

- [2] P. Erdős, A. Hajnal and R. Rado, *Partition relations for cardinals*, Acta. Math. Acad. Sci. Hungar. 16 (1965), pp. 159–229.
- [3] F. Galvin and S. Shelah, *Some counterexamples in the partition calculus*, Jour. of Combinatorial Theory, Ser. A., 15 (1973), pp. 167–174.
- [4] E. M. Kleinberg, *The equiconsistency of two large cardinal axioms*, to appear.
- [5] — and R. A. Shore, *Weak compactness and square bracket partition relations*, J. Symbolic Logic 37 (1972), pp. 673–676.
- [6] R. B. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic 4 (1972), pp. 229–308.
- [7] — and K. Kunen, *Some combinatorial properties of L and V*, unpublished.
- [8] F. Rowbottom, *Some strong axioms of infinity incompatible with the axiom of constructibility*, Ann. of Math. Logic 3 (1971), pp. 1–44.

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On monotonic generalizations of Moore spaces, Čech complete spaces and p -spaces

by

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Abstract. In this paper the generalized base of countable order theory of H. H. Wicke and J. M. Worrell, Jr. is simplified. This theory allows to understand what the role of paracompactness is in the theories of metrizable and complete metrizable. We consider four classes of spaces and prove that they are closed under the action of perfect mappings and open and compact mappings. Furthermore we give a mapping characterization of these classes and show that in the class of paracompact spaces they are equal to the classes of metrizable spaces, completely metrizable spaces, paracompact p -spaces and paracompact Čech complete spaces. Besides, we construct an example of a locally completely metrizable metacompact space which is not Čech complete.

The aim of this paper is to introduce new characterizations of some classes of spaces investigated by H. H. Wicke and J. M. Worrell, Jr. Our characterizations allow us to simplify the theory of these authors.

We shall use the terminology and notation from [12]. By a *mapping* we always mean a continuous function. All spaces are assumed to be regular. If \mathfrak{M} and \mathfrak{N} are families of subsets of a certain space X and if for an arbitrary $N \in \mathfrak{N}$ there exists an $M \in \mathfrak{M}$ such that $M \subseteq N$, then we write that $\delta\mathfrak{M} < \mathfrak{N}$. The letter \mathfrak{A} will always denote a centred family; centred families of closed sets will be denoted by \mathfrak{F} .

Let $\mathfrak{B} = \{\mathfrak{B}_n\}_{n=1}^{\infty}$ be a sequence of bases of a given space X . We recall that if for each sequence $\{B_n\}_{n=1}^{\infty}$ such that $B_n \in \mathfrak{B}_n$ one of the following conditions is satisfied:

- (d) $\bigcap_{n=1}^{\infty} B_n \ni x$, then $\{B_n\}_{n=1}^{\infty}$ is a base at x ,
- (p) $\bigcap_{n=1}^{\infty} B_n \ni x$ and $\delta\mathfrak{F} < \{B_n\}_{n=1}^{\infty}$, then $\bigcap \mathfrak{F} \neq \emptyset$,
- (c) $\delta\mathfrak{F} < \{B_n\}_{n=1}^{\infty}$, then $\bigcap \mathfrak{F} \neq \emptyset$,

then X is a Moore space or a p -space [17], or a Čech complete space [12, Theorem 3.8.2], respectively. If the conditions (d) and (c) are satisfied simultaneously, then X is a complete Moore space.

To get our monotonic generalizations we shall restrict ourselves to the sequences $\{B_n\}_{n=1}^\infty$ such that $B_{n+1} \subseteq B_n$.

1. Monotonic properties. A property \mathfrak{P} of sequences of sets will be called *monotonic* if the following condition is satisfied

$$\{H_m\}_{m=1}^\infty \in \mathfrak{P} \quad \text{and} \quad \delta\{W_n\}_{n=1}^\infty < \{H_m\}_{m=1}^\infty, \quad \text{then} \quad \{W_n\}_{n=1}^\infty \in \mathfrak{P}.$$

Let $X \subseteq X'$; the sequence $\mathfrak{G} = \{\langle G_n, A_n, \pi_n \rangle\}_{n=1}^\infty$ will be called a *sieve of X in X'* if, for an arbitrary n , $G_n = \{G_a\}_{a \in A_n}$ ⁽¹⁾ is a covering of X open in X' and $\pi_n: A_{n+1} \rightarrow A_n$ is such that

$$1) a \in A_n, \text{ then } X \cap G_a = \bigcup \{X \cap G_{a'} : \pi_n(a') = a\},$$

$$2) a \in A_{n+1}, \text{ then } G_a \subseteq G_{\pi_n(a)}.$$

A sequence $\{G_n\}_{n=1}^\infty$, where $a_n \in A_n$ and $\pi_n(a_{n+1}) = a_n$, will be called a *thread of \mathfrak{G}* . We say that \mathfrak{G} is a *strong sieve* if each thread of \mathfrak{G} is a strongly decreasing sequence, which means that $\bar{G}_{n+1} \subseteq G_n$.

If each thread of \mathfrak{G} satisfies a monotonic property (m), then \mathfrak{G} will be called a (strong) (m)-sieve.

A sequence $\{\mathfrak{B}_n\}_{n=1}^\infty$ of bases of X in X' will be called an (m)-sequence if each decreasing sequence $\{W_n\}_{n=1}^\infty$ such that $W_n \in \mathfrak{B}_n$ satisfies (m).

