On the equivalence of certain coincidence theorems
and fixed point theorems

by
T. B. Muenzenberger and R. E. Smithson (Laramie, Wyoming)

Abstract. The properties that are preserved when certain multifunctions on trees are composed are used to show that some coincidence theorems for multifunctions on compact spaces into trees are equivalent to some classical fixed point theorems on trees. These methods are also used to derive some new coincidence theorems.

1. Introduction. In [3] Helga Schirmer proved a multipart coincidence theorem for certain multifunctions into trees and derived some fixed point theorems from this theorem. In section 2 of this paper we prove the fixed point theorem by using the order structure of a tree and then by applying this theorem and two classical fixed point theorems on trees we obtain the coincidence theorem referred to above. This accomplishes two purposes. First it provides a new proof of the coincidence theorem, and secondly it shows that the coincidence theorem and the fixed point theorems are equivalent.

In [4] Schirmer extended results of Schweigert [5] and Ward [9] on homeomorphisms and monotone maps which leave an endpoint fixed to a coincidence theorem for u.c. biconnected multifunctions. In section 3 of this paper we prove some results for biconnected (but not necessarily u.c.) multifunctions which leave an endpoint fixed and use these to prove a fixed point theorem for biconnected multifunctions of the Schweigert-Ward type which was a corollary of Schirmer’s theorem. We then use this theorem to prove Schirmer’s coincidence theorem. A corollary to the results on biconnected multifunctions is a slightly improved version of a theorem of Smithson [7]. Finally, in the last section we use these methods to derive two new coincidence theorems.

A multifunction \( F: X \to Y \) is a point to set correspondence (i.e., \( F(x) \) is a nonempty subset of \( Y \) for each \( x \in X \)). If \( A \subseteq X \), then \( F(A) = \bigcup \{ F(x) : x \in A \} \) and if \( B \subseteq Y \), then \( F^{-1}(B) = \{ x : F(x) \cap B \neq \emptyset \} \) where \( \emptyset \) is the empty set. The multifunction \( F \) is said to be upper semi-continuous (u.s.c.) in case \( F^{-1}(A) \) is closed for each closed \( A \subseteq Y \). Also \( F \) is lower semi-continuous (l.s.c.) in case \( F^{-1}(V) \) is open for each open \( V \subseteq Y \) and \( F \) is continuous in case it is both u.s.c. and l.s.c. Further, \( F \) is point closed
(connected, etc.) in case \( F(x) \) is closed (connected, etc.). A multifunction \( F \) is connected in case \( F(A) \) is connected for each connected \( A \subset X \), and \( F \) is monotone in case \( F^{-1}(y) \) is connected for each \( y \in Y \). Finally, \( F \) is biconnected in case \( F \) and \( F^{-1} \) are connected.

A well known result that we shall need later is that each u.s.c. or l.s.c. point connected multifunction is connected.

A tree is a continuum \( X \) in which each distinct pair of points in \( X \) can be separated by the removal of a third point. A partial order, \( \preceq \), is defined on \( X \) as follows: Let \( e \in X \) be fixed, then \( x \preceq y \) if \( e = y \) or \( e \) separates \( x \) and \( y \). Now set \( L(x) = \{ y \in X : y \preceq x \} \) and \( M(x) = \{ y \in X : x \preceq y \} \). Then Ward [10] characterized trees in terms of this partial order as follows:

**Lemma 1.1.** A continuum \( X \) is a tree iff it admits a partial order \( \preceq \) satisfying:

(i) \( L(x) \) and \( M(x) \) are closed for every \( x \in X \),

(ii) if \( x \preceq y \), then there is a \( z \in X \) such that \( x \preceq z \preceq y \),

(iii) \( L(x) \cap L(y) \) is nonempty and totally ordered for each \( x, y \in X \),

(iv) \( M(x) \setminus \{ x \} \) is open for each \( x \in X \).

A chain is a totally ordered subset of a partially ordered set. If \( X \) is a tree, the element \( e \) used in the definition of the partial order is the least element in \( X \), and in the sequel we refer to the above partial order as the partial order on \( X \) with least element \( e \), and will assume each tree is endowed with this partial order. A point \( x \in X \) (a tree) is an endpoint in case \( X \preceq x \) is connected. Some further properties of this partial order are:

**Lemma 1.2.** Every nonempty closed subset of \( X \) contains a maximum, and each nonempty chain has a least upper bound.

