Maximally almost periodic groups and varieties of topological groups

by

Sidney A. Morris (Kensington, N.S.W.)

Abstract. It has been shown by Kakutani, Nakayama and Gelbaum that if $\underline{V}$ is the variety of all topological groups or the variety of all abelian topological groups and X is a Tychonoff space, then the free topological group $F(X, \underline{V})$ is maximally almost periodic (MAP). It is shown here that for any non-abelian (respectively, abelian) variety $\underline{V}$, $F(X, \underline{V})$ exists and is MAP for each Tychonoff space X if and only if $\underline{V}$ contains an arcwise connected non-abelian (respectively, abelian) MAP group. This clearly implies the previously known results and also yields: If $\underline{V}$ is a variety containing a connected compact non-abelian group, then $F(X, \underline{V})$ is MAP for each Tychonoff space X. It is also shown that if for some connected space X, $F(X, \underline{V})$ is MAP, then the underlying variety of groups of $\underline{V}$ is either the class of all groups or the class of all abelian groups.

Preliminaries. A non-empty class $\underline{V}$ of topological groups (not necessarily Hausdorff) is said to be a *variety* if it is closed under the operations of taking subgroups, quotient groups, arbitrary cartesian products and isomorphic images. (See [2] and [9]-[16].)

We note that a variety $\underline{V}$ (of topological groups) determines a variety $\overline{\underline{V}}$ of groups [18]; the latter is simply the class of all groups which with some topology appear in the former.

The smallest variety containing a class Ω of topological groups is said to be the *variety generated by* Ω and is denoted by $\underline{V}(\Omega)$.

If $\underline{V}$ is a variety, X is a topological space and F is a member of $\underline{V}$, then F is said to be a *free topological group of* $\underline{V}$ *on* X, denoted by $F(X, \underline{V})$, if it has the properties:

(a) X is a subspace of F.

(b) X generates F algebraically.

(c) For any continuous mapping γ of X into any group H in $\underline{V}$, there exists a continuous homomorphism Γ of F into H such that $\Gamma | X = \gamma$.

The following results on free topological groups are proved in [9]: (i) $F(X, \underline{V})$ is unique (up to isomorphism) if it exists, (ii) $F(X, \underline{V})$ exists if and only if there is a member of $\underline{V}$ which has X as a subspace, (iii) $F(X, \underline{V})$ is the free group of the underlying variety $\overline{\underline{V}}$ of groups on the set X.

A topological group G is said to be *maximally almost periodic* (MAP) if there exists a continuous monomorphism of G into a compact group. (See [4].)

Results. Our first theorem extends Theorem 2.5 (iv) of [2].

THEOREM 1. *If G is a connected* MAP *group then $\overline{\underline{\underline{V}}(G)}$ is either the variety of all groups or all abelian groups.*

Proof. Let f be a continuous monomorphism of G into a compact group H. Without loss of generality we can assume $f(G)$ is dense in H, and hence H is connected. We note that since f is a monomorphism, G and $f(G)$ satisfy precisely the same laws [18]. Further since $f(G)$ is dense in H, any law of $f(G)$ is a law of H; the converse statement is trivially true. Thus G and H satisfy the same laws.

It is clear from [1], that H satisfies either no (non-trivial) laws or just the commutative law. Thus $\overline{\underline{\underline{V}}(H)}$, which is also $\overline{\underline{\underline{V}}(G)}$, is either the class of all groups or all abelian groups.

THEOREM 2. *Let $\underline{\underline{V}}$ be a variety such that for some non-totally disconnected space X, $F(X, \underline{\underline{V}})$ is* MAP. *Then $\underline{\underline{V}}$ is either the class of all groups or all abelian groups.*

Proof. Let Y be a connected subspace of X with more than one element. For any element y in Y, the subgroup G of $F(X, \underline{\underline{V}})$ algebraically generated by $y^{-1}Y$ is a connected MAP group. In view of Theorem 1, it will suffice to show that $\overline{\underline{\underline{V}}(G)} = \underline{\underline{V}}$.

