d. Also, if \( b \) is increasing, \( a \wedge b = 0 \) implies that \( d(a) \wedge d(b) = 0 \). It follows that \( a \) has a pseudocomplement \( d(a) \) in \( L \).

Conversely suppose that each \( a \in L \) has a pseudocomplement \( a^* \) in \( L \). To show that \( d(a) \) is open we show that \( d(a) = 1 \). \( a \wedge a^* = 0 \) and \( a^* \) is increasing, so \( a^* \subseteq d(a) \). If \( X \) does not belong to the closed decreasing set \( d(a) \), then total order disconnectedness of \( X \) provides a clopen increasing set \( b \) such that \( x \ll b \subseteq d(a) \). \( a \wedge b = 0 \), so \( b \subseteq a^* \).

Conversely suppose that each \( a \in L \) has a pseudocomplement \( a^* \) in \( L \). A compact t.o.d. space \((X, \mathcal{J}, \ll)\) will be called an SLD-space if \( a \in \mathcal{U} \) implies \( d(a) \in \mathcal{U} \).

**Proposition 2.** \( X \) is an SLD-space if and only if \( X \) is the dual space of a Stone lattice.

**Proof.** Suppose that the lattice \( L \) of clopen increasing subsets of a compact t.o.d. space \( X \) is a Stone lattice. \( L \) is then pseudocomplemented and, for all \( a \in L \), \( a^* \wedge a^* = 1 \). The last condition gives \( d(a) = a^* \), which is open increasing. Any open increasing set is the union of its clopen increasing subsets and the operator \( d \) preserves unions. Thus \( d(a) \in \mathcal{U} \) whenever \( a \in \mathcal{U} \).

For the converse, note that, if \( a \in L \) implies that \( d(a) \in \mathcal{U} \), then \( a^* = \{ d(\{ d(a) \}) \} = \{ d(a) \} \).

**Proposition 3.** Let \( X \) be the dual space of a pseudocomplemented distributive lattice and let \( \mathcal{M}_X \) denote the set of points in \( X \) which are maximal with respect to the partial order on \( X \). Then the following statements are equivalent:

(i) \( X \) is an SLD-space;

(ii) for each \( x \in X \) there exists a unique \( m(x) \in \mathcal{M}_X \) with \( x \ll m(x) \);

(iii) for any increasing sets \( p, q \subseteq X \), \( d(p \cap q) = d(p) \wedge d(q) \).

**Proof.** (i) \( \Rightarrow \) (ii): A Zorn's lemma argument shows that each \( x \in X \) is majorized by at least one element in \( \mathcal{M}_X \). Suppose \( x \ll y, z \), where \( y, z \in \mathcal{M}_X \), \( y \ll z \). There exist clopen increasing sets \( a, b \) with \( y \ll a \ll \wedge \tau b \), \( x \ll b \ll \tau a \), \( x \in d(a) \vee d(b) \), which is an increasing set by (i). Hence \( y \in d(a) \vee d(b) \), and so \( y \) is maximal, \( y \ll \tau a \), a contradiction.

(ii) \( \Rightarrow \) (iii): Let \( p \) and \( q \) be increasing, and let \( x \in d(p) \vee d(q) \). (iii) holds trivially if \( d(p), d(q) \) are disjoint. \( x \) is majorized by maximal points \( y \in p, z \in q \). (ii) forces \( y = z = m(x) \), so \( x \in d(p \vee q) \). Trivially \( d(p \vee q) = d(p) \vee d(q) \).

(iii) \( \Rightarrow \) (i): We show that \( d(p) \) is increasing for each increasing set \( p \).

Let \( x \in d(p) \), \( y \in p \). Then \( x \ll y \ll z \in \mathcal{M}_X \). Also \( x \ll z \), \( x \in d(p) \), \( x \in d(z) = d(\{ z \}) \), by (iii). Hence \( z \) is maximal, \( y \ll \tau z \).

The equivalence (i) \( \Rightarrow \) (ii) is related to the characterization of Stone lattices as those pseudocomplemented distributive lattices in which, for all lattice elements \( a, b, a^* \cup b^* = (a \wedge b)^* \) (see [8]). (i) \( \Rightarrow \) (iii) is the dual of the theorem of Varlet [32] stating that a pseudocomplemented distributive lattice is a Stone lattice if and only if every prime ideal contains a unique minimal prime ideal. An equivalent form of this theorem was earlier obtained by Grätzer and Schmidt [11]: a pseudocomplemented distributive lattice is a Stone lattice if and only if the join or any two distinct minimal prime ideals in \( L \) is \( L \). That the restriction to pseudocomplemented lattices is necessary was shown by an example by Katrňák [13]. Examples of a different type can be constructed as follows.

Let \( X \) be the dual of any \((0,1)\)-distributive lattice which is not pseudocomplemented. Then there is a clopen increasing subset \( a \in X \) such that \( d(a) \) is not open. Let \( X \) be the \( (0,1) \)-distributive lattice which is not pseudocomplemented. Then there is a clopen increasing subset \( a \in X \) such that \( d(a) \) is not open.
by requiring that the given order relations be induced on $X_1$ and $X_2$ and that

\[ y \succ x \in X_1 \text{ if and only if } x \in a, \]
\[ z \succ x \in X_2 \text{ if and only if } x \in a. \]

$X$ is compact and t.o.d. and satisfies condition (ii) of Proposition 3. The set $a \cup \{y\}$ is clopen increasing in $X$ and the smallest decreasing set in $X$ containing it is $d(a) \cup \{y\}$, which is not open in $X$. Hence the lattice of clopen increasing subsets of $X$ is not pseudocomplemented, but is such that any prime ideal contains a unique minimal prime ideal.