The following lemma will play a fundamental role in our considerations:

LEMMA 1.1. Let (m) be a monotonic property and let $X \subseteq X'$. The following conditions are equivalent:

- (a) X has an (m)-sieve in X' ,
- (b) for every sequence $\{\mathfrak{B}_n\}_{n=1}^\infty$ of bases of X in X' there exists a sequence $\{\mathfrak{W}_n\}_{n=1}^\infty$ of bases of X in X' such that $\mathfrak{W}_n \subseteq \mathfrak{B}_n$ and each sequence $\{W_n\}_{n=1}^\infty$ such that $W_n \in \mathfrak{W}_n$ and $W_{n+1} \subseteq \bar{W}_n$ ⁽²⁾ satisfies (m),
- (c) X has an (m)-sequence of bases in X' ,
- (d) X has a strong (m)-sieve in X' .

Proof. It is obvious that (b) implies (c), (c) implies (d) and (d) implies (a). We have to prove that (a) implies (b).

Let $\mathfrak{G} = \{\langle G_n, A_n, \pi_n \rangle\}_{n=1}^\infty$ be an (m)-sieve of X in X' . We can well-order each A_n such that, for $a, a' \in A_{n+1}$, $a < a'$ implies $\pi_n(a) \leq \pi_n(a')$. If $x \in X$ we put $a_n(x) = \min\{a \in A_n : x \in G_a\}$, then it follows that $\pi_n(a_{n+1}(x)) = a_n(x)$. To simplify the notation we shall write $G_n(x)$ instead of $G_{a_n(x)}$. Let $\mathfrak{W}_n(x) = \{W \in \mathfrak{B}_n : x \in W \subseteq \bar{W} \subseteq G_n(x)\}$ and let $\mathfrak{W}_n = \bigcup \{\mathfrak{W}_n(x) : x \in X\}$. Suppose that $\{W_n\}_{n=1}^\infty$ is such that $W_n \in \mathfrak{W}_n$ and $W_{n+1} \subseteq \bar{W}_n$. For each m take the first $a_m \in A_m$ such that $\delta\{\bar{W}_n\}_{n=1}^\infty < \{G_{a_m}\}_{m=1}^\infty$.

⁽¹⁾ It is possible that for $a, a' \in A_n$ we have $a \neq a'$ and $G_a = G_{a'}$.

⁽²⁾ The closure is taken in X' . When we do not assume regularity, our proof shows the equivalence of (a) and (c).

Since (m) is a monotonic property and $\delta\{W_n\}_{n=1}^\infty < \{G_{a_m}\}_{m=1}^\infty$, the proof will be complete if we show that $\{G_{a_m}\}_{m=1}^\infty$ is a thread of \mathfrak{G} . For an arbitrary m take $n > m$ such that $\bar{W}_n \subseteq G_{a_m}$ and $\bar{W}_n \subseteq G_{a_{m+1}}$. Let $x \in X$ be such that $W_n \in \mathfrak{W}_n(x)$. From the definition of $\mathfrak{W}_n(x)$ and the fact that $\{G_n(x)\}_{n=1}^\infty$ is a thread of \mathfrak{G} we have

$$\bar{W}_n \subseteq G_n(x) \subseteq G_{m+1}(x) \subseteq G_m(x).$$

Thus $G_{m+1}(x)$ and $G_m(x)$ do not precede $G_{a_{m+1}}$ and G_{a_m} , respectively. On the other hand,

$$x \in W_n \subseteq G_{a_{m+1}} \quad \text{and} \quad x \in W_n \subseteq G_{a_m};$$

hence $G_{a_{m+1}} = G_{m+1}(x)$, $G_{a_m} = G_m(x)$ and we deduce that $\{G_{a_m}\}_{m=1}^\infty$ is a thread of \mathfrak{G} ⁽³⁾.

Let us notice that if a sequence $\{\mathfrak{B}_n\}_{n=1}^\infty$ is constant, then $\mathfrak{W}_{n+1} \subseteq \mathfrak{W}_n$ and it follows that each subsequence of a sequence $\{\mathfrak{W}_n\}_{n=1}^\infty$ is an (m)-sequence.

COROLLARY 1.2. If X has an (m)-sieve and an (mm)-sieve in X' , then X has a strong (m) and (mm)-sieve in X' .

COROLLARY 1.3. If $X = X'$ is κ_c -Lindelöf ⁽⁴⁾ (hereditarily κ_c -Lindelöf) and has an (m)-sieve, then X has a (strong) (m)-sieve $\mathfrak{G} = \{\langle G_n, A_n, \pi_n \rangle\}_{n=1}^\infty$ such that each G_n has cardinality not greater than κ_a .

We will often use the following, well-known lemma (see [12, Theorem 3.2.10])

LEMMA 1.4. The inverse limit of a sequence $\{\langle A_n, \pi_n \rangle\}_{n=1}^\infty$ of non-empty finite sets is non-empty.