Noting that Y is a connected Tychonoff space we see that it contains at least $2^{\aleph_0}$ elements. Since the subgroup H of $F(X, \underline{\underline{V}})$ algebraically generated by Y is a free group of $\underline{\underline{V}}$ on the set Y, we see by 15.62 of [18] that $\overline{\underline{\underline{V}}(H)} = \underline{\underline{V}}$. It is also readily seen that G and H satisfy the same laws and thus $\overline{\underline{\underline{V}}(G)} = \overline{\underline{\underline{V}}(H)} = \underline{\underline{V}}$ and the proof is complete.

The following example shows that the condition "non-totally disconnected" cannot be removed in the above theorem.

EXAMPLE. Let Ω be the class of all discrete nilpotent groups of class c, for some positive integer c. Then by 17.75 and 32.22 of [18], for any discrete space X, $F\big(X, \underline{\underline{V}}(\Omega)\big)$ is MAP. However $\overline{\underline{\underline{V}}(\Omega)}$ is not the class of all groups or all abelian groups.

THEOREM 3. *For any non-abelian (respectively, abelian) variety $\underline{\underline{V}}$, $F(X, \underline{\underline{V}})$ exists and is* MAP *for each Tychonoff space X if and only if $\underline{\underline{V}}$ contains a non-abelian (respectively, abelian) arcwise connected* MAP *group.*

Proof. Let X be an arcwise connected space. If $F(X, \underline{\underline{V}})$ exists and is MAP then the subgroup G of $F(X, \underline{\underline{V}})$ algebraically generated by $x^{-1}X$, for some x in X, is an arcwise connected MAP group. Further, if $\underline{\underline{V}}$ is non-abelian then so too is G.

Conversely, let G be an arcwise connected MAP group in $\underline{\underline{V}}$ (where G is non-abelian if $\underline{\underline{V}}$ is). If X is any Tychonoff space, then Lemma 5, p. 116 of [7] implies that X can be embedded in a product of copies of G and therefore $F(X, \underline{\underline{V}})$ exists. Since G is MAP, there exists a continuous monomorphism of G into a compact group. Therefore, to show that $F(X, \underline{\underline{V}})$ is MAP, we only have to find for each $a \in F(X, \underline{\underline{V}})$, $a \neq e_1$, a continuous homomorphism Γ of $F(X, \underline{\underline{V}})$ into G such that $\Gamma(a) \neq e$, where e_1 and e are the identity elements of $F(X, \underline{\underline{V}})$ and G, respectively.

By Theorem 1, G satisfies only those laws which define $\underline{\underline{V}}$. This, together with the fact that $F(X, \underline{\underline{V}})$ is the free group of $\underline{\underline{V}}$ on the set X, implies that there exists a (not necessarily continuous) homomorphism Φ of $F(X, \underline{\underline{V}})$ into G such that $\Phi(a) \neq e$. Let $a = x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}$, where $x_i \in X$ and ε_i is an integer for $i = 1, \ldots, n$. By Theorem 3.6 of [5], there exists a continuous map γ of X into G such that $\gamma(x_i) = \Phi(x_i)$ for $i = 1, \ldots, n$. Therefore there exists a continuous homomorphism Γ of $F(X, \underline{\underline{V}})$ into G such that $\Gamma | X = \gamma$. Clearly $\Gamma(a) = \Phi(a) \neq e$ and the proof is complete.

COROLLARY. *If $\underline{\underline{V}}$ is any variety containing a connected compact non-abelian group, then for each Tychonoff space X, $F(X, \underline{\underline{V}})$ is* MAP.

Proof. This is a consequence of Theorem 3 and the fact (§ 4.6 of [8]) that any connected compact non-abelian group has a quotient which is a compact connected non-abelian Lie group (which is of course arcwise connected).

Open question. If $\underline{\underline{V}}$ is a variety containing a connected non-abelian MAP group, is $F(X, \underline{\underline{V}})$ (necessarily) a MAP group for each Tychonoff space X. (Indeed, does $F(X, \underline{\underline{V}})$ necessarily exist?)

References

[1] S. Balcerzyk and J. Mycielski, *On the existence of free subgroups in topological groups*, Fund. Math. 44 (1957), pp. 303–308.

