Simplicial approximation of small multifunctions *

by

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Abstract. A multifunction $\varphi\colon |K|\to|L|$ from a polyhedron $|K|$ to a polyhedron $|L|$ is called *small* if $\varphi(x)\subset \mathrm{st}(w; L)$ for all $x\in|K|$, where $w = w(x)$ is a suitable vertex of L and $\mathrm{st}(w; L)$ is the star of w in L. It is shown that a small, usc and point-closed multifunction φ can be approximated by a simplicial map. This simplicial approximation is unique up to homotopy, and can be used to prove that small homotopy classes of usc multifunctions are in one-to-one correspondence with ordinary homotopy classes of maps. Applications are given to extension and lifting of small usc multifunctions.

1. Small multifunctions. This paper stems from the idea that a multifunction $\varphi\colon X\to Y$ with "small" point-images shows many resemblances to a single-valued function, an idea which has already been found fruitful in previous work (see [3], [4], [5]). The concept "small" can be defined in various ways, and the resemblance explored in different directions.

If Y has a metric, then φ has been called *ε-small* if, for all $x\in X$, the diameter of $\varphi(x)$ is $\leqslant\varepsilon$ [4], [5]. If Y is an n-sphere S^n, then the term small has been used to denote a multifunction for which every $\varphi(x)$ is contained in a hemi-sphere [3]. In the present paper we assume that Y is a polyhedron, and define φ as small if every $\varphi(x)$ is contained in the star of a vertex of Y (see Definition 1.1 below).

The resemblance of small multifunctions to single-valued functions has been investigated in several ways, e.g. in connection with continuous selections [4] and with near fixed points [5]. For $Y = S^n$ it has been shown that each semi-continuous small multifunction is related, via a small homotopy, to a single-valued map, and that homotopy classes of small multifunctions are in one-to-one correspondence with homotopy classes of single-valued maps [3]. Here we construct small homotopies between small upper semi-continuous multifunctions and single-valued maps if Y is a polyhedron, using as a tool a generalization to multifunctions of the concept of a simplicial approximation.

We use $|K|, |L|, \ldots$ to denote the polyhedron underlying the simplicial complexes $K, L, \ldots$, and assume that all such complexes are finite. If v is a vertex of K, then $\mathrm{st}(v; K)$ denotes the *star* of v in K, which is the open

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subset of $|K|$ consisting of the union of the interiors of all simplexes which have v as a vertex.

A *multifunction* $\varphi: X \to Y$ from a space X into a space Y is a correspondence which assigns to each $x \in X$ a non-empty subset $\varphi(x)$ of Y. Recall that φ is *point-closed* if each $\varphi(x)$ is closed in Y, and usc (*upper semi-continuous*) if for every open set V in Y with $\varphi(x) \subset V$ there exists an open neighbourhood U of x such that $\varphi(U) \subset V$. The term *map* is reserved for single-valued continuous functions.

All multifunctions in this paper are small in the following sense.

DEFINITION 1.1. Let $\varphi: X \to |L|$ be a multifunction from a space X into a polyhedron $|L|$. Then φ is called *small* (with respect to the simplicial structure L of $|L|$) if $\varphi(x) \subset \mathrm{st}(w; L)$ for all $x \in X$, where $w = w(x)$ is a suitable vertex of L.

Note that for the case $|L| = S^n$ this definition of small is more restrictive than the one used in [3], and that the results of [3] do not follow from the ones given here. The more precise, but also more cumbersome, term "star-small" could be used in Definition 1.1 to point out the difference, but confusion seems unlikely as "small" is used in only one meaning throughout the rest of this paper.

Note also that the composite of two small multifunctions need not be small. A rather extreme example can be obtained as follows: Let $X = |L|$ be the perimeter of an equilateral triangle, and let L consist of three vertices and three one-simplexes in the obvious way. Define $\varphi: |L| \to |L|$ by choosing as $\varphi(x)$ the closed portion of the perimeter which has x as its midpoint and is of a length equal to $\frac{1}{4}$ of that of the perimeter. Then φ is small, but $\varphi^4 = \varphi \circ \varphi \circ \varphi \circ \varphi$ is the constant multifunction given by $\varphi^4(x) = |L|$ for all $x \in |L|$. This example makes it clear that results involving the composition of maps can usually not be extended to small multifunctions.