2. Monotonic spaces. We shall consider the following monotonic properties of sequences of subsets of a given space X :

- (md) $\bigcap_{n=1}^\infty B_n \ni x$, then $\{B_n\}_{n=1}^\infty$ is a base at x ,
- (mp) $\bigcap_{n=1}^\infty B_n \ni x$ and $\delta\mathfrak{W} < \{B_n\}_{n=1}^\infty$, then $\bigcap \{\bar{A} : A \in \mathfrak{W}\} \neq \emptyset$,
- (mc) $\delta\mathfrak{W} < \{B_n\}_{n=1}^\infty$, then $\bigcap \{\bar{A} : A \in \mathfrak{W}\} \neq \emptyset$,
- (l) there exists an $x \in X$ such that if U is open and $x \in U$, then, for a certain n , $B_n \subseteq U$.

⁽³⁾ This method is often used in the papers of H. H. Wicke and J. M. Worrell, Jr. We can prove that (a) and (d) are equivalent and deduce Corollaries 1.2 and 1.3 without using this method but this would be a technical proof, and without the equivalence of (a) and (c) we would not obtain the equivalence of our theory with the theory of Wicke and Worrell.

⁽⁴⁾ A space X is κ_a -Lindelöf if every open cover of X has a subcover of cardinality not greater than κ_a .

DEFINITION 2.1. A space X is said to be *monotonically developable* (a *monotonic p -space*, a *monotonically Čech complete space*, a space with a λ -base) if it has an (md) -sieve $((mp)$, (mc) , (λ) -sieve respectively) ⁽⁵⁾.

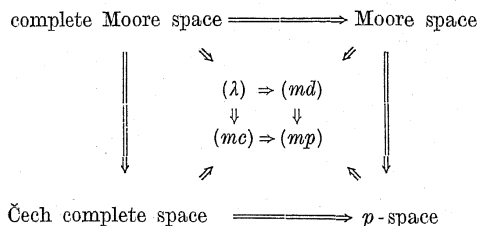
Note that, by virtue of Corollary 1.2, X has a (λ) -base if and only if X is monotonically developable and monotonically Čech complete.

For a given space X and a monotonic property (m) it is much easier to construct an (m) -sieve than to construct a corresponding monotonically contracting sequence [22]. For example the following propositions are obvious (compare with [32] and [23, Theorem 7.3])

PROPOSITION 2.2. If a space X has, locally, one of the properties listed in Definition 2.1, then X has that property.

PROPOSITION 2.3. Each property listed in Definition 2.1 is hereditary with respect to closed subsets and the property of being monotonically developable is hereditary with respect to arbitrary subsets.

The connections between the monotonic and non-monotonic properties are illustrated in the following diagram:



All the implications are obvious. To show that the monotonic properties do not imply the corresponding non-monotonic properties we shall give an example of a space X which has a λ -base and is not a p -space.

EXAMPLE 2.4. Let $Y = \{a: a < \omega_1\} \times \{a: a \leq \omega_0\}$. Obviously Y is locally compact. Let

$$X = Y \setminus \{a+1: a < \omega_1\} \times \{\omega_0\}.$$

The space X is locally metrizable in a complete manner; hence, by virtue of Proposition 2.2, X has a λ -base. To show that X is not a p -space it is sufficient to prove that X does not have any feathering in Y . Suppose that X has a feathering in Y ; then the set L of the limit ordinals in $V = \{a: a < \omega_1\}$ has a feathering in V . Assume that $\{U_n\}_{n=1}^{\infty}$ is a feathering

of L in V and each U_n consists of intervals. If there exists a $v \in L$ such that for every n $v \in \text{St}(\mu_n, U_n)$ for a certain limit ordinal $\mu_n > v$, then for $\mu = \min\{\mu_n\}_{n=1}^{\infty}$ we have $v+1 \in \bigcap_{n=1}^{\infty} \text{St}(\mu, U_n)$. Thus there exists an uncountable subset L' of L and n' such that for every $v \in L'$ and every limit ordinal $\mu > v$ we have $v \in \text{St}(\mu, U_{n'})$, but this is, of course, a contradiction.

Before giving conditions under which monotonic properties imply corresponding non-monotonic properties we introduce some new monotonic properties.

DEFINITION 2.5. Let $C \subseteq X$. We say that C is a W_δ -set in X if it has a (W) -sieve in X , where $\{B_n\}_{n=1}^{\infty} \in (W)$ iff $\bigcap_{n=1}^{\infty} B_n \subseteq C$ [26]. We say that C has a *monotonic feathering* in X if it has an (mf) -sieve in X , where $\{B_n\}_{n=1}^{\infty} \in (mf)$ iff $\bigcap_{n=1}^{\infty} B_n \cap C \neq \emptyset$ implies that $\bigcap_{n=1}^{\infty} B_n \subseteq C$ [22].

Let us notice that, by virtue of Lemma 1.1, these properties are monotonic equivalents of the property of being a G_δ -subset and having a feathering respectively.

DEFINITION 2.6. A sequence $\{B_n\}_{n=1}^{\infty}$ satisfies (Δ) iff $\bigcap_{n=1}^{\infty} B_n$ contains at most one point.

Since, by virtue of Lemma 1.1, if the diagonal Δ is a W_δ -set in $X \times X$, then we can choose a (W) -sieve from a base $\{U \times U: U \text{ open in } X\}$ of Δ in $X \times X$, we get (compare with [8])

PROPOSITION 2.7. Let $\Delta \subseteq X \times X$ be the diagonal; then Δ is a W_δ -set in $X \times X$ iff X has a (Δ) -sieve.