**Proposition 4.** In an SLD-space $X$, $X_m$ is 3-closed and the map

\[ m: (X, \cup, \cap) \rightarrow (X_m, \cup, \cap), \]

assigning to each point $x \in X$ the unique maximal point majorizing $a$, is continuous.

**Proof.** Let $a = a_i \cap X_m$, where $a_i$ is open increasing in $X$.

\[ m^{-1}(a) = d(a_i \cap X_m) = d(a_i) \cap d(X_m) = d(a), \]

which is 3-open. Hence $m$ is continuous when $X$ has the topology $\cup$, $X_m$ the related $\cup$-topology. It follows that $X_m$ is $\cup$-compact. Take $a \in X_m$.

For each $y \in X_m$, choose a clopen increasing set $a_y$ with $y \in a_y$, $a \in a_y$. $X_m$ can be covered by a finite number of the sets $a_y$. The intersection of the complements of these sets is a 3-open neighbourhood of $a$ disjoint from $X_m$, so $X_m$ is 3-closed.

Finally, if $b$ is closed in $X_m$ (in the relative 3-topology), then $b$ is 3-closed in $X$, $m^{-1}(b) = d(b)$, which is closed.

We remark that compactness of $X_m$ (in either $\cup$- or $\cup$-topology) is implicitly obtained by Speed in [21].

A Stone algebra is a Stone lattice in which 0, 1 are regarded as nullary operations and $\ast$ as a unary operation. Morphisms in the category of Stone algebras are consequently lattice homomorphisms which preserve these operations. If $L, L'$ are $\{0, 1\}$-distributive lattices with dual spaces $X, X'$, there is ([18], [20]) a one-to-one correspondence between $\{0, 1\}$-preserving lattice homomorphisms $\Phi: L \rightarrow L'$ and continuous increasing maps $f: X \rightarrow X'$ such that $f^{-1}(a) = \Phi(a)$ for all $a \in L$.

**Proposition 5.** If $X, X'$ are SLD-spaces and $f: X \rightarrow X'$ is continuous and increasing, then the dual map $\Phi: L \rightarrow L'$ is a Stone algebra morphism if and only if $f$ maps $X_m$ into $(X_m)'$.

**Proof.** For $a \in L$,

\[ \Phi(a') = \Phi(a)^\ast \text{ if and only if } f^{-1}(d(a)) = d(\Phi(a)). \]

If $x \preceq y \in \Phi(a)$, then $f(x) \preceq f(y) \in a$, and so $d(\Phi(a)) \subseteq f^{-1}(d(a))$ always. Suppose now that $f$ preserves maximality, so that, for all $x \in X$, $f(m(x))$ is the unique maximal point majorizing $f(x)$ in $X'$. This implies that, if $x \preceq f^{-1}(d(a))$, $m(x) \preceq f^{-1}(d(a)) = \Phi(a)$. Thus $x \preceq d(\Phi(a))$, so $\Phi$ preserves $\ast$.

Conversely, let $x \in X_m$. Then

\[ f(x) \preceq a \iff x \in \Phi(a) \iff x \preceq d(\Phi(a)) \iff f(x) \preceq d(a). \]

If $f(x)$ were non-maximal and majorized by $y \preceq f(x)$, there would exist a clopen increasing set $a$ such that $f(x) \in a$, $y \in a$. But then $f(x) \in d(a)$, which is impossible.

3. The dual triple associated with a Stone lattice. Two subsets of a Stone lattice $L$ are of special importance: the centre of $L$,

\[ C(L) = \{a^* : a \in L\}, \]

and the dense set,

\[ D(L) = \{a : a^* = 0\}. \]

Any element $a$ of $L$ may be represented as

\[ a = a^* \cap (a \cup a^*). \]

$a^{**} \in C(L)$ and $a \cup a^* \in D(L)$. Thus if $C(L)$ and $D(L)$ are given and the relation between $C(L)$ and $D(L)$ is known, then $L$ is completely determined. The map used by Chen and Grätzer in [5] to link $C(L)$ and $D(L)$ is $\Phi(L)$, which is the $(0, 1)$-homomorphism mapping $C(L)$ into the lattice of filters of $D(L)$ defined by

\[ (\Phi(L))_a = \{b \in D(L) : b \supseteq a^*\}. \]

Chen and Grätzer show in §3 of [5] that $L$ is determined up to isomorphism by its triple $(C(L), D(L), \Phi(L))$ (or see [9], p. 163). One would therefore expect the dual space $X$ of $L$ to be determined by the dual spaces $X(L)$, $Z(X)$ of $D(L)$, $C(L)$ and an appropriate connecting map $m(X)$.

$a^*$ is given by $a^* = \mathcal{d}(a)$ in $X$. It follows that $C(L)$ is the Boolean algebra of those clopen subsets of the dual space $X$ which are atomic (i.e. simultaneously increasing and decreasing), while $D(L)$ consists of those clopen increasing subsets of $X$ which contain $X_m$. $C(L)$ and $D(L)$ are isomorphic and appropriate connecting spaces of $X(L)$.

We now identify the dual of $C(L)$. This is (homeomorphic to the) the isometrically ordered quotient space $X_R$ where $a \preceq y$ if and only if, for every clopen atomic set $a, x \subseteq a$ if and only if $y \subseteq a$. If $m(a) = m(y)$, then any atomic set containing $a$ contains $y$, and vice versa, so $a \preceq y$. On the other hand, if $m(a) \neq m(y)$, then a clopen increasing set $a$ can be found with $m(a) \in a$, $m(y) \notin a$. $d(a)$ is clopen atomic and $x \in d(a)$. 