**Proof.** See [9, Th. 1].

**Lemma 1.3.** Every nonempty subcontinuum of \( X \) contains a least element.

**Proof.** See [1, Lemma 2].

**Lemma 1.4.** If \( A \subset X \) is connected, then \( A \cap M(y) \neq \emptyset \), and if \( A \cap L(x) \neq \emptyset , \) then \( y \in A \).

**Proof.** See [7, Lemma 4].

If \( F : X \to Y \) is a multifunction on a set \( X \) into itself, then \( F \) has a fixed point in case there exists an \( x \in X \) such that \( x \in F(x) \). If \( F, G : X \to Y \) are two multifunctions on a set \( X \) into a set \( Y \), then \( F, G \) have a coincidence point (or simply a coincidence) in case there exists an \( x \in X \) such that \( F(x) \cap G(x) \neq \emptyset \). A multifunction \( F : X \to Y \) is coincidence producing for a family of multifunctions in case \( F \) has a coincidence with every member of the family.

### 2. Coincidences and fixed points of multifunctions

In [3] Schirmer proved the following theorem.

**Theorem A.** Any point closed u.s.c. multifunction \( F : Z \to X \) from a compact, Hausdorff space \( Z \) onto a tree \( X \) is coincidence producing for all multifunctions \( G : Z \to X \) which are either continuous or u.s.c. and point connected if it is either open or monotone.

By permuting the hypotheses we see that there are four parts to Theorem A. We shall show that two of these can be derived from two well known fixed point theorems. The other two parts we shall obtain by giving a separate proof of Theorem 5.2 of [3] which was obtained from Theorem A. Then prove Theorem A by using this theorem. For this we shall need the following lemma whose proof is omitted (this lemma is similar to Lemma 5.1 in [3]).

**Lemma 2.1.** Let \( F, G : X \to Y \) be two multifunctions on a set \( X \) into \( Y \) with \( G \) onto. Then \( F, G \) have a coincidence iff \( F \circ G^{-1} : Y \to Y \) has a fixed point.

Each multifunction \( F : X \to Y \) can be considered to be the following subset of \( X \times X : F = \{(x, y) : y \in F(x), x \in X\} \).

Using this interpretation of \( F \) we obtain the following lemma.

**Lemma 2.2.** Let \( F : X \to Y \) be a point closed, u.s.c. multifunction on the space \( X \) into the regular space \( Y \). Then \( F \) is a closed subset of \( X \times Y \).

**Proof.** Let \( (x, y) \in F \). Then since \( F(x) \) is closed and \( Y \) is regular there exist disjoint open sets \( U, V \) in \( Y \) with \( y \in U \), \( F(x) \subseteq V \). Further, since \( F \) is u.s.c., there exists an open set \( W \) containing \( x \) such that \( F(W) \subseteq V \). Thus \( W \times U \) is open and contains \( (x, y) \) and \( x \times U \subseteq \emptyset \).

Now we can prove the following lemma.

**Lemma 2.3.** Let \( F : X \to Y \) be an u.s.c., point closed multifunction on a compact, \( T_2 \)-space onto a regular space \( Y \). Then \( F^{-1} \) is point closed and u.s.c.

**Proof.** Let \( A \subset X \) be closed and \( \pi_1, \pi_2 \) be projections on \( X \times Y \) onto \( Y \) respectively. Then \( \pi_2 \circ \pi_1^{-1} : \pi_1(A) \to \pi_2(A) \) which is closed since the projection parallel to a compact space is closed.

**Corollary 2.4.** If \( F : X \to Y \) is u.s.c. and if both \( X, Y \) are compact, \( T_2 \)-spaces, then \( F \) is u.s.c. and point closed iff \( F^{-1} \) is u.s.c. and point closed.

It is also easy to show that the composition of point closed u.s.c. multifunctions on compact \( T_2 \)-spaces is point closed and u.s.c. and that the composition of l.s.c. multifunctions is l.s.c.

For the remainder of this section assume that each multifunction is point closed, that \( Z \) is a compact \( T_2 \)-space and that \( X \) is a tree.
In 1941 Wallace ([12] showed that a tree had the fixed point property for point connected u.s.c. multifunctions. Now if $F: Z \to X$ is a u.s.c., monotone multifunction on the compact $T_0$-space $Z$ onto the tree $X$, then $F^{-1}$ is u.s.c. and connected, and if $G$ is point closed, point connected and u.s.c., then $G \circ F^{-1}: X \to X$ is point closed, point connected and u.s.c. Thus $G \circ F^{-1}$ has a fixed point and hence, $F$, $G$ have a coincidence.

