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**Theorem 3.** Let the function $g$ fulfills hypothesis (H) and let conditions (3)–(8) be fulfilled for $f$ defined by (28). Then $g$ has a completely monotonic iteration group if and only if condition (C) is fulfilled (for $f$ defined by (28)).

**Example 3.** Let

$$g(x) = (2-x)x-x^2$$

for $x \in (0, 1)$. The function $g$ is completely monotonic. Suppose that there exists a completely monotonic function $X$ such that

$$X(x) = g(x).$$

Then $\varphi(x) = 1-x-x(1-x-x)$ is an absolutely monotonic solution of (24), where

$$f(x) = 1-x-g(1-x-x) = x^2 + x^2.$$

But this is impossible (cf. Example 2). Therefore equation (30) with $g$ given by (29) has no completely monotonic solution.

Theorem 3 and Example 3 answer in the negative U. T. Bödevadt’s conjecture [1] that for a completely monotonic $g$ the equation

$$\varphi^{(n)}(x) = g(x)$$

always has a unique completely monotonic solution for every positive integer $n$.

**References**


*Reçu par la Edition le 29/9/1972*
space, which as a set can be identified with $\mathbb{E}^n$, is however not metrizable
and thus is not a normed linear space. Our idea of proving Theorem 1 is to
define a topology $\mathcal{T}$ on $\mathbb{E}^n$ such that: 1) the "Henderson map"
f: $X \times (\mathbb{E}^n, \mathcal{T}) \to (\mathbb{E}^n, \mathcal{T})$ remains still a homeomorphism (h), and 2) the
space $(\mathbb{E}^n, \mathcal{T})$ is homeomorphic to a normed linear space.

The proof that $X \times X_n \simeq X + \mathbb{E}^n$. We shall fix here a normed linear
space $\mathbb{E}^n$ and a retraction $r: \mathbb{E}^n \to X \subset \mathbb{E}$ which is regular in the
metric induced by $\parallel \cdot \parallel$. Define

$$u(\lambda) = \sup \{ \parallel r(\delta \xi) - \delta \parallel : \xi \in X, \parallel \xi \parallel < \lambda \}, \ \lambda \in (0, \infty)$$

Because of the regularity of $r$, $u$ is a non-decreasing function with
$\lim u(\lambda) = 0$; in particular $u$ is bounded by $1$ on an interval $[0, \delta) \subset [0, 1]$.

If we set

$$K = \text{conv} \{ \{ (\lambda, \mu) \in [0, \delta) \times [0, 1] : \mu \leq u(\lambda) \} \cup \{ 1 \} \times \{ 0, 1 \} \}$$

and

$$u(\lambda) = \begin{cases} \sup \{ \mu : (\lambda, \mu) \in K \} & \lambda \in [0, 1], \\ \frac{1}{1 + \lambda} & \lambda \in (1, \infty), \end{cases}$$

then $u$ will satisfy the conditions

(0) $u(\lambda) \leq (\lambda, 1)$ if $\lambda \in [0, 1]$ and $u(\lambda) \leq \lambda$ if $\lambda \geq 1$,

(1) $\parallel r(\delta \xi) - \delta \parallel \leq u(\parallel \xi \parallel)$ for all $\xi \in X$ and $t \in \mathbb{E}^n$ with $\parallel \xi \parallel < \delta$.

Setting finally $w = u$, $u^{t+1} = u * u^t$ and $v = \sum_{i < \alpha} u^t$, we get a homeomorphism $v$ of $[0, \infty)$ onto itself such that

$$v = \sum_{i < \alpha} u^t \leq 2v$$

and, because $v(0) = 0$ and $v$ is a concave function,

$$v(\lambda) \leq \sum_{i < \alpha} v(\lambda_i) \quad \text{and} \quad v(2\lambda) \leq 2v(\lambda), \quad \lambda, \lambda_1, \lambda_2 \in [0, \infty)$$

Now let us define Henderson's [8] map $f: X \times \mathbb{E}^n \to \mathbb{E}^n$; here $\mathbb{E}^n$
denotes the set $\{ (\xi, v) \in \mathbb{E}^n : \xi = 0 \text{ for almost all } \xi \}$. (and at the moment)
no topology on $\mathbb{E}^n$ or on $X$ is considered. We agree that, from now till