Thus $x, y$ belong to different $E$-equivalence classes. We conclude that the elements of $X / E$ are the sets $d(z)$, where $z \in X_\wedge$.

**Proposition 6.** The dual space of $O(L)$ is homeomorphic to $X_\wedge$ (with the relative 3-topology).

**Proof.** The map $g : X_\wedge \to X / E$ defined by $g(z) = d(z)$ is one-to-one and onto. A set of equivalence classes $d(z)$ where $z \in X_\wedge$ is open in $X / E$ if and only if $d(z)$ is open in $X$. When $d(z)$ is open, $z = g(d(z)) = X_\wedge$ is open in $X_\wedge$ and $g$ is continuous. Since $X_\wedge$ is compact and $X / E$ Hausdorff, $g$ is a homeomorphism.

Next we turn our attention to $D(L)$. An equivalence relation $E'$ can be defined on $X$ by letting $xE'y$ if and only if either $x = y$ or $x, y$ both belong to $X_\wedge$. $X / E'$ is a compact t.o.d. space under the quotient topology and quotient order (cf. [20], Proposition 14 (ii)), and it is clear that the lattice of clopen increasing subsets of $X / E'$ is isomorphic to $D(L)$. Hence the dual space of $D(L)$ can be identified with $X / E'$.

$X / E'$ is homeomorphic and order isomorphic to the space obtained by taking the one-point compactification of $X_\wedge$ and ordering it by requiring that the given order be induced on $X_\wedge$ and that the adjoined point majorize every point of $X_\wedge$. (See [20], p. 510 and also [16], [16].) That the dual of $D(L)$ takes the form of a one-point order compactification reflects the fact that $D(L)$ is obtained from $D(L)$ by the adjunction of $0$. Since the prime ideals of $D(L)$ are the prime ideals of $D(L)$ with $0$ added to each, together with the zero ideal, $X_\wedge$ is the dual space of $D(L)$, according to the definition in [20], § 10. We sum up these results in

**Proposition 7.** The dual space of $D(L)$ is the ordered quotient space obtained from $X$ by identifying the points of $X_\wedge$ or, equivalently, is a one-point order compactification of $X_\wedge$. The dual space of $D(L)$ is $X_\wedge$.

We now define the dual triple associated with an SLD-space $X$ to be $(X, Z(X), m(X))$, where $X = X \setminus X_\wedge, Z(X) = X_\wedge$ (both with relative topology and order from $X$) and $m(X) : Y(X) \to Z(X)$ is the continuous increasing map defined by the restriction of $m$ to $X_\wedge$. We shall denote by $\tilde{Y}(X)$ the one-point order compactification of $Y(X)$ in which $X_\wedge$ is adjoined to $Y(X)$ as a universal maximal point.

Before describing, in Theorem 9, how $X$ is determined by its dual triple, we explain the relation between $m(X)$ and Chen and Grätzer's structure map $\Phi(L)$. Recall that $\Phi(L)$ is defined by

$$\Phi(L)(a) = \{ b \in D(L) : b \geq a \}.$$

$\Phi(L)(a)$ is a proper filter in $D(L)$ and may be identified with the non-empty closed increasing subset $F(a)$ of $Y(X)$ obtained by taking the intersection of the sets in $\Phi(L)(a)$ (regarded as clopen increasing subsets of $Y(X)$) (cf. [12], § 19, [19], [20]). Reverting to $D(L)$ and $Y(X)$,

we see that $\Phi(L)(a)$ may be identified with the closed increasing subset of $Y(X)$ obtained by deleting the point $X_\wedge$ from $F(a)$. With this interpretation of $\Phi(L)(a)$ we have

**Proposition 8.**

$$\Phi(L)(a) = (m(X))^{-1}(a)$$

for each set $a$ in $Z(X)$.

$$(m(X))^{-1}(a) = \{ x \in Z(X) : a \leq m(X) \}$$

for all $x \in Y(X)$.

(a' denotes the Boolean complement of $a$ in $C(L)$.)

**Proof.** The first assertion is an immediate consequence of the definitions; the second can be deduced using compactness and total disconnectedness of $Z(X)$.

**Theorem 9 (Dual structure theorem).** If $X$ is an SLD-space with dual triple $(Y(X), Z(X), m(X))$, then $X$ is homeomorphic and order isomorphic to the increasing subspace

$$\approx = m^{-1}(X_\wedge) \cup \{(Z(X)) \times Z(X)\}$$

of $Y(X) \times Z(X)$.

($m$ denotes the graph of $m(X)$.)

**Proof.** A map $g$ from $X$ onto $\approx$ is defined by

$$g(z) = \begin{cases} (x, m(z)), & x \in Y(X) = X_\wedge \setminus X_\wedge, \\ (X_\wedge, z), & x \in Z(X) = X_\wedge. \end{cases}$$

$Y(X)$ and $Z(X)$ can be regarded as ordered quotients of $X$; $g$ is then the product of the associated quotient maps and so is continuous.

The proof is completed by showing that $g$ is an order isomorphism. If $x_1 < x_2$ in $X$, $m(x_2) = m(x_1)$. It follows that $g(a_1) < g(a_2)$. Now let $g(a_2) = (y_i, z_i), i = 1, 2, z_i = m(z_i)$; $y_i = z_i$ if $z_i \notin X_\wedge$. Otherwise suppose that $g(a_1) < g(a_2)$. This implies that $y_1 < y_2$ and $m(z_1) = m(z_2)$. If $x_1 < x_2$ in $X$, $x_1 < z_2$ trivially and if $x_1 < x_2$, $x_1 < z_2$, $x_1 < z_1$. Again $x_2 \notin X_\wedge$ is impossible, while $x_1 < X_\wedge, x_1 \notin X_\wedge$ gives $x_1 < m(z_1) = m(z_2) = z_2$.