2. Simplicial approximation of a small usc **multifunction.** This paragraph contains the construction of a simplicial approximation to a small usc multifunction (Theorem 2.2), and shows that a simplicial approximation is homotopic to the given multifunction (Theorem 2.4).

We first give the definition of a simplicial approximation, which generalizes one frequently used for maps (see e.g. [2], p. 46).

DEFINITION 2.1. Given simplicial complexes K and L, a small multifunction $\varphi: |K| \to |L|$, and a simplicial map $f: |K| \to |L|$, then f is called a *simplicial approximation* of φ if $\varphi(\mathrm{st}(v; K)) \subset \mathrm{st}(f(v); L)$ for all vertices $v \in K$.

We see that the definition depends on both the simplicial structures K and L of $|K|$ and $|L|$. Smallness only requires from a multifunction $\varphi: |K| \to |L|$ that $\varphi(v) \subset \mathrm{st}(w; L)$, and therefore it is not necessarily true that $\varphi(\mathrm{st}(v; K)) \subset \mathrm{st}(w; L)$. But the proof of Theorem 2.2 will show that this condition can always be fulfilled for a suitable subdivision of K if φ is usc.

THEOREM 2.2. *Let K, L be simplicial complexes and $\varphi: |K| \to |L|$ be an* usc *multifunction which is small with respect to L. Then there exists a subdivision K' of K such that $\varphi: |K'| \to |L|$ has a simplicial approximation $f: |K'| \to |L|$.*

Proof. The multifunction φ is small, hence we can find for each $x \in |K|$ a vertex $w = w(x)$ of L such that $\varphi(x) \subset \mathrm{st}(w; L)$. As φ is usc and as $\mathrm{st}(w; L)$ is open, there exists an open set $U(x)$ containing x with $\varphi(U(x)) \subset \mathrm{st}(w; L)$. Let δ be a Lebesgue number of the covering $\{U(x) |\ x \in |K|\}$ of the compact space $|K|$, and let K' be a subdivision of K such that the maximum of the diameters of the stars of its vertices is $< \delta$. Then there exists for every vertex v of K' a vertex $w = w(v)$ of L such that $\varphi(\mathrm{st}(v; K')) \subset \mathrm{st}(w; L)$.

Define a vertex map f from K' to L by

(1) $$f(v) = w(v) \quad \text{for all vertices } v \text{ of } K'.$$

In order to see that f can be extended to a simplicial map from $|K'|$ to $|L|$, take any simplex $\sigma = [v_0, v_1, \ldots, v_r]$ of dimension $\geqslant 1$ of K', and let $\mathrm{Int}\,\sigma$ denote its interior. As

(2) $$\varphi(\mathrm{Int}\,\sigma) \subset \varphi(\mathrm{st}(v_i; K')) \subset \mathrm{st}(f(v_i); L) \quad \text{for} \quad i = 0, 1, \ldots, r,$$

we have

$$\bigcap_{i=0}^{r} \mathrm{st}(f(v_i); L) \neq \emptyset .$$

Hence $f(v_0), f(v_1), \ldots, f(v_r)$ span a simplex of L, and f can be extended over σ. So (1) determines a simplicial map f from $|K'|$ into $|L|$, and (2) shows that f is a simplicial approximation of φ.

In the single-valued case a simplicial approximation is homotopic to the given map. A similar result is true here if the homotopy is suitably defined.

DEFINITION 2.3. Let $|L|$ be a polyhedron and $\varphi_0, \varphi_1: X \to |L|$ be smalls usc and point-closed multifunctions. A *small homotopy* is a small, usc and point-closed multifunction $\Phi: X \times I \to |L|$ with $\Phi(x, 0) = \varphi_0(x)$ and $\Phi(x, 1) = \varphi_1(x)$ for all $x \in X$.

We write $\varphi_0 \overset{s}{\simeq} \varphi_1$ for a small homotopy, and $f_0 \simeq f_1$ for a single-valued one. That $\overset{s}{\simeq}$ determines an equivalence relation follows exactly as in the single-valued case.