THEOREM 2.8. Let X be a θ -refinable ⁽⁶⁾ space; then

- (a) X being monotonically developable implies that X is a Moore space,
- (b) X being a monotonic p -space implies that X is a p -space,
- (c) X having a W_δ -diagonal implies that X has a G_δ -diagonal,
- (d) F being a closed W_δ -subset of X implies that F is a G_δ -subset of X .

Part (a) of Theorem 2.8 is proved in [32], and part (b) is announced in [23], Theorem 7.4. It can be noticed that the proof from [32] can be applied to all cases listed in Theorem 2.8. This, roughly speaking, is due

⁽⁶⁾ A space X is said to be θ -refinable if for each open cover \mathcal{U} of X there exists a sequence $\{U_n\}_{n=1}^{\infty}$ of open covers of X refining \mathcal{U} such that for each $x \in X$ one of the covers U_n contains only a finite number of elements containing x [32]. Clearly meta-compact (weakly paracompact) spaces are θ -refinable, and also subparacompact (F_δ -screenable) and hence Moore spaces are θ -refinable [32].

⁽⁵⁾ These classes were investigated, for example, in [22]. We introduce here a new terminology: in particular, a space is monotonically developable iff it has a base of countable order (see [1] and [32]). For the sake of simplicity we shall not consider spaces which are called in [22] β_c and λ_c -spaces. All our results can be extended to these spaces.

to the fact that we consider properties which contain all sequences with the empty intersection ^(*).

Theorem 2.8 does not hold for the properties (1) and (mc) even if X is assumed to be a metacompact Moore space. To construct an example we shall put together a method from [28] and a method from [19]. Let us notice that the example, constructed in [13, Theorem 9], of a locally completely metrizable Moore space which is not Čech complete is locally separable and therefore is not metacompact [20].

EXAMPLE 2.9. We shall construct a metacompact Moore space X which is locally completely metrizable but not Čech complete.

Let $V = \{a: a < \omega_1\}$ and let Y be a maximal family of monotonically increasing functions from the set N of natural numbers into V such that $y, y' \in Y$ implies $y(N) \cap y'(N)$ is a finite subset of V . We shall introduce a topology for the set $X = Y \cup V''$, where V'' is a set of all ordered pairs (α, φ) satisfying $\alpha \in V$ and φ is a finite subset of Y .

For $y \in Y$ and $n \in N$ let

$$\begin{aligned} D(y, n) &= y(\{m \in N: m \geq n\}) \subseteq V, \\ U(y, n) &= \{y\} \cup \{(\alpha, \varphi) \in X: y \in \varphi \text{ and } \alpha \in D(y, n)\} \quad \text{and} \\ \mathcal{U}_n &= \{U(y, n): y \in Y\} \cup \{(v): v \in V''\}. \end{aligned}$$

It is now easy to check that the sequence $\{\mathcal{U}_n\}_{n=1}^\infty$ of point finite coverings of X is a development for the set X with the topology introduced by a base $\mathfrak{B} = \bigcup_{n=1}^\infty \mathcal{U}_n$. Clearly X is metacompact, completely regular and locally completely metrizable. It remains to show that X is not Čech complete.

Suppose that X is Čech complete and let $\{\mathfrak{H}_k\}_{k=1}^\infty$ be a sequence of coverings of X such that $\mathfrak{H}_k \subseteq \mathfrak{B}_k = \bigcup_{n=k}^\infty \mathcal{U}_n$ and if $H_k \in \mathfrak{H}_k$; then $\{H_k\}_{k=1}^\infty$ satisfies (mc). For each k let

$$D_k = \{D(y, n): U(y, n) \in \mathfrak{H}_k\}.$$

We shall show that $D = \bigcap_{k=1}^\infty \bigcup D_k \subseteq V$ is non-empty. If this is not the

case, then we can find for each $\alpha \in V$ a number $k(\alpha)$ such that $\alpha \notin \bigcup D_{k(\alpha)}$ and consequently there exists a monotonically increasing function z from N into V and a number k_0 such that $z(N) \cap \bigcup D_{k_0} = \emptyset$. From the assumption of the maximality of Y it follows that for a certain $y \in Y$ the set $y(N) \cap z(N)$ is infinite and it is easy to check that $y \notin \bigcup \mathfrak{H}_{k_0}$. The contradiction shows that D is non-empty. Let β be a fixed element of D . For each k we can find a point $y_k \in Y$ and a number $n_k \geq k$ such that $\beta \in D(y_k, n_k)$ and $U(y_k, n_k) = H_k \in \mathfrak{H}_k$. It suffices to prove that $\{H_k\}_{k=1}^\infty$ does not satisfy (mc). To do this, let us notice that the conditions imposed on n_k and y_k ensure that the set $\{y_k\}_{k=1}^\infty$ is infinite. Hence the family $\{F_k\}_{k=1}^\infty$, where $F_k = \{(\beta, \varphi_j): j \geq k\}$ and $\varphi_j = \{y_i\}_{i \leq j}$, has the empty intersection. On the other hand, each F_k is closed in X and $F_k \subseteq H_k$. Thus X is not Čech complete.

It is easy to notice that the space X constructed above is an open finitely multiple image of a complete metric space; hence Example 2.9 solves Problem 7.14 of [16].

We shall show in the next section (Remark 3.9) that in the class of paracompact spaces the monotonic properties are equivalent to the corresponding non-monotonic properties.