Henceforth, if $X$ and $L$ are dual, with $X$ an SLD-space and $L$ a Stone lattice, we shall refer to $(Y(X), Z(X), m(X))$ and $(C(L), D(L), \Phi(L))$ as being dual to one another and to each as being associated with $X$ and with $L$. In this situation, it can be inferred from [9], Theorem 12.2, that the dual space of the free $(0, 1)$-directed product of $D(L)$ and $C(L)$ is the ordered cartesian product of $Y(X)$ and $Z(X)$. The formation of a closed subspace of a dual space corresponds to the formation of the
quotient by a congruence relation of the corresponding lattice. Theorem 9 may then be interpreted as asserting that

\[ L \cong (D(L) \times C(L))/\theta, \]

where \( \theta \) is a congruence (cf. [17] and [9], p. 193).

In § 4 of [5], Chen and Grätzer abstract the idea of a triple associated with a Stone lattice. They define a triple \((Z, D, \Phi)\) to consist of (i) a Boolean algebra \( Z \), (ii) a distributive lattice \( D \) with \( 1 \), (iii) a \((0, 1)\)-homeomorphism \( \Phi \) from \( Z \) into the lattice of filters of \( D \), and they show how a Stone lattice \( L \) can be built out of this triple. \( L \) is defined by

\[ L = ([0, 1]; c \in C, b \in \Phi(c)). \]

The difficult part of Chen and Grätzer's Construction Theorem is the proof that \( L \) can be partially ordered so as to form a distributive lattice. This difficulty can be by-passed if one constructs initially not a Stone lattice but an SLD-space. The collection of clopen increasing subsets of this space then forms automatically a Stone lattice.

4. The dual construction theorem. An abstract dual triple \((Y, Z, m)\) consists of

(i) an ordered space \((Y, \mathcal{Y}, \leq)\) in which the topology of open decreasing sets has a \( \mathcal{Y}\)-compact base;

(ii) a non-empty compact totally disconnected space \((Z, \mathcal{Z})\);

(iii) an increasing map \( m: Y \to Z \), continuous with respect to the topologies \( \mathcal{Y}, \mathcal{Z}\).

The condition in (i) allows us to form by one-point compactification a compact t.o.d. space \( Y \), where \( Y = Y \cup \{Z\} \) and \( Z \geq y \) for all \( y \in Y \). Guided by Theorem 9, we prove

**Theorem 10** (Dual construction theorem). Let \((Y, Z, m)\) be an abstract dual triple. Then there exists an SLD-space \( Y \) such that, up to homeomorphism and order isomorphism, \( Y = Y(X), Z = Z(X) \) and \( m = m(X)\).

**Proof.** Take

\[ X = G_m \cup ([Z] \times Z) \subseteq Y \times Z \]

If \((y, z) \in (Y \times Z)\setminus X\), then \( y \neq Z \) and \( z \neq m(y) \). Take an open subset \( a \in Z \) containing \( m(y) \) but not \( z \), \( z \prec m(y) \) is open in \( Y \) and so also in \( Z \). Consequently \( m^{-1}(a) \) is an open neighbourhood of \((y, z)\), disjoint from \( X \) is thus a closed subset of the compact t.o.d. space \( Y \times Z \), and so is itself compact and t.o.d.

Let \( a \) be a non-empty increasing subset of \( X \):

\[ a = \{y, m(y); y \in b \subseteq Y\} \cup ([Z] \times z); z \in c \subseteq Z\],

where \( m(b) \subseteq c \) since \( y \leq Z \) for all \( y \in X \). Then a non-empty implies \( c \) non-empty, and we conclude that, in \( X \), \( d(a) = (Y \times c) \cap X \), which is atomic. Suppose in addition that \( a \) is open in \( X \) and that \( z \in c \). It is possible to find \( p \) open in \( Y \), \( q \) open in \( Z \) with

\[ (Z, z) \in (y \times q) \cap X \subseteq a, \]

\( z \in q \subseteq a \), which shows that \( c \) is open in \( Z \). Then it follows that \( d(a) \) is open increasing in \( Y \) whenever \( a \) is open increasing in \( X \), so that \( X \) is an SLD-space.

We now identify the triple \((Y(X), Z(X), m(X)), (y, z) \leq (y', z') \in Y \times Z \) if and only if \( y \leq y' \) and \( z \leq z' \). Thus \( Xm = ([Z] \times Z); z \in Z \).

Clearly \( Z \) is homeomorphic to \( Xm = Z(X) \) under the map \( z \mapsto (x, z) \).

Further, the map \( y \mapsto (y, m(y)) \) is a homeomorphism from \( Y \) onto \( Xm = Y(X); \tau \) is also an order isomorphism by condition (iii) above. Finally,

\[ (m(X))(y, m(y)) = (Z, m(y)), \]

i.e.

\[ (m(X))(\tau(y)) = m(y). \]

A morphism from a triple \((Y, Z, m)\) to a triple \((Y_1, Z_1, m_1)\) is a pair \((f, g)\) where \( f \) is a continuous increasing map from \( Y \) into \( Y_1 \), \( g \) is a continuous map from \( Z \) into \( Z_1 \) and the diagram

\[ \begin{array}{ccc}
Y & \xrightarrow{m} & Z \\
| & | & | \\
| & \downarrow{f} & | \\
Y_1 & \rightarrow & Z_1 \\
\end{array} \]

commutes. Analogous to Theorem 1 of [5] we have Proposition 11. We omit the routine proof.