THEOREM 2.4. *Let $\varphi: |K| \to |L|$ be a small,* usc *and point-closed multifunction and $f: |K'| \to |L|$ be a simplicial approximation of φ. Then there exists a small homotopy Φ between φ and f.*

Proof. We construct Φ inductively on the r-skeleton K'_r of K', for $r = 0, 1, \ldots, \dim K'$.

If v is a vertex of K', then $f(v) = w$ is a vertex of L such that $\varphi(v) \subset \mathrm{st}(w; L)$. Hence we can define a set $\Phi(v; t)$ for $0 \leqslant t \leqslant 1$ linearly by

$$\begin{aligned}\Phi(v, t) &= tf(v) + (1-t)\varphi(v)\\ &= \{tf(v) + (1-t)y \mid y \in \varphi(v)\}.\end{aligned}$$

Assume now that Φ has been defined over $|K'_r| \times I$ for $0 \leqslant r \leqslant n-1$, and let $\sigma = [v_0, v_1, \ldots, v_n]$ be an n-simplex of K'_n. The vertices $w_i = f(v_i)$, for $i = 0, 1, \ldots, n$, span a simplex τ of L. Hence we have, for any point $x \in \mathrm{Int}\,\sigma$,

$$\varphi(x) \subset \varphi(\mathrm{st}(v_i; K')) \subset \mathrm{st}(f(v_i); L) \quad \text{for} \quad i = 0, 1, \ldots, n,$$

and thus

$$\varphi(x) \subset \bigcap_{i=0}^{n} \mathrm{st}(f(v_i); L) = \bigcap_{i=0}^{n} \mathrm{st}(w_i; L).$$

As $f(x) \in \tau$, we can again define a set $\Phi(x, t)$ in a linear way by

$$\Phi(x, t) = tf(x) + (1-t)\varphi(x)$$

for all $x \in \mathrm{Int}\,\sigma$ and $0 \leqslant t \leqslant 1$. As $\Phi(x, t)$ is already defined if x lies on the boundary of σ, this defines Φ on all of σ and therefore inductively on $|K'| \times I$. It is easy to see that Φ is point-closed and usc, and it is small as $\Phi(x, t) \subset \mathrm{st}(w; L)$ for any vertex w of the carrier-simplex of $f(x)$.

The construction of Φ shows that Φ is a special homotopy in the sense of [1], i.e. that $\Phi(x, t)$ is homeomorphic to $\Phi(x, 0)$ for all $x \in |K|$ and $0 \leqslant t < 1$. In particular we have

THEOREM 2.5. *The homotopy Φ in Theorem 2.4 can be chosen so that it is single-valued for all $0 \leqslant t \leqslant 1$ whenever $\varphi(x)$ is single-valued.*

3. Homotopy classes of small usc **multifunctions.** We show in this paragraph that the simplicial approximation of a small usc multifunction is unique up to homotopy, and that small homotopy classes of usc multifunctions are in one-to-one correspondence with ordinary homotopy classes of maps. These results are easy consequences of the following theorem.

THEOREM 3.1. *Let K and L be simplicial complexes, and K', K'' be subdivisions of K. Let further φ_0, φ_1: $|K| \to |L|$ be small,* usc *and point-closed multifunctions, and let f_0: $|K'| \to |L|$ be a simplicial approximation of φ_0 and f_1: $|K''| \to |L|$ be a simplicial approximation of φ_1. If φ_0 and φ_1 are related by a small homotopy, then f_0 and f_1 are related by an ordinary homotopy.*

Proof. We are given a small homotopy

$$\Phi'\colon |K| \times I \to |L| \quad \text{with } \Phi'(x, 0) = \varphi_0(x) \text{ and } \Phi'(x, 1) = \varphi_1(x),$$

and we also know from Theorem 2.4 that there exist small homotopies

$$\Phi_i\colon |K| \times I \to |L| \quad \text{with } \Phi_i(x, 0) = \varphi_i(x) \text{ and } \Phi_i(x, 1) = f_i(x) \text{ for } i = 0, 1.$$

If we define a correspondence Φ: $|K| \times I \to |L|$ by

$$\Phi(x, t) = \begin{cases} \Phi_0(x, 1-3t) & \text{for } 0 \leqslant t \leqslant \frac{1}{3},\\ \Phi'(x, 3t-1) & \text{for } \frac{1}{3} \leqslant t \leqslant \frac{2}{3},\\ \Phi_1(x, 3t-2) & \text{for } \frac{2}{3} \leqslant t \leqslant 1,\end{cases}$$

then Φ is a small homotopy between f_0 and f_1.