(1) We shall write $f: X \times \{ Y, \mathcal{T} \} \to (\mathbb{E}, \mathcal{T})$ (respectively $f: X \times \{ Y, \mathcal{T} \} \to (\mathbb{E}, \mathcal{T})$)
if we want to stress the topologies (resp. the metrics) in which the spaces in question
are treated. The space $X$ will be always treated in the topology induced by $\parallel \cdot \parallel$.

the end of the next section, $t = (t_1)$ and $s = (s_2)$ will stand for the
two points lying in $E^\alpha$. Set

$$f(\xi, t) = (\xi + t_1, \xi + t_2 - r(\xi + t_1))$$

It is easy to see that $f$ has the map

$$s \mapsto (g(s), \gamma(s))$$

as inverse, where in the definition above

$$g(s) = s_1 \quad \text{and} \quad \gamma(s) = s_{n+1} + r(s)$$

and $g_\infty = \lim g(s)$.

Proposition 1. The formula

$$q(t) = \sum_{i < \alpha} 2^{-\eta_i} \quad \text{and} \quad d(s, t) = g(s-t), \quad s, t \in \mathbb{E}^n$$
defines a metric $d$ on $\mathbb{E}^n$, and both $f$ and $g$ are continuous when considered
as maps between $X \times (\mathbb{E}, d)$ and $(\mathbb{E}, d)$.

Proof. The metric inequality follows immediately from (3). Given
$t \in \mathbb{E}^n$ let us now denote $m(t) = \inf \{ j : t_j = 0 \text{ if } j \geq j \}$.
We have

$$\| x + x \| \leq \| x \| + \| x \|$$

whenever $x, y \in \mathbb{E}^n$ and $\xi_i \in \mathbb{E}^n$ satisfy $\| \xi_i \| < \delta$. Hence

$$\| x + y \| \leq \| x \| + \| y \|$$

because of the inequalities (2) and (3). Since moreover

$$q(t-s) \geq \sum_{i < \alpha} (\| x_i \| - \| s_i \|) \geq \sum_{i < \alpha} (\| t_i \| - \| s_i \|) \text{ if } q(t-s) \leq 1$$

we infer, for every fixed $(x, t) \in X \times \mathbb{E}^n$, that $d(f(x, t), y) \leq \tau(x, t)$
where $\tau(x, t)$ is a continuous function of $(x, t)$ and $\tau(x, t) = 0$.
Hence $f$ is continuous at the (arbitrarily given) point $(x, t) \in X \times \mathbb{E}^n$. 

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Now let us consider the function $g$. If $s \in \Sigma E$ and $i_0 \in N$ fulfill $|s| < \delta$ for all $i \geq i_0$, then

\begin{align}
\text{(11)} \quad |g(s) - g_{i_0}(s)| & = |[\sigma_{-i_0} + g_{i_0}(s)] - g_{i_0}(s)| < u(||s||), \quad i \geq i_0, \\
\text{(12)} \quad |g(s) - g_i(s)| & < |s_i + g_{i_0}(s) - g_{i_0}(s)| \\
& < |s_i| + u(||s||) < 2u(||s||), \quad i \geq i_0, \\
\text{(13)} \quad ||g_{i_0}(s) - g_i(s)|| & = ||g_{i_0}(s) - g_i(s)|| \\
& < \sum_{i = i_0}^{n} |g_{i}(s) - g_{i_0}(s)| = 2u(||s||),
\end{align}

Therefore, given another point $t \in \Sigma E$ we get for $i \geq \max(i_0, m(t))$: \begin{align}
\text{(14)} \quad |g(s) - g_i(t)| & = |s_i + g_{i_0}(s) - g_{i_0}(t)| \\
& < |s_i| + \sum_{i = i_0}^{n} |g_{i}(s) - g_{i_0}(s)| + |g_{i_0}(s) - g_{i_0}(t)| \\
& < |g_{i_0}(s) - g_{i_0}(t)| + \sum_{i = i_0}^{n} u(||s||) \\
& < |g_{i_0}(s) - g_{i_0}(t)| + d(s, t).
\end{align}