**Proposition 11.** Let \( X, X_1 \) be SLD-spaces with dual triples \((Y, Z, m), (Y_1, Z_1, m_1)\). Then if \((f, g)\) is a triple morphism, the map

\[ f \times g: Y \times Z \rightarrow Y_1 \times Z_1 \]

induces by restriction an SLD-space morphism \( h: X \rightarrow X_1 \). Conversely, an SLD-space morphism \( h: X \rightarrow X_1 \) induces maps \( f: Y \rightarrow Y_1, g: Z \rightarrow Z_1 \) (regarding \( Y, Z \) as quotient spaces of \( X, X_1, Z_1 \) as quotient spaces of \( X_1 \)). The resulting pair \((f, g)\) is a triple morphism. The correspondence so established between SLD-space morphisms and triple morphisms is one-to-one and onto.

If \( h \) is onto and only if each of \( f, g \) is onto. If \( f(y) = Z_1 \) implies \( y = Z \), then \( h \) is an order isomorphism if and only if \( f \) is an order isomorphism and \( g \) is one-to-one.

We remark that dual versions of the "fill-in" theorems in [6], § 6, can be obtained by straightforward diagram-chasing.
Given a triple $(G, D, \Phi)$ associated with a Stone lattice $L$, the reduced triple associated with $L$ is $(G_r, D, \Phi_r)$, where $G_r$ is the quotient of $G$ by $\theta$, the congruence determined by $\Phi$, and $\Phi_r([a][b]) = \Phi(a)$ for all $a \in G ([6], \S 5$; see also [7]). Dually, given a dual triple $(Y, Z, m)$ with associated SLD-space $X$, we define the reduced dual triple to be $(X, Z, m)$ where $Z = cl_m(Y)$.

**Proposition 12.** If $(G, D, \Phi)$ and $(Y, Z, m)$ are dual to one another, so too are the reduced triples $(G_r, D, \Phi_r)$ and $(X, Z, m)$.

**Proof.** It is enough to show that $G_r$ and $Z$ are mutually dual. For $a, b$ clopen in $Z$,

$$a \sim b \Leftrightarrow \Phi(a) = \Phi(b) 
\Leftrightarrow m^{-1}(a) = m^{-1}(b) 
\Leftrightarrow a' \cap m(Y) = b' \cap m(Y) 
\Leftrightarrow a \cap m(Y) = b \cap m(Y)$$

(dashes denote complements in $Z$). Since $a$ is open,

$$a \cap c[X(m(Y)] \subseteq c[a \cap m(Y)],$$

and because $a$ is closed,

$$c[a \cap m(Y)] \subseteq a \cap c[Y(m(Y))].$$

It follows that $a \sim b$ if and only if $a \cap c[Y(m(Y))] = b \cap c[Y(m(Y))]$. We deduce that $G_r = G/\theta$ is isomorphic to the Boolean algebra of sets $a \cap Z_r$, where $a$ is clopen in $Z$ and $Z_r = c[Y(m(Y))]$. An application of Lemma 17.1 of [12] completes the proof.

5. A description of $L$ in terms of $(Y, Z, m)$. Proposition 2 and Theorem 10 together show that Stone lattices may be built from dual triples; the lattice $L$ of clopen increasing subsets of $X = G_m \cup (Z \times Z)$ is a Stone lattice for any abstract dual triple $(Y, Z, m)$.

Theorem 14 characterizes order and topological properties of subsets of $X$. It provides an explicit description of $L$ in terms of its dual triple; this is used in Theorem 16, which concerns completeness of $L$. In addition Theorem 14 yields an explicit, though unwieldy, description of the ideals and filters of $L$ (respectively open increasing, closed increasing, subsets of $X$) and of the minimal Boolean extension of $L$ (clopen subsets of $X$).

A typical subset of $X$ is

$$a = [y, m(y)] = \{z \in X \cap m(Y) : z \in c \subseteq Z\};$$

more concisely, $a = G_m \cup (Z \setminus Z)$. For each $b \subseteq Y$, we define

$$\delta = \{z \in Z : z \subseteq \lambda \in b\}$$

for some net $\lambda \subseteq b \in Z$ in $Y$.

$\delta = \emptyset$ whenever $b \subseteq \emptyset$.

Proposition 13. For each $b \subseteq Y$, $\delta$ is closed in $Z$.

**Proof.** Let $e \in \partial b$ and let $p$ be any open neighborhood of $Z$ in $Y$, $q$ any open neighborhood of $z$ in $Z$. There exists a point $z_p, a \in b \cup p$ such that $m(y_{a,b}) \in q$ $(y_{a,b})$ forms a net in $Y$ when the indices are directed by pairwise set inclusion. By compactness of $Y$, we can find a convergent subnet $(y_{a,\alpha})$ of $(y_{a,b})$. Necessarily $y_{a,\alpha} \to Z$ and $m(y_{a,\alpha}) \to z$.

**Theorem 14.** Let $a = G_m \cup (Z \times c)$.

(i) $a$ is increasing if and only if $b$ is increasing in $(Y, Z, m)$ and $m(b) \subseteq c$;

(ii) $a$ is decreasing if and only if $b$ is decreasing in $(Y, Z, m)$ and $m^{-1}(c) \subseteq b$;

(iii) $cl_a(b) = [y, m(y)] = \{z \in \partial b \cap cl_{y_a}(c)\}$;

(iv) $a$ is open if and only if $b$ is open in $(Y, Z, m)$ and $\Delta(\emptyset, \emptyset) \subseteq c$;

(v) $b \subseteq c$ whenever $c$ is closed and $m(b) \subseteq c$; $(\Delta(\emptyset, \emptyset) \subseteq c$ whenever $c$ is open and $m^{-1}(c) \subseteq b$.