Let us finish this section with some easy propositions.

PROPOSITION 2.10. *If a sequence $\{B_n\}_{n=1}^\infty$ is strongly decreasing, then $\{B_n\}_{n=1}^\infty$ satisfies (mc) iff $B = \bigcap_{n=1}^\infty B_n$ is a non-empty compact set and $\{B_n\}_{n=1}^\infty$ is a base at B , $\{B_n\}_{n=1}^\infty$ satisfies (mp) iff $\bigcap_{n=1}^\infty B_n$ is empty or $\{B_n\}_{n=1}^\infty$ satisfies (mc).*

PROPOSITION 2.11 [26]. *A completely regular space X is monotonically Čech complete iff X is W_δ in some (or, equivalently, in each) of its compactifications.*

Proposition 2.11 and Example 2.9 show that Theorem 2.8(d) is not true for arbitrary subsets of X . The remark following the proof of Lemma 1.1 allows us to prove that if X is a Moore space and $A \subseteq X$ is a W_δ -set in X , then A is a G_δ -set in X [26] (see also [13]).

PROPOSITION 2.12 [22]. *A completely regular space X is a monotonic p -space iff it has a monotonic feathering in some (or, equivalently, in each) of its compactifications.*

PROPOSITION 2.13. *All the properties listed in Definition 2.1 are hereditary with respect to W_δ -subsets.*

PROPOSITION 2.14 ([23], Theorem 4.1). *Every closed subset of a monotonically developable space X is a W_δ -subset of X .*

(*) For the sake of completeness we shall indicate here a method of proving Theorem 2.8 which seems to be more natural than that from [32]. We shall outline the proof of part (a), and the same method can be applied in the other cases. Let $\{\mathfrak{B}_n\}_{n=1}^\infty$ be an (md)-sequence of bases of X . We shall construct a development $\{\mathcal{U}(t)\}_{t \in T}$, where T denotes a family of finite sequences of natural numbers. Let $\mathcal{U}(\emptyset) = \{X\}$ and assume that $\mathcal{U}(t)$ is defined for $t \in N^m$; then take $\{\mathcal{U}(t, k)\}_{k=1}^\infty$ as a θ -refinement of $\mathfrak{B}(t)$, where $\mathfrak{B}(t)$ is an arbitrary covering of X contained in \mathfrak{B}_{m+1} and refining $\mathcal{U}(t)$. From Lemma 1.4 it follows that $\{\mathcal{U}(t)\}_{t \in T}$ is a development of X .

3. Mappings and monotonic spaces. Let us first introduce a notion of monotonically complete mappings (compare with [11], [23] and [22]).

DEFINITION 3.1. A mapping $f: X \rightarrow Y$ is said to be *monotonically complete* if X has an (mcf) -sieve, where (mcf) is a monotonic property such that $\{B_n\}_{n=1}^\infty$ satisfies (mcf) iff for each $y \in Y$ a sequence $\{B_n \cap f^{-1}(y)\}_{n=1}^\infty$ satisfies (mc) in $f^{-1}(y)$.

Obviously every mapping on a monotonically Čech complete space is monotonically complete and every compact mapping⁽⁸⁾ is monotonically complete.

From Corollary 1.2 we easily get

THEOREM 3.2 ([25], [26], [22]). *Let $f: X \rightarrow Y$ be an open and monotonically complete mapping of X onto Y ⁽⁹⁾. If X has one of the properties listed in Definition 2.1, then Y has that property. If $A \subseteq X$ is W_δ in X , then $f(A)$ is W_δ in Y .*

THEOREM 3.3 (compare with [25], Theorem 7, and [9]). *If $f: X \rightarrow Y$ is an open (or closed) monotonically complete mapping of a monotonic p -space X onto a monotonically Čech complete space Y , then X is a monotonically Čech complete space.*

Proof. To show that X is monotonically Čech complete we have to find an (mp) -sieve on X such that each thread of that sieve has the non-void intersection.

Let \mathcal{G} be a strong (mc) -sieve on Y and let (m) be a monotonic property such that a sequence $\{B_n\}_{n=1}^\infty$ satisfies (m) iff for a certain thread $\{G_{\alpha_n}\}_{n=1}^\infty$ of \mathcal{G} we have $\delta\{f(B_n)\}_{n=1}^\infty < \{G_{\alpha_n}\}_{n=1}^\infty$. It is easy to see that f induces an (m) -sieve on X . By Corollary 1.2 X has a strong sieve \mathcal{G}' satisfying simultaneously the conditions (m) , (mcf) and (mp) . It is easy to check that the closedness of f implies that each thread of \mathcal{G}' has the non-void intersection. If f is open, we have to construct a strong sieve \mathcal{G}'' on X satisfying, in addition, a certain non-monotonic condition, namely for each thread $\{G_{\alpha_n}\}_{n=1}^\infty$ of \mathcal{G}'' the sequence $\{f(G_{\alpha_n})\}_{n=1}^\infty$ has to be strongly decreasing. This can be done by induction since, by Lemma 1.1, X has an (m) , (mcf) and (mp) -sequence of bases and f is an open mapping.

Let us recall that a perfect image of a Moore space (Čech complete space) is a Moore space (Čech complete space) [29] ([12], Problem 3.Y) and a perfect image of a p -space need not be a p -space [28]. For monotonic properties we have the following

THEOREM 3.4. *All the properties listed in Definition 2.1 are invariant under perfect mappings.*

⁽⁸⁾ A mapping is said to be *compact* if all inverses of points are compact.