[2] Su-shing Chen and S. A. Morris, *Varieties of topological groups generated by Lie groups*, Proc. Edin. Math. Soc. 18 (1972), pp. 49–53.

[3] B. R. Gelbaum, *Free topological groups*, Proc. Amer. Math. Soc. 12 (1961), 737–743.

[4] S. Grosser and M. Moskowit, *Compactness conditions in topological groups*, J. Reine Angew, Math. 246 (1971), pp. 1–40.

[5] E. Hewitt and K. A. Ross, *Abstract harmonic analysis I*, New York 1963.

[6] S. Kakutani, *Free topological groups and infinite direct product topological groups*, Proc. Imp. Acad. Tokyo 20 (1944), pp. 595–598.

[7] J. L. Kelly, *General topology*, New York 1955.

[8] D. Montgomery and L. Zippin, *Topological transformation groups*, New York 1955.

[9] S. A. Morris, *Varieties of topological groups*, Bull. Austral. Math. Soc. 1 (1969), pp. 145–160.
[10] — *Varieties of topological groups II*, Bull. Austral. Math. Soc. 2 (1970), pp. 1–13.
[11] — *Varieties of topological groups III*, Bull. Austral. Math. Soc. 2 (1970), pp. 165–178.
[12] — *Varieties of topological groups generated by maximally almost periodic groups*, Fund. Math. (to appear).
[13] — *On varieties of topological groups generated by solvable groups*, Coll. Math. 25 (1972), pp. 225–226.
[14] — *Locally compact abelian groups and the variety of topological groups generated by the reals*, Proc. Amer. Math. Soc. 34 (1972), pp. 290–292.
[15] — *Varieties of topological groups generated by solvable and nilpotent groups*, Coll. Math. 27 (1973), pp. 211–213.
[16] — *Locally compact groups and β-varieties of topological groups*, Fund. Math. 78 (1973), pp. 23–26.
[17] T. Nakayama, *Note on free topological groups*, Proc. Imp. Acad. Japan 19 (1943), pp. 471–475.
[18] H. Neumann, *Varieties of groups*, Ergebnisse der Mathematik unter ihrer Grenzgebiete, Band 37, Berlin 1967.

UNIVERSITY OF NEW SOUTH WALES
Kensington, N. S. W., Australia

Reçu par la Rédaction le 22. 5. 1972

Set existence principles of Shoenfield, Ackermann, and Powell

by

W. N. Reinhardt (Colorado)

Abstract. The author proposes a formalization of an informal set existence principle of Shoenfield. Some consequences of the axioms are developed and comparisons are made with other axiomatic theories which have been proposed. The author also makes some general remarks about the problem of axiomatic principles in mathematics.

Introduction. Shoenfield has formulated the following principle $\mathcal{S}$ for the existence of sets. The principle assumes that sets are built up in cumulative stages, and that there is an ordering on the stages as they are built up.

$\mathcal{S}$ If P is a property of stages, and if we can *imagine* a situation in which all the stages having P have been built up, then there *exists* a stage s beyond all the stages which have P.

We remark at the outset that one can read $\mathcal{S}$ in either (i) a more or (ii) a less constructive way, namely (i) that the stage s exists mathematically because of (or in) the act of imagination, which is thus a sort of construction of s, or (ii) that what can be imagined is but an indication of what has mathematical existence, so that the latter can retain a certain changeless Platonic impregnability or Cantorian absoluteness. It is (ii) which seems appropriate to this author in the context of classical set theory. (Evidently this does not preclude consideration of processes, constructions, and the like, only they are not to be regarded as more primitive than existence.)

Although $\mathcal{S}$ is certainly vague, Shoenfield has used it rather convincingly to derive a number of the usual axioms of set theory [10]. The purpose of this paper is to propose a formalization of this principle, (§ 1, § 5) and to deduce some of its consequences, the most striking being the existence of measurable cardinals (see § 5, Theorem 5.12). The formalization proposed will bear a close relation to two other set theories, one due to Ackermann and one to Powell, (see § 2, § 5 Remark 5.13 and § 6). In a sense, adding arbitrary properties of sets to Ackermann's theory yields measurable cardinals (see 5.13).