**Proof.** (i) and (ii) are elementary. To prove (iii) let

$$cl_a(b) = G_m \cup (Z \times c).$$

If $z \in cl_a(b)$, $(y, m(y)) = cl_z(a)$, by continuity of $m$. Hence $cl_a(b) \subseteq c$. On the other hand, if $x = [y, m(y)] \in cl{(a)}$ then it is possible to choose a net $[y_{a,b}(m(y))]$ of points in $a$ converging to $x$. Hence $y \in cl_{a}(b)$.

It is clear that each point $(Z, c)$ with $x \in cl_{a}(c)$ is in $cl_{a}(c)$ and that, by definition of $\delta$, so too is each point $(Z, c)$ for which $x \in b$. Now take $x = (Z, c)$; $cl_{a}(c)$ and $(x_{a,b})$ a net of points in $a$ with $x_{a,b} \to x$. We may write $A = A_1 \cup A_2$, where $x_{a,b} \in A$ for $x \in A_1$, $x_{a,b} \in (Z \times Z)$ for $x \in A_2$. At least one of $A_1, A_2$ is cofinal. If $A_1$ is cofinal, $x \in b$; if $A_2$ is coinitial, $x \in cl_{a}(c)$.

We conclude that $c = b \cup cl_{a}(c)$.

(iv) is proved by applying (iii) to $a$. Finally, (v) is obvious from (i),

(ii) and the definitions.

**Corollary 15.** $a = G_m \cup (Z \times c)$ if and only if

(i) $b$ is clopen increasing in $Y$;

(ii) $c$ is clopen in $Y$;

(iii) $(Y, c) \subseteq c$;

(iv) $m(b) \subseteq c$.
To illustrate Corollary 13 we consider the lattice $B^{00}$. By definition (\cite{2}, \cite{9}), for any Boolean algebra $B$, $B^{00}$ is the sublattice 

$\{(b, c): b, c \in B, b \subseteq c\}$

of $B \times B$. $B^{00}$ is a Stone lattice with pseudocomplement given by $\{b, c\}^* = \{c', c\}$. Clearly $C(B^{00}) \cong B$, $D(B^{00}) \cong B$. If $X$ is the dual space of $B$, then the dual triple for $B^{00}$ is $(X, \mathcal{X}, f)$, $f$ being the identity map on $X$.

The dual space of $B^{00}$ is

$X^{00} = \{(x, y): x \in X \cup \{(X, x): x \in X\}\}.$

A subset $\mathcal{X}$ of $X^{00}$ is described by a pair $b, c$ of subsets of $X$, $a$ is clopen increasing if and only if $b$ and $c$ are clopen in $X$ and $b = \{d\} \subseteq c$ (condition (iii) of Corollary 15 is vacuous here). Thus we recover

$B^{00} \cong \{(b, c): b, c \in B, b \subseteq c\}$.

We remark finally that $X^{00}$ can be identified with the ordered topological sum $X_1 \oplus X_2$, where $X_1 = X_2 = X$ and $x \leq y$ if and only if $x = y$ in $X_1 \oplus X_2$ or $x \leq X_1, y \leq X_2$ and $s \leq y$. This interpretation of the dual of $B^{00}$ will be used in §6.

Completeness of a Stone lattice $L$ is discussed using the triple for $L$ in §4 of \cite{6}. We conclude this section with an analogous theorem using $L$'s dual triple. We recall that a $\{0, 1\}$-distributive lattice is complete if and only if its dual space $X$ is extremally order disconnected (i.e., for each set $a$ open and decreasing in $X$, $\mathcal{F}(a)$ is open (290), Proposition 16).

Theorem 16. Let $L$ be a Stone lattice with dual triple $(Y, Z, m)$. Then $L$ is complete if and only if

(i) $Z$ is extremally disconnected;

(ii) $Y$ is extremally order disconnected;

(iii) for each $b$ open decreasing in $Y$, $b$ is open in $Z$.

Proof. The first step is to show that (ii) is equivalent to (ii') $d(\mathcal{F}(b))$ is open in $Y$ whenever $b$ is open in $Y$ and $Z \in \mathcal{F}(b)$. (ii) holds if and only if, for all $b$ open in $Y$, $d(\mathcal{F}(b)) = \mathcal{F}(\mathcal{F}(b))$ is open in $Y$. Suppose $b$ is open in $Y$. If $Z \in \mathcal{F}(b)$, $d(\mathcal{F}(b)) = Y$. So assume $Z \notin \mathcal{F}(b)$. Then

$d(\mathcal{F}(b)) = d(\mathcal{F}(b)) \cap Y = d(\mathcal{F}(b))$

which is open in $Y$ if and only if it is open in $Y$. Hence (ii) is equivalent to (ii').

Let $a$ be open decreasing in $X$. By \cite{20}, Lemma 2,

$\mathcal{F}(a) = d(\mathcal{F}(a))$

is (x, m(x)): x \in \mathcal{F}(b) \cup \mathcal{F}(C) \cup \mathcal{F}(Z), x \in \mathcal{F}(b) \cup \mathcal{F}(C)$,

by Theorem 14. $\mathcal{F}(a)$ is open if and only if

(a) $d(\mathcal{F}(b)) \cup \mathcal{F}(C) \cup \mathcal{F}(Z)$ is open in $Y$ and

(b) $b \cup \mathcal{F}(a)$ is open in $Z$.

(Therefore, (iv), (v)).