It follows from Theorem 2.2 that Φ has a simplicial approximation F: $|(K \times I)'| \to |L|$, and from Theorem 2.4 that there exists a small homotopy ψ: $|K \times I \times I| \to |L|$ such that $\psi(x, t, 0) = \Phi(x, t)$ and $\psi(x, t, 1) = F(x, t)$ for all $x \in |K|$ and $0 \leqslant t \leqslant 1$. Now define G: $|K \times I| \to |L|$ by

$$G(x, t) = \begin{cases} \psi(x, 0, 3t) & \text{for } 0 \leqslant t \leqslant \frac{1}{3},\\ \psi(x, 3t, 1) & \text{for } \frac{1}{3} \leqslant t \leqslant \frac{2}{3},\\ \psi(x, 1, 3-3t) & \text{for } \frac{2}{3} \leqslant t \leqslant 1.\end{cases}$$

Then G is a single-valued and (as it is usc) continuous homotopy between $G(x, 0) = f_0(x)$ and $G(x, 1) = f_1(x)$, and Theorem 3.1 is proved.

Taking $\varphi_0 = \varphi_1$ in Theorem 3.1, we obtain the uniqueness (up to homotopy) of the simplicial approximation.

COROLLARY 3.2. *If f_0: $|K'| \to |L|$ and f_1: $|K''| \to |L|$ are two simplicial approximations of the small,* usc *and point-closed multifunction φ: $|K| \to |L|$, then f_0 and f_1 are homotopic.*

A further corollary of Theorem 3.1 will be used in § 4.

COROLLARY 3.3. *If f_0, f_1: $|K| \to |L|$ are two maps which are related by a small homotopy, then they are also related by an ordinary homotopy.*

Proof. Let f'_0, f'_1 be simplicial approximations of f_0, f_1. Then Theorem 3.1 implies $f'_0 \simeq f'_1$, and Theorem 2.4 implies $f_0 \simeq f'_0$ and $f_1 \simeq f'_1$. Hence $f_0 \simeq f_1$.

By the *small homotopy class* $[\varphi]$ of the small, usc and point-closed multifunction φ we mean the set of all multifunctions φ' which are related to φ be a small homotopy. Theorem 3.1 shows that a correspondence from the set of small homotopy classes of multifunctions to the set of ordinary homotopy classes of maps can be defined by assigning to each multifunction its simplicial approximation, and Theorem 2.4 shows that this correspondence is one-to-one. More precisely, we have

THEOREM 3.4. *If K, L, are simplicial complexes, then the set of small homotopy classes of small,* usc *and point-closed multifunctions from $|K|$ into $|L|$ is in one-to-one correspondence with the set of ordinary homotopy classes of maps from $|K|$ into $|L|$.*

4. Extension and lifting problems. As a further illustration of the close relation between small multifunctions and their simplicial approximations we finish with some results on extension and lifting of multifunctions.

THEOREM 4.1. (Extension problem). *Let X, Y be polyhedra, and let A be a subpolyhedron of X. Let $\varphi\colon A \to Y$ be a small,* usc *and point-closed multifunction, and $f\colon A \to Y$ be a simplicial approximation of φ. If f can be extended to a map from X to Y, then φ can be extended to a small,* usc *and point-closed multifunction from X to Y.*

Proof. If $g\colon X \to Y$ is an extension of f, then its restriction to A equals f. Hence Theorem 2.4 shows the existence of a small homotopy $\Phi\colon A \times I \to Y$ with $\Phi(a, 0) = g(a)$ and $\Phi(a, 1) = \varphi(a)$ for all $a \in A$. Let $r\colon X \times I \to (X \times 0) \cup (A \times I)$ be a retraction, define $\Phi'\colon (X \times 0) \cup (A \times I) \to Y$ by

$$\Phi'(x, t) = \begin{cases} g(x) & \text{if} \quad t = 0, \\ \Phi(x, t) & \text{if} \quad t > 0, \end{cases}$$

and define $\Psi\colon X \times I \to Y$ by $\Psi = \Phi' \circ r$. Then Ψ is small, usc and point-closed, and $\Psi(x, 1) = \varphi(x)$ if $x \in A$. Therefore $\psi(x) = \Psi(x, 1)$ is an extension of φ.