By (14) and (10), $d(s, t) < \min(\delta, \epsilon/2)$ and $|g_{i_0}(s) - g_{i_0}(t)| < \epsilon/2$, together imply $|g(s) - g(t)| < \epsilon$ for all $i \geq m(t)$: this shows that the functions $g_i$ are equicontinuous at (every point) $t \in \Sigma E$. Therefore $g_{i_0} = \lim_{i \to i_0} g_i$ is a continuous function. Denoting by $\delta$ the product metric on $X \times \Sigma E$ (the sum of the norm $||\cdot||$ on $X$ and of the metric $d$ on $\Sigma E$), we further get for all $s, t \in \Sigma E$ with $d(s, t) < \delta$ (use (10), (11), (3) and (2)):

\begin{align}
\delta(g(s), g(t)) - |g_{i_0}(s) - g_{i_0}(t)| & = \sum_{i = i_0}^{n} 2^i \|g_{i}(s) - g_{i}(t)| + g_{i_0}(s) - g_{i_0}(t))|| \\
& < \sum_{i = i_0}^{n} 2^i \|g_{i}(s) - g_{i}(t)| < \sum_{i = i_0}^{n} 2^i \sum_{i = i_0}^{n} u(||s||) \\
& < 2\delta(s, t).
\end{align}

Hence $g$ is continuous at every point $t \in \Sigma E$, which completes our proof.

Now, the first assertion of Theorem 1 follows from the lemma below:

**Lemma 1.** Let $(E, ||\cdot||)$ be a normed linear space, let $v : [0, \infty) \to [0, \infty)$ be a homeomorphism and let $d$ be the metric on $\Sigma E$ defined by (7). Then the formulas

\begin{align}
\text{(15)} \quad A(t) = \left(2^v(||d||) \frac{t_1}{|d_0|}, \quad \text{where} \quad d = 0,
\end{align}

defines a homeomorphism of $(\Sigma E, d)$ onto $(\Sigma E, ||\cdot||)$ (recall that $||\cdot||$)
\begin{align}
= \sum_{i < 0} |d_i|, \quad t \in \Sigma E.
\end{align}

**Proof.** We have

\begin{align}
||A(s) - A(t)'|| = \sum_{i < 0} 2^v(||d||) \frac{t_1}{|d_0|} - \frac{v(||d||)}{|d_0|} t_0 \\
= \sum_{i < 0} 2^v(||d||) \frac{t_1}{|d_0|} < d(s, t).
\end{align}

Since the map $s \mapsto 2^v(||d||) s_1/s_0^{-1} - \frac{v(||d||)}{|d_0|} t_0$ is continuous, $A$ must be continuous at the given point $t \in \Sigma E$. Moreover

\begin{align}
\text{(16)} \quad A^{-1}(s) = \left(2^{-v}(-s_1) \frac{s_1}{|d_0|}
\end{align}

and

\begin{align}
\delta(A^{-1}(s), A^{-1}(t)) = \sum_{i < 0} 2^v(-v(2^{-v}(-s_1)/s_0 - 2^{-v}(-t_1)/t_0)) \\
= \sum_{i < 0} 2^v(-v(-s_1)) < ||s - t||.
\end{align}

Hence $A^{-1}$ is also a continuous map.

**Proof of the second part of the theorem.** Under our previous notation, let us denote by $Z$ the set $t = (t_1) \in E^\infty$, $\sum_{i < 0} 2^v(||d||) < \infty$. Because of (3), $Z$ is a linear space; we shall consider it under the metric $\delta(s, t) = \delta(s - t)$, where $\delta(s) = \sum_{i < 0} 2^v(||d||)$. Observe that

\begin{align}
\text{(10a)} \quad \delta(s, t) \geq \sum_{i < 0} u(||s_i - t_i||) = \sum_{i < 0} u_i \quad \text{if} \quad \delta(s, t) < 1.
\end{align}

Let $f : X \times Z \to E^\infty$ be the natural extension of the map $f$, defined by the same formula (4).

**Proposition 1a.** The map $f$ has its range contained in $Z$. If moreover $X$ is complete in the norm $||\cdot||$, then $f$ is a homeomorphism of $X \times (Z, \delta)$ onto $(Z, \delta)$.