Since, when $a$ is open decreasing, $b$ is open decreasing in $Y$ and $C$ is open in $Z$, the sufficiency of conditions (i)–(iii) is now clear.

For the converse, assume that a open decreasing in $Y$ implies $\mathcal{F}(a)$ is open. Let $c$ be open in $Z$.

$a = \mathcal{F}(m(a)) \cup \mathcal{F}(Z) \in \mathcal{F}(c)$. Hence, by (b) above, $\mathcal{F}(c)$ is open, so (i) holds.

Now take $b$ open decreasing in $Y$, $a = \mathcal{F}(b)$ is open decreasing in $Y$.

By (ii), $b$ is open in $Z$, whence (ii) holds. Now suppose also $Z \in \mathcal{F}(b)$. Then $b = \emptyset$. By (ii), $d(\mathcal{F}(b))$ is open in $Y$. Therefore (ii') holds and the proof is complete.

Comparing Theorem 16 with Theorem 5 of \cite{6}, we see that our conditions (i), (ii) are the duals of conditions (1), (2) in Chen and Grätzer's completeness theorem. The relation of their condition (3) to our condition (iii) is however not clear. Finally we note that the corollaries to Chen and Grätzer's theorem are equally easy consequences of Theorem 16.

6. Free and injective Stone algebras. The Stone space of a free Stone algebra $L$ has been used by Grätzer and Lakser \cite{10} to examine the structure of $L$ (see also \cite{1}, \cite{5}, \cite{9}, \cite{16}). Our approach is a variant on that adopted by Grätzer and Lakser, emphasizing the order structure.

The dual space of $\mathcal{F}(a)$, the free Stone algebra on $a$, is the ordered cartesian product of $a$ copies of a discretely topologized 3-element space $S = \{a_0, a_1, a_2\}$ in which $x_i \geq y_i$ if and only if $i = j$, $i = 1$, $j = 2$ (cf. \cite{9}. Theorem 17.5). $S$ is the disjoint union of the partially ordered sets $S_0 = \{a_0\}$ and $S_1 = \{a_1, a_2\}$.

If $n$ is finite, the $3$-topology on $X(n)$, the dual of $\mathcal{F}(a)$, is discrete and only the order concerns us. We see (cf. \cite{9}. Theorem 17.6) that

$X(n)$ is order isomorphic to $\sum_{i=n}^{\infty} 3^{Or(S_i)^n}$,

where summation denotes the disjoint union of partially ordered sets, $S_i^n = S_i$, and, for $r \geq 1$, $(S_r)^n$ is the cartesian product of $r$ copies of $S_i$, and the binomial coefficient $wOr$ indicates the number of copies of $(S_r)^n$ occurring in the sum. The dual triple for $\mathcal{F}(a)$, $(Y(n), Z(n), m(n))$,
is easily recognized. For $r \geq 1$, $(S_1)'$ has a unique maximal point $(x_1, \ldots, x_r)$. Hence $Z(s)$ has $2^r$ elements. Let $T_r$ be the partially ordered set obtained by deleting the maximal element from $(S_1)'$. It is clear that, to within order isomorphism,

$$Y(n) = \sum_{r=1}^{n} n! r T_r.$$ 

We may write

$$Y(n) = \sum_{r=1}^{n} W_r,$$

where, for $r \geq 1$, $W_r = T_r$ for $r \geq 0$, $W_0 = \{0\}$, $\sum_{r=1}^{n} n! r < s < \sum_{r=0}^{n} n! r$;

$$Z(n) = \{z: s = 0, \ldots, 2^n - 1\}.$$ 

Then $m(n)$ is given by

$$(m(n))(s) = s \text{ if and only if } x \in W_s.$$ 

$O(n)$ is the lattice of subsets of $Z(n)$; $D(n)$ is the lattice of all increasing subsets of $Z(n)$, as is the lattice of filter bases of $D(n)$. We sum up our results in Proposition 17 (cf. [8], Problem 1), using the notation of [9].

**Proposition 17.** Let FSL$(n)$ have triple $(O(n), D(n), \Phi(n))$. Then $O(n) = F_{\beta \aleph_0}(n)$ and $D(n) = \prod_{\aleph_0} (\beta \aleph_0 \setminus \{0\})$. If $\Phi(n)$ is a p-element set, then $\Phi(n)(a)$ is an increasing subset of $Y(n)$, isomorphic as a partially ordered set to the dual of a product of $2^{n-p} - p$ or of $2^{n-p} - (p-1)$ lattices, each of which is a free $(0,1)$-distributive lattice with zero deleted.

Proof. The statement concerning $\Phi(n)$ is obtained from Proposition 8.

In answer to a question posed by Chen and Grätzer ([6], Problem 2) we state

**Proposition 18.** Let $(O(n), D(n), \Phi(n))$ be the reduced triple associated with FSL$(n)$. Then $O(n) = F_{\beta \aleph_0}(n-1)$. If $a$, $O(n)$, $(\Phi(n))(a)$ can be represented by a product of $2^{n-p} - p$ lattices, each of which is a free $(0,1)$-distributive product with zero deleted.

We have been unable to find any worthwhile description of the dual triple for FSL$(n)$ when $n$ is a general cardinal (although it is clear that $Z(n)$ is the Cantor space $2^n$).

Finally we present a new proof of Theorem 2 of [2], which characterizes injective Stone algebras as Stone lattices expressible in the form $B_1 \times B_2$, where $B_1$, $B_2$ are complete Boolean algebras (see also [14]).

**Theorem 19.** If $\mathcal{X}$ is a projective SLD-space then the dual triple for $\mathcal{X}$ takes the form $(X_1, X_2 \oplus X_2, j)$, where $X_1, X_2$ are compact, discretely ordered and extremely disconnected and $j: X_1 \to X_2 \oplus X_2$ the embedding map.