Remark. It is also easy to prove the converse, i.e. the fact that the existence of an extension of φ implies the existence of an extension of f.

THEOREM 4.2. (Lifting problem). *Let X, E, B be polyhedra and $\varphi\colon X \to B$, $\pi\colon E \to B$ be small,* usc *and point-closed multifunctions. Let $f\colon X \to B$ and $p\colon E \to B$ be simplicial approximations of φ and π. If f can be lifted to a map $g\colon X \to E$ with $p \circ g \simeq f$, then φ can be lifted to a small,* usc *and point-closed multifunction $\psi\colon X \to E$ with $\pi \circ \psi \overset{s}{\simeq} \varphi$.*

Proof. Take $\psi = g$, so that $\pi \circ \psi = \pi \circ g$. Let $\Pi\colon E \times I \to B$ be a small homotopy from π to p (see Theorem 2.4), and let $1_I\colon I \to I$ be the identity map. Then $\pi \circ (g \times 1_I)\colon X \times I \to B$ is a small homotopy from $\pi \circ g$ to $p \circ g$. As $p \circ g \simeq f$ and $f \overset{s}{\simeq} \varphi$ (Theorem 2.4), we have $\pi \circ \psi \simeq \varphi$.

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Stone lattices: a topological approach

by

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Abstract. A $\{0, 1\}$-distributive lattice L can be represented as the lattice of clopen increasing subsets of an appropriate ordered topological space X. It is shown that, when L is a Stone lattice, the dual space X is characterized by subspaces $Y(X)$, $Z(X)$ of X and a continuous increasing map $m(X)\colon Y(X) \to Z(X)$. This enables the structure of Stone lattice dual spaces to be analysed in terms of simpler components and leads to a construction theorem for such dual spaces which is in the same spirit as, but not directly dual to, Chen and Grätzer's triple construction theorem for Stone lattices.

1. Introduction. Chen and Grätzer show in [5] that a Stone lattice L can be studied by investigating an associated triple of simpler components, $(C(L), D(L), \Phi(L))$, where $C(L)$, $D(L)$ are appropriate subsets of L and $\Phi(L)$ a connecting map. In this paper the duality between $\{0, 1\}$-distributive lattices and compact totally order disconnected spaces developed in [19] and [20] is applied to Stone lattices and the dual space X of a Stone lattice L is shown to be characterized by subspaces $Y(X)$, $Z(X)$ of X and a continuous increasing map $m(X)\colon Y(X) \to Z(X)$. The ordered spaces $Y(X)$, $Z(X)$ are the duals of the lattices $D(L)$, $C(L)$; $m(X)$ and $\Phi(L)$ are related, but are not mutually dual maps.

The Construction Theorem in [5] asserts that, given a suitably defined triple (C, D, Φ), there exists a Stone lattice L with $C(L) = C$, $D(L) = D$, $\Phi(L) = \Phi$. Problem 55 of [9] seeks a less computational proof of this theorem than that given in [5]. Motivated by this problem, we show how to construct a space dual to a Stone lattice from a "dual triple" (Y, Z, m) and hence obtain a new method of constructing Stone lattices from simpler components.

Dual triples also provide new information on free Stone algebras and a short proof of Theorem 2 of [2], characterizing injectives.

2. The dual space of a Stone lattice. We refer to [19], [20] for the ordered topological space concepts needed, recalling only two crucial definitions concerning a set X endowed with a partial order $\leqslant$ and a topology $\mathfrak{J}$. A subset E of X is *decreasing* (*increasing*) if $x \leqslant y \in E$ ($x \geqslant y \in E$) implies