In 1956 Plunkett ([2] showed that trees have the fixed point property for continuous (point closed) multifunctions. Thus let $F: Z \to X$ be point closed, u.s.c. and open. Then $F^{-1}$ is continuous and point closed, and if $G: Z \to X$ is continuous, then $G \circ F^{-1}: X \to X$ is continuous. Hence $G \circ F^{-1}$ has a fixed point and so $F$, $G$ have a coincidence.

Now in order to prove the remaining two parts of the Theorem we prove the following two fixed point theorems.

**Theorem 2.9.** Let $F: X \to X$ be a multifunction on the tree $X$ itself and suppose there exists a compact $T_0$-space $Z$ and multifunctions $F_1: X \to Z$ and $F_2: Z \to X$ such that $F_1$ is continuous, $F_2$ is point connected and u.s.c. and $F = F_2 \circ F_1$. Then $F$ has a fixed point.

**Theorem 2.10.** In Theorem 2.9 let $F_1$ be point connected and u.s.c. and $F_2$ continuous. Then $F$ has a fixed point.

In order to prove these theorems we need the following two lemmas. The proof of Lemma 2.7 is routine and is omitted. Lemma 2.8 follows from the fact that the partial order $\lesssim$ is a closed subset of $X \times X$ (see Ward [10]) and from the fact that the graph of $M$ is the graph of $\lesssim$ and that $M(z)$ is closed for each $z \in X$.

**Lemma 2.7.** If $F, G: X \to Y$ are two point closed, u.s.c. multifunctions on the space $X$ into the normal space $Y$, then the set $\{x \in X : G(x) \cap \mathcal{M}(z) \neq \emptyset\}$ is closed.

**Lemma 2.8.** Let $X$ be a tree. Then the multifunction $M: X \to X$ defined by $M(x) = \{y \in X : \exists x \lesssim y \}$ is point closed and u.s.c.

**Proof of Theorem 2.9.** The set $E = \{x \in X : M(x) \cap F(x) \neq \emptyset\}$ is closed by Lemmas 2.7 and 2.8. Let $x_0 \in E$ and assume that $x_0 \not\in F(x_0)$. We shall show that $E$ contains an $x_1$ with $x_0 < x_1$. Let $y_0 \in F(x_0) \cap M(x_1)$ and let $x_0 \not\in F(x_1)$ be such that $y_0 \not\in F(x_1)$. Now $F(x_1)$ is connected and contains $y_0$ and so contains a minimal element $y_1$. Pick $x' \in X$ such that $x_0 < x_1 < x'$. Then $F(x_1)$ is contained in the open set $M(x' \lesssim \cdot)$. Thus $F(x_1)$ contains an open set $U \subset Z$ such that $x_0 \not\in F(x_1) \cap F(U) \subset M(x' \lesssim \cdot)$. Also, there is an open set $V \subset X$ with $x_0 \not\in V$, such that $F(x_1) \cap U \subset F(V) \subset M(x' \lesssim \cdot)$. Since $F$ is closed, $F$ contains a maximal element, and thus $F$ has a fixed point.

**Proof of Theorem 2.10.** As above the set $E$ is closed and let $x_0 \in E$. Assume that $x_0 \not\in F(x_0)$, and let $y \in F(x_0) \cap M(x_0)$. Since $F(x_0)$ is closed there is an $x_1, x_2 < x_0 < y$, such that $\{x_2 : x_2 \preceq x \} \cap F(x_2) = \emptyset$. Now $F(x_0)$ is a closed, connected subset of $Z$. Let

$$A = \{x \in F_1(x_0) : F_1(x) \cap M(x_0) \not\supset x_0 \}.$$ 

Since $F_1$ is u.s.c., $A$ is a relatively open subset of $F_1(x_0)$, and since $F_1$ is u.s.c. and $x_0 \in F_1(x_0)$, $A$ is a closed subset of $F_1(x_0)$. Thus $A = F_1(x_0)$. Hence, there is an open subset $U \subset Z$ such that $F_1(x_0) \cap U$ and $x \in U$ implies that $F_1(x_0) \cap M(x_0) \not\supset x_0$. Thus there is an open set $V \subset X$, $x_0 \in V$, such that $F_1(V) \subset U$. Thus there is a $x_1 \not\in E$ and $x_2 < x_0$, and so above $F$ has a fixed point.