Proof. Let $(X, \mathcal{I}, \leq)$ be projective, with dual triple $(Y, Z, m)$. The first step is to show that the order on $X$ is discrete (cf. the proof of Proposition 11 of [30]). Suppose not, and take $x, y \in X$, $x \neq y$, $y \not\leq y$, $y \not\geq y$, so we can find a clopen increasing set $X_1$ such that $x \in X_1$, $y \not\in X_1 = X \setminus X_1$. We may assume that $\mathcal{X} \subseteq Z$. Define an order relation $\leq'$ on $X$ by

$$(x \leq' y \text{ if and only if (i) } x = y, \text{ or (ii) } x, y \in X_1, x \leq y, \text{ or (iii) } x \leq X_1, y \in Z, x \leq y;$$

this gives a genuine partial order. We claim that $(X, \mathcal{I}, \leq')$ is t.o.d. Take $x \leq' y$ and note that a set increasing $(\leq)$ is increasing $(\leq')$. In either of the cases (i) $x, y \in X_1$, (ii) $x \in X_2$, $y \in Z$, $x \leq y$ and so there is a clopen increasing $(\leq')$ set containing $x$ but not $y$. If $x \in X_1$, $y \in X_2$, $X_1$ is clopen increasing $(\leq')$ and contains $x$ but not $y$. Finally let $x \in X_1, y \in X_2 \cap Y$. There is a clopen decreasing $(\leq')$ set $p$ disjoint from $Z$ and containing $y$, $p \cap X_1$ is decreasing $(\leq')$ and contains $y$ but not $x$.

Let $d'(a)$ be the smallest decreasing $(\leq')$ set containing the clopen increasing $(\leq')$ set $a$. Then $a \cap X_1$ is increasing $(\leq')$. Take $x \leq' y$ and suppose first that $x, y \in X_1$, $x \leq y$. Since $X_1$ is increasing $(\leq')$,

$$x \leq d'(a) \cap X_1 \subseteq d'(a).$$

Secondly, if $x \in X_1$, $y \in Z$, $x \leq y$,

$$x \leq d'(a) \cap X_1 \subseteq d'(a).$$

(to obtain this inclusion, note that if $z \in d'(a) \cap X_1$, $z \leq x \leq a \cap Z$, because $a$ is increasing). Thus $d'(a)$ is open, and it follows from Proposition 3 that $(X, \mathcal{I}, \leq')$ is an SLD-space, since each point in $X$ is majorized in the order $\leq'$ by a unique maximal point.

The identity map from $(X, \mathcal{I}, \leq')$ onto $(X, \mathcal{I}, \leq)$ is an SLD-space morphism. Projectivity of $X$ implies that $\leq'$ and $\leq$ coincide. But this is false since $y_1 \not\leq y_2$, $y_1 \not\geq y_2$. Therefore $Y$ has the discrete order.

Let $\mathcal{F}$ denote the discrete topology on $X$. The identity map from $(X, \mathcal{I})$ to $(X, \mathcal{F})$ extends to a continuous map $\alpha$ from $\beta X$ (the Stone-Cech compactification of $(X, \mathcal{I})$) to $(X, \mathcal{F})$. $\beta X$ and $\beta(\mathcal{I}, \mathcal{M}(X))$ may be regarded as totally disconnected subspaces of $\beta X$. Let $\mathcal{X} = X_1 \times X_2 \oplus X_2$, where $X_1 = X_2 = X$ and $X_2 = \mathcal{I}, \mathcal{M}(X)$. We can construct a continuous map $\alpha$ on $\mathcal{X}$ onto $(X, \mathcal{F})$ by stipulating

$$\alpha | X_1 = a \beta Y, \quad \alpha | X_2 = a \beta(\mathcal{I}, \mathcal{M}(X)), \quad \alpha | X_2 = \pi(a \beta Y),$$
where \( \pi: X \to Z \) is the projection map. \( \hat{X} \) may be ordered by defining \( x \preceq y \) if and only if \( x = y \in \hat{X} \) or \( x \in X \) and \( x = y \) in \( \beta Y \). \( \hat{X} \) is then an SLD-space with dual triple \((X_1, X_2 \oplus X_3), \alpha)\), with \( \alpha(x) = x \) for all \( x \in X_1 \). Further, \( \alpha \) is easily seen to be an SLD-space morphism. By projectivity of \( \hat{X} \), the identity map from \( X \) into \( \hat{X} \) is an SLD-space morphism. We conclude that \( Y, X, \hat{X}, m \)compact and that each point in \( m \) is a point in \( Y \). Thus the dual triple of \( Y \) is of the form \((Y_1, Y_2 \oplus Y_3), j\), with \( Y_1, Y_2, Y_3 \) Boolean spaces and \( j \) the natural embedding. That \( Y_1, Y_2 \) are extremally disconnected is clear since \( X \) is, a priori, projective in the category of Boolean spaces \([17] \), cf. also \([20]\).

**Corollary 20** (\([2] \), Theorem 2). A Stone algebra is injective if and only if it can be represented in the form \( B_i \times B_i^0 \), where \( B_i, B_i^0 \) are complete Boolean algebras.

Proof. Necessity follows from Theorem 19 since the dual triple for \( B_i \times B_i^0 \) is of the form \((X_1, X_2 \oplus X_3), j\) (see the discussion of \( B_i^0 \) in \( 5 \)). For sufficiency we refer to Lakser's paper \([14]\), remarking that a dual version of Lakser's proof can easily be constructed to prove directly the converse of Theorem 19.

**References**