⁽⁹⁾ One can assume that the restriction of f to a certain $X' \subseteq X$ such that $f(X') = Y$ is open and monotonically complete.

It is proved in [31] that the perfect image of a monotonically developable space is monotonically developable. The invariantness of the other monotonic properties is announced in [27] and [22].

Proof. Let us first assume that $f: X \rightarrow Y$ is a perfect mapping of a monotonic p -space X onto a space Y . We shall prove that Y is a monotonic p -space.

By Lemma 1.1 X has an (mp) -sequence $\{\mathcal{B}_n\}_{n=1}^\infty$ of bases. Hence we can construct a sieve $\mathcal{G} = \{\langle \mathcal{G}_n, A_n, \pi_n \rangle\}_{n=1}^\infty$ on Y such that $A_n \subseteq Y \times \{n\}$, and for each $(y, n) \in A_n$ there exists a finite subfamily $\mathcal{B}(y, n)$ of \mathcal{B}_n such that the following conditions are satisfied:

- 1) $G_{(y,n)} = Y \setminus f(X \setminus \bigcup \mathcal{B}(y, n))$,
- 2) $y \in G_{(y,n)}$,
- 3) $B \in \mathcal{B}(y, n)$ implies $B \cap f^{-1}(y) \neq \emptyset$,
- 4) if $\pi_n(y_{n+1}, n+1) = (y_n, n)$, then the closures of elements of the family $\mathcal{B}(y_{n+1}, n+1)$ refine $\mathcal{B}(y_n, n)$.

Now assume that $\{G_{(y,n)}\}_{n=1}^\infty$ is a thread of \mathcal{G} , $y_0 \in Y$ is contained in its intersection and $\delta \mathfrak{A} < \{G_{(y,n)}\}_{n=1}^\infty$ for a certain centred family \mathfrak{A} . We have to show that $\bigcap \{\bar{A} : A \in \mathfrak{A}\} \neq \emptyset$. Since each $\mathcal{B}(y_n, n)$ is finite, we can deduce from 1 that for some $B \in \mathcal{B}(y_n, n)$ the family $\{B\} \cup \{f^{-1}(A) : A \in \mathfrak{A}\}$ is centred. By Lemma 1.4 there exists a sequence $\{B_n\}_{n=1}^\infty$, where $B_n \in \mathcal{B}(y_n, n) \subseteq \mathcal{B}_n$ and $\bar{B}_{n+1} \subseteq B_n$; such that $\mathfrak{A}' = \{B_n\}_{n=1}^\infty \cup \{f^{-1}(A) : A \in \mathfrak{A}\}$ is centred. Since $\{B_n\}_{n=1}^\infty$ satisfies the condition (mp) , the proof will be finished if we show that the intersection of $\{B_n\}_{n=1}^\infty$ is non-empty.

Using conditions 1 and 4 we can find, applying Lemma 1.4, a sequence $\{W_n\}_{n=1}^\infty$ such that $f^{-1}(y_0) \cap W_n$ is non-empty, $W_n \in \mathcal{B}(y_n, n)$ and $\bar{W}_{n+1} \subseteq W_n$. Thus $W = \bigcap_{n=1}^\infty W_n$ is non-empty and by virtue of Proposition 2.10 the set $Z = W \cup \bigcup_{n=1}^\infty (\bar{W}_n \cap f^{-1}(y_n))$ is compact. It is easy to check that condition 3 implies that each B_n intersects the compact set $f^{-1}(f(Z))$. Hence $\bigcap_{n=1}^\infty \bar{B}_n = \bigcap_{n=1}^\infty B_n \neq \emptyset$ and the proof is finished.

Using the same method, we can prove that a perfect image of a monotonically developable space is monotonically developable⁽¹⁰⁾.

In the case of monotonically Čech complete spaces the proof is simple, for we need not show that $\bigcap_{n=1}^\infty B_n \neq \emptyset$.

Since a space has a λ -base if and only if it is monotonically de-

⁽¹⁰⁾ The proof from [31] does not involve the regularity of X but it cannot be extended to monotonic p -spaces.

velopable and monotonically Čech complete, the proof of Theorem 3.4 is complete ⁽¹¹⁾.

One can easily prove the following two propositions.

PROPOSITION 3.5. *Let $f: X \rightarrow Y$ be a perfect mapping of X onto Y . If $f^{-1}(A)$ is W_δ in X , then A is W_δ in Y .*

PROPOSITION 3.6. *The inverse image of a monotonically Čech complete space (a monotonically p -space) under a perfect mapping is a monotonically Čech complete space (a monotonically p -space).*

Let us recall a mapping characterization for monotonic spaces [22] (see also [25], [21] and [27]).