Now to complete the proof of Theorem A observe that if $F$ is u.s.c. and open and $G$ u.s.c. and point connected, then Theorem 2.9 applies to $G \circ F^{-1}$ and so by Lemma 2.1 $F$ and $G$ have a coincidence. Finally if $F$ is u.s.c. and monotone and $G$ is continuous, then Theorem 2.6 applies.

3. Biconnected multifunctions. In this section we prove some results about biconnected multifunctions on a tree which leave an endpoint fixed. We then use these results to prove the main theorem of [4]. We develop the proof of these results through a series of lemmas. In this section $X$ denotes a tree and $e \times X$ is the minimal element for the partial order $\lesssim < X$. If $x < y$, we set $[x, y] = \{z : x \leq z \leq y\}$.

**Lemma 3.1.** Let $F$ be a point closed, connected multifunction on $X$. If $x < y$ implies $F(x) \subset M(y)$, and if $x_0 \not\in F(x_0)$, then $F(x_0) \subset M(y_0)$.

**Proof.** Suppose $F(x_0) \cap (X \setminus M(y_0)) = \emptyset$ and let $y = \min F(x_0)$. Then $z_0 = \sup F(x_0) \cap L(y)$ and let $x_0 < x_0 < z_0 = \max F(x_0) < y$. Since $F$ is connected, $A = F([x_0, z_0])$ is connected and meets $M(y_0)$ and contains $y$. Thus if $x < x' < z_0 < x_0$, $x \not\in A$. But $x_0 < x < \max F(x_0)$ implies that $F(x) \subset M(y_0)$ and so $x \not\in F(x_0)$ and so we conclude that $F(x_0) \subset M(y_0)$.

**Lemma 3.2.** Let $F: X \to X$ be a point closed, monotone, connected multifunction which does not have a fixed point. If $F(x_0) \subset M(x_0)$, then there exists a $x_1 \in F(x_0) \cap M(x_0)$ such that $x_1 < x \not\in F(x_0)$ implies $x \not\in M(x_0)$.

**Proof.** Let $y_0 = \min F(x_0)$ and let $x_0 < y_0 < y_1$. Then let $x_0 < y_0 < y_1$. Then $y_0 \not\in F(x_0)$ and $F(y_1) \subset M(y_0)$. Then $F([x_0, x_1])$ is a connected set which meets $M(y_0)$ and its complement. Thus $F^{-1}(y_0) \cap [x_0, x_1] = \emptyset$. Similarly $F^{-1}(y_1) = \emptyset$. Since $F$ is monotone, $x \not\in F^{-1}(y_0)$ for all $x \in [x_0, x_1]$. Thus, since $F$ is monotone, $x \not\in F^{-1}(y_1)$ for all $x \in [x_0, x_1]$. As a corollary to Lemmas 3.1 and 3.2 we get the following strengthened version of the main result of [7].
Corollary 3.3. Each connected, point closed, monotone multifunction on a tree into itself has a fixed point.

Before stating the main theorem of this section, we need one more lemma.

Lemma 3.3. Let \( F : T \to T \) be connected and monotone, and let \( s_0 \in T \) be such that \( F(s_0) \subseteq M(s_0) \). If \( F \) does not have a fixed point in \([s_0, s_T]\), then \( F^n(s) \subseteq M(s) \) for all \( s \in L(s_0) = [s_0, s_T] \).

Proof. Let \( s = (s \in L(s_0); F(s) \subseteq M(s)) \). If \( s \neq \emptyset \), then \( \text{lub} \ s = s_T \) exists in \( L(s_0) \). First suppose that \( F(s) \subseteq M(s) \). But then \( F([s_0, s_T]) \) is connected and meets both \( M(s) \) and its complement and hence, contains \( s_0 \). This implies that \( s_0 \in F(s) \). Thus \( F(s_0) \subseteq M(s_0) \), and let \( s < s' < s_0 \) (note \( s_0 \neq \emptyset \) as \( s \neq \emptyset \)) with \( s \in S \). The set \( F([s, s_0]) \) is connected and meets \( M(s) \) and its complement. Hence, \( F^{-1}(s) \subseteq [s, s_0] \neq \emptyset \). Since \( F \) is monotone, there is an element \( s' \in [s, s_0] \) such that \( s' < s < s_0 \). This contradicts either the hypothesis of the lemma or the choice of \( s_0 = \text{lub} \ s \). Thus \( S = \emptyset \), and the lemma follows.

We are now ready to state the main theorem of this section. (In [4] Theorem 3.4 appeared as a corollary to the main theorem.)

Theorem 3.4. Let \( F : T \to T \) be a point closed, point connected, monotone, u.s.c. multifunction on \( T \) onto \( T \). If \( s \) is an endpoint, if \( s \in F(s) \), and if \( F(s) \neq \emptyset \), then \( F \) has a fixed point not \( s \).

Proof. First note that the hypotheses on \( F \) imply that \( F \) is biconnected, and that both \( F \) and \( F^{-1} \) are point closed. Next we assert that there is an \( s < t < s_0 \) in \( T \) such that either \( F(s) \subseteq M(s) \) or \( F^{-1}(s) \subseteq M(s) \) or \( F \) has a fixed point not \( s \). For this let \( s_0 \) be a maximal element of \((\text{lub} \ L(s_0)) \) and let \( s_1 \in T \) be such that \( s_1 \in F(s_0) \). Then \( s_1 = \text{lub} \ L(s_1) = L(s_0) \). Suppose that \( s_1 \neq s_T \), and let \( s_2 = \max \ L(s_0) \). If \( s_1 = s_T \), we are done. Thus suppose \( s_1 \neq s_T \) and that \( F(s_0) \subseteq M(s_0) \). Then \( F([s_0, s_T]) \) meets \( M(s_0) \) and its complement. Hence, there is an \( x \in [x_0, x_T] \) such that \( x \in F(s_0) \) but then \( x \in F^{-1}(s_0) \) and so \( s_0 \) is the required fixed point or \( F^{-1}(s_0) \subseteq M(s_0) \).

For convenience suppose that \( F(s_0) \subseteq M(s_0) \) for some \( s' \neq s_0 \). Then by Lemma 3.3 if \( s < t < s' \), \( s' \subseteq F(s) \subseteq M(s) \). Thus let \( s = (s \in L(s_0); s' \subseteq F(s) \subseteq M(s)) \). Then \( s = \text{lub} \ L(s_0) \), which satisfies (i) \( [s, s'] \subseteq S \) (ii) \( L(s_0) \subseteq \emptyset \) and \( t \in s < s' \); then \( F(s) \subseteq M(s) \). Then \( s_0 = \text{lub} \ L(s_0) \) and, by Lemma 3.2 the maximality of \( s_0 \) and \( s_0 \) is a fixed point, and \( s_T \neq s_0 \).

Remark. The properties of \( F \) used in the proof of Theorem 3.4 are that \( F \) is biconnected and both \( F \) and \( F^{-1} \) are point closed. However, Smithson showed in [6] that these conditions imply that \( F \) is u.s.c.

We now apply this Theorem 3.4 to obtain Schirmer's result [4].

Theorem 3.5 (Schirmer). Let \( F, G : T \to T \) be two point closed monotone connected, u.s.c. multifunctions on a tree \( T \) onto the tree \( T \) which have an endpoint \( e \) of \( T \) as a coincidence. If both \( F(e) \neq T \neq G(e) \) and if either \( F(e) \neq T \) or \( G(e) \) is a singleton, then \( F \) and \( G \) have a coincidence distinct from \( e \).

Proof. Suppose that \( F(e) = \{e\} \) is a singleton. If \( G^{-1} \cdot F : T \to T \) satisfies, \( G^{-1} \cdot F(e) \neq T \), then Theorem 3.4 applies, and \( G^{-1} \cdot F \) has a nonempty fixed point \( e \). Then \( F \) and \( G \) have a coincidence not \( e \). Now suppose \( G^{-1} \cdot F(e) = T \). Then \( G^{-1} \cdot F(e) = T \) so \( e \neq F(t) \) for all \( t \), and if \( e \) is not an endpoint of \( T \), there exists a \( t \in T \) such that \( e \neq F(t) \) and \( t \neq e \). Hence, in this case, \( F \) and \( G \) have a coincidence not \( e \). Now assume that \( e \) is an endpoint of \( T \) and \( G^{-1} \cdot F(e) = T \). If \( F^{-1}(e) \) is not a singleton, then there exists a \( t \in T \), \( t \neq e \), such that \( e \neq F(t) \) and \( t \in T \) is a coincidence of \( F \) and \( G \). Thus assume \( e \neq F^{-1}(e) \). Then \( G \cdot F^{-1}(e) \neq T \), and Theorem 3.4 applies as above.