THEOREM 3.7. *For an arbitrary space X there exist a metric space M , a space $X' \subseteq X \times M$ and an open monotonically complete mapping f' of X' onto X such that*

- (a) *if X is monotonically developable, then X' is metrizable,*
- (b) *if X has a λ -base, then X' is metrizable in a complete manner,*
- (c) *if X is a monotonically p -space, then X' is a paracompact p -space,*
- (d) *if X is a monotonically Čech complete space, then X' is a paracompact Čech complete space.*

From part (d) of Theorem 3.7 and Theorem 8 from [18] we get

COROLLARY 3.8. *If X is a paracompact monotonically Čech complete space, then X is Čech complete.*

Remark 3.9. Theorem 2.8 together with a metrization theorem from [4], Theorem 4.3.11 from [12] and Corollary 3.8 imply that for a paracompact space X we have

- (a) *X is monotonically developable iff X is metrizable [1],*
- (b) *X has a λ -base iff X is metrizable in a complete manner [25],*
- (c) *X is a monotonically p -space iff X is a p -space,*
- (d) *X is monotonically Čech complete iff X is Čech complete.*

4. Applications. The theory of monotonic spaces allows us to understand the real sense of some well-known theorems. For example from Proposition 2.2 and Remark 3.9 we can deduce that a locally metrizable paracompact space is metrizable [12, Problem 5. J], and a locally Čech complete paracompact space is Čech complete [12, Problem 5. P]. From Theorem 3.4 and the invariantness of paracompactness under closed mappings [12, Problem 5. B], it follows that a perfect image of a metrizable space is metrizable [12, Problem 4. S].

Using Theorem 2.8, Theorem 3.4 and the fact that metacompactness (subparacompactness) is invariant under closed mappings [30] ([5]), we get a generalization of a theorem from [14] (see also [28]).

⁽¹¹⁾ Our method allows us to show that a perfect image of a Hausdorff space with a λ -base has a λ -base.

THEOREM 4.1. *A perfect image of a metacompact (subparacompact) p -space is a metacompact (subparacompact) p -space.*

Let us notice that by virtue of Proposition 2.7 and Lemma 1.1 a monotonic p -space with a W_δ -diagonal is monotonically developable [24]. This, together with Theorem 2.8, generalizes Theorem 3.3 from [15]. We shall give another, more general method of proving this fact.

THEOREM 4.2. *Let \mathfrak{Q} be a class of spaces such that if $X' \subseteq X \times M$, where X is from \mathfrak{Q} and M is a metric space, then X' belongs to \mathfrak{Q} . If paracompact p -spaces from \mathfrak{Q} are metrizable, then monotonic p -spaces from \mathfrak{Q} are monotonically developable.*

Proof. Theorem 4.2 follows directly from Theorem 3.2 and Theorem 3.7(c).

The following classes are hereditary with respect to arbitrary subsets and closed under the product with a metric space:

- (a) spaces with a point-countable base,
- (b) spaces with a point-countable separating open cover ⁽¹²⁾,
- (c) quasi-developable spaces,
- (d) spaces with a G_δ -diagonal,
- (e) spaces with a W_δ -diagonal.

Theorem 4.2 generalizes a number of theorems. For example, for spaces with a point-countable base we get a generalization of Theorem 2.7 from [7] and Theorem 2.10 from [6].

By virtue of Theorems 3.2 and 3.4 we can apply our results to the investigation of classes $\text{OCP}(\mathfrak{M})$ defined in [16] as minimal classes closed under open compact and perfect mappings and containing the class \mathfrak{M} ⁽¹³⁾. Namely each space in $\text{OCP}(\text{Moore spaces})$ is monotonically developable and each space in $\text{OCP}(p\text{-spaces})$ is a monotonic p -space. Thus Theorem 2.8 is a generalization of Theorems 6.1 and 6.4 from [16] and gives a positive answer to Problem 7.15 from [16].

The following theorem can be used in the proof of Theorem 4.1 from [16] ⁽¹⁴⁾.

THEOREM 4.3 (compare with [10]). *If $f: X \rightarrow Y$ is an open monotonically complete mapping from a monotonic p -space X onto Y , then f is compact-covering ⁽¹⁵⁾.*

⁽¹²⁾ A covering \mathfrak{U} of X is said to be *separating* if for distinct points $x, x' \in X$ there exists a $U \in \mathfrak{U}$ such that $x \in U$ and $x' \notin U$.

⁽¹³⁾ As in [16], we assume that all mappings in the definition of $\text{OCP}(\mathfrak{M})$ have a regular domain and a regular range.

⁽¹⁴⁾ Theorem 4.3 together with Theorem 3.7 give a solution of Problem 7.16 from [16].

⁽¹⁵⁾ A mapping $f: X \rightarrow Y$ is said to be *compact-covering* if for each compact subset K of Y there exists a compact subset Z of X such that $f(Z) = K$.

Proof. Obviously we can assume that Y is compact and it suffices to show that $f(Z) = Y$ for a certain compact subset Z of X . From Theorems 3.3 and 3.7(d) it follows that there exist a paracompact Čech complete space X' and an open mapping f' of X' onto X . By virtue of Theorem 1.2 from [3] we can find a compact subset Z' of X' such that $f'(f''(Z')) = Y$. The compact set $Z = f'(Z') \subseteq X$ satisfies the condition $f(Z) = Y$.