4. We conclude this paper by using the methods of the previous sections to prove two new coincidence theorems.

In [11] Ward proved that each u.s.c. continuum valued multifunction on a treewise connected, hereditarily unicoherent, continuum into itself has a fixed point. From this we get:

Theorem 4.1. Let \( X \) be an hereditarily unicoherent treewise connected metric continuum and let \( Z \) be a compact, \( T_0 \) space. If \( F : Z \to X \) is a monotone, point closed, u.s.c. multifunction onto \( X \), then \( F \) has a coincidence with every u.s.c. continuum valued multifunction \( G : Z \to X \).

Proof. The multifunction \( G \cdot F^{-1} : X \to X \) is u.s.c. and continuum valued and therefore has a fixed point. Hence, \( F \) and \( G \) have a coincidence.

Note that Theorem 4.1 implies Ward's theorem and so the fixed point theorem and the coincidence theorem are equivalent.

In [5] Smithson proved that every l.s.c. point connected multifunction on a tree into itself has a fixed point. Hence, we get:

Theorem 4.2. Let \( F : T \to T \) be an open, monotone multifunction on the space \( T \) into the tree \( T \). Then \( F \) is coincidence producing for every l.s.c. point connected multifunction \( G : X \to T \).

Finally note that, as above, Theorem 4.2 is equivalent to the fixed point theorem used in its proof.

References
Square bracket partition relations in $L$

by

Richard A. Shore (Chicago, Ill.)

Abstract. We settle some questions about partition relations proposed by Erdős and Hajnal under the assumption that $V = L$: 1) $\kappa \rightarrow \left[\kappa\right]^{\omega}_{\alpha}$ if and only if $\kappa \rightarrow \left[\kappa\right]^{\omega}_{\alpha}$ for all ordinals $\alpha$ and integers $\omega$.

Introduction. Our goal in this paper will be to settle some of the open questions about partition relations proposed by Erdős and Hajnal [1] under the assumption that $V = L$. We will do this by combining some combinatorial lemmas with known results (particularly those of Jensen) on the fine structure of $L$. We begin by defining the square bracket partition relation $\kappa \rightarrow \left[\kappa\right]^{\omega}_{\alpha}$ of Erdős, Hajnal and Rado [2] for $\kappa$, $\gamma$, $\delta$ and $\alpha$ cardinals:

For every function $F: [\kappa]^\omega \rightarrow \alpha$ (called a partition of $[\kappa]^\omega$, the size $\delta$ subsets of $\kappa$ into $\alpha$ pieces) there exists a set $C \subseteq \kappa$ (called homogeneous for $F$) such that $\bar{U} = \gamma$ and $F''[\bar{U}] \neq \alpha$, i.e., some element of the range is omitted when $F$ is restricted to the size $\delta$ subsets of $C$. If this is not the case we write $\kappa \rightarrow \left[\kappa\right]^{\omega}_{\alpha}$.

These partition relations naturally split into two quite different types according to whether $\delta$ is infinite or finite. In the former case most of the relations are known to contradict the axiom of choice. Thus for example, $\text{AC}$ implies that $\kappa \rightarrow \left[\kappa\right]^{\omega}_{\alpha}$ for every $\kappa$ (Erdős and Rado). The more important and interesting questions about these partition relations are therefore concerned with finite $\delta$. In particular, the most fundamental question seems to be to determine the relationships among various apparent strengthenings and weakenings of the simplest non-trivial relation $\kappa \rightarrow \left[\kappa\right]^{\omega}_{\alpha}$ (the well-known equivalent of weak compactness).

When one first considers this problem it seems that increasing the superscript should strengthen the relation while increasing the subscript should weaken it. At it turns out the strengthening achieved by increasing the superscript is illusory (as long as it remains finite). Indeed one can prove that $\kappa \rightarrow \left[\kappa\right]^{\omega}_{\alpha}$ if $\forall x < \omega \ (\kappa \rightarrow \left[\kappa\right]^{\omega}_{\alpha})$ using an alternate formulation of $\kappa \rightarrow \left[\kappa\right]^{\omega}_{\alpha}$ in terms of trees.

On the other hand, increasing the subscript does have some real effect. $\kappa \rightarrow \left[\kappa\right]^{\omega}_{\alpha}$ is strictly weaker than $\kappa \rightarrow \left[\kappa\right]^{\omega}_{\alpha}$ in that there are singular