From Theorem 3.4 and Proposition 2.10 we get (compare with Theorem 5.4 in [16])

PROPOSITION 4.4. *If X is a monotonic p -space, then X is of countable type.*

Corollary 1.3 implies

PROPOSITION 4.5. *If a monotonically developable space X is κ_α -Lindelöf, then X has a base of cardinality not greater than κ_α .*

PROPOSITION 4.6. *If a monotonic p -space X has a net of cardinality κ_α , then X has a base of cardinality not greater than κ_α .*

Proof. By virtue of Corollary 1.3, X has a strong (mp) -sieve $\mathfrak{G} = \{\langle \mathfrak{G}_n, A_n, \pi_n \rangle\}_{n=1}^\infty$ such that each A_n is of cardinality not greater than κ_α . Since a space M from Theorem 3.7 can be constructed as the inverse limit of a sequence $\{\langle A_n, \pi_n \rangle\}_{n=1}^\infty$, where each A_n is a discrete space, $X \times M$ has a net of cardinality not greater than κ_α . Thus X is an open image of a paracompact p -space X' which has a net of cardinality not greater than κ_α . From [2] X' has a base of cardinality not greater than κ_α and the proof is finished.

Propositions 4.5 and 4.6 give a positive answer to Problems 7.13 and 7.10 from [16].

References

- [1] A. V. Arhangel'skiĭ, *Certain metrisation theorems*, Usp. Mat. Nauk 18 (1963), pp. 139–145 (Russian).
- [2] — *A class of spaces which contains all metric and all locally compact spaces*, Mat. Sb. (N. S.) 67 (1965), pp. 55–88 (Russian).
- [3] — *Open and close-to-open mappings. Relations among spaces*, Tr. Mosk. Mat. Obšč. 15 (1966), pp. 181–223 (Russian).
- [4] R. H. Bing, *Metrisation of topological spaces*, Can. J. Math. 3 (1951), pp. 175–186.
- [5] D. K. Burke, *On subparacompact spaces*, Proc. Amer. Math. Soc. 23 (1969), pp. 655–663.
- [6] — *On p -spaces and ωA -spaces*, Pac. Journ. of Math. 35 (1970), pp. 285–296.
- [7] — and R. A. Stoltenberg, *A note on p -spaces and Moore spaces*, Pac. Journ. of Math. 30 (1969), pp. 601–607.
- [8] J. G. Ceder, *Some generalizations of metric spaces*, Pac. Journ. of Math. 11 (1961), pp. 105–125.
- [9] J. Chaber, *Remarks on open-closed mappings*, Fund. Math. 74 (1972), pp. 197–208.

- [10] M. M. Čoban, *Multivalued mappings and some associated problems*, Dokl. Akad. Nauk SSSR 190 (1970), pp. 293–296 (Russian).
- [11] — *On the theory of p -spaces*, Dokl. Akad. Nauk SSSR 194 (1970), pp. 528–531 (Russian).
- [12] R. Engelking, *Outline of General Topology*, Amsterdam 1968.
- [13] M. E. E. Still, *Concerning abstract spaces*, Duke Math. Journ. 17 (1950), pp. 317–327.
- [14] V. V. Filippov, *On the perfect image of a paracompact p -space*, Dokl. Akad. Nauk SSSR 176 (1967), pp. 533–535 (Russian).
- [15] D. E. Kullman, *Developable spaces and p -spaces*, Proc. Amer. Math. Soc. 27 (1971), pp. 154–160.
- [16] K. Nagami, *Minimal class generated by open compact and perfect mappings*, Fund. Math. 78 (1973), pp. 227–264.
- [17] Ch. M. Pareek, *Characterizations of p -spaces*, Canad. Math. Bull. 14 (1971), pp. 459–460.
- [18] B. Pasynkov, *Open mappings*, Dokl. Akad. Nauk SSSR 175 (1967), pp. 292–295 (Russian).
- [19] F. D. Tall, *A counterexample in the theories of compactness and of metrisation*, Indag. Math., to appear.
- [20] D. R. Traylor, *Concerning metrizable of pointwise paracompact Moore spaces*, Canad. J. Math. 16 (1964), pp. 407–411.
- [21] H. H. Wicke, *The regular open images of complete metric spaces*, Pac. Journ. of Math. 23 (1967), pp. 621–625.
- [22] — *Open continuous images of certain kinds of M -spaces and completeness of mappings and spaces*, Gen. Top. Appl. 1 (1971), pp. 85–100.
- [23] — *Base of countable order theory and some generalizations*, Proc. of the Univ. of Houston Point Set Top. Conf. (1971), pp. 76–95.
- [24] — *On spaces whose diagonal is a set of interior condensation*, Notices Amer. Math. Soc. 19 (1972), p. A. 657.
- [25] — and J. M. Worrell, Jr., *Open continuous mappings of spaces having bases of countable order*, Duke Math. Journ. 34 (1967), pp. 255–271.
- [26] — — *On the open continuous images of paracompact Čech complete spaces*, Pac. Journ. of Math. 37 (1971), pp. 265–276.
- [27] — — *On topological completeness of first countable Hausdorff spaces I*, Fund. Math. 75 (1972), pp. 209–222.
- [28] J. M. Worrell, Jr., *A perfect mapping not preserving the p -space property*, forthcoming.
- [29] — *Upper semicontinuous decompositions of developable spaces*, Proc. Amer. Math. Soc. 16 (1965), pp. 485–490.
- [30] — *The closed continuous images of metacompact topological spaces*, Port. Math. 25 (1966), pp. 175–179.
- [31] — *Upper semicontinuous decompositions of spaces having bases of countable order*, Port. Math. 26 (1967), pp. 493–504.
- [32] — and H. H. Wicke, *Characterizations of developable topological spaces*, Canad. J. Math. 17 (1965), pp. 820–830.

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