**Proof.** Given points $(x, t), (y, s) \in X \times Z$ we have for every integer $m$ which is so large that $s_i + u_i < \delta$.
\[ (9a) \quad 2\bar{v}(y, t) = 2\bar{v}(\|y + t e_1 - s e_2\|) - 2\bar{v}(\|y + t e_1 - s e_2\|) - \sum_{k=1}^{m} 2\bar{v}(\|y + t e_1 - s e_2\| + \|y + s e_2 - r(x + s e_2 - i)\|) \]
\[ \leq \sum_{k=1}^{m} 2\bar{v}(\|y + t e_1 - s e_2\| + \|y + s e_2 - r(x + s e_2 - i)\|) \]
\[ \leq \sum_{k=1}^{m} 2\bar{v}(d_k + \|\|e_1\| + \|e_2\|\) + \|\|e_2\| + \|\|e_1\|\) \]
\[ \leq 2 \sum_{k=1}^{m} 2\bar{v}(d_k + \|\|e_1\| + \|\|e_2\|\) \]

Seeing here \( y = y, s = 0, 0, \ldots \) and \( m \) an integer which is large enough we infer that image(\( f \)) \( \subset Z \). If further \( (s, t) \in X \times Z \) and \( (s, t) \in X \times Z \) are fixed, and \( m \) is an integer with \( \sum_{k=1}^{m} 2\bar{v}(d_k) \leq 30 \), then \( \bar{v}(s, t) \leq 30 \) implies
\[ \sum_{k=1}^{m} 2\bar{v}(d_k) \leq \sum_{k=1}^{m} 2\bar{v}(d_k + \|t e_1 - s e_2\|) < 15 \]
and, consequently,
\[ 2\bar{v}(y, t) = 2\bar{v}(\|y + t e_1 - s e_2\|) = \sum_{k=1}^{m} 2\bar{v}(\|y + t e_1 - s e_2\| + \|y + s e_2 - t e_1 - i\|) + \|t e_1 - s e_2\| + \|t e_1 - s e_2\| + \|t e_1 - s e_2\| \]

Thus \( f \) is continuous at the point \((s, t) \in X \times Z\).

Now assume \( X \) to be \( \|\| \)-complete and for each \( \delta > 1 \) denote by \( \bar{v}_\delta : Z \to B \) the natural extension onto \( Z \) of the map \( \bar{v}, \delta \), defined by (6). Given \( s \in Z \) \( \delta \in \mathbb{R} \) is by (11) and (10a) a Cauchy sequence; define \( \bar{v}_\delta(s) = \lim_{\delta \to \infty} \bar{v}_\delta(s) \in X \). By (12) also the sequence \( \{\bar{v}_\delta(s)\} \) converges to \( \bar{v}(s) \).

Arguing as in the proof of Proposition 1 (the necessary changes, similar to those given above, are left to the reader) one shows that all the \( \bar{v}_\delta \)'s are equicontinuous and that the map
\[ \delta \to \bar{v}_\delta(s), \quad \delta \to \bar{v}_\delta(s) - \bar{v}_\delta(s) \in X \times B \]
has its range contained in \( X \times Z \) and is continuous when considered as a map of \((Z, \delta) \) into \( X \times (Z, \delta) \). Since further \( \delta B \) is \( \delta \)-dense in \( Z \) \( \bar{v} \) and \( \delta \) must be inverse to each other.

To finish the proof of Theorem 1 it remains to demonstrate

**Lemma 1a.** Under the notation of Proposition 1a the spaces \((Z, \delta) \) and \((B, \|\|) \) are homeomorphic.

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**Proof.** Let us consider the map \( \pi : Z \to B \) defined by (15). Then \( \|\|\pi(s)||| \leq \bar{v}(s) \leq \delta \) for all \( s \in Z \), and therefore image(\( \pi \)) \( \subset B \). Moreover for every \( s, t \in Z \) and \( m \geq 1 \) we have
\[ \|\|\pi(s) - \pi(t)||| \leq \sum_{k=1}^{m} \|\|\pi(d_k)||| + \|\|\pi(s) - \pi(t)||| \]
\[ \leq \sum_{k=1}^{m} \|\|\pi(d_k)||| + \|\|\pi(s) - \pi(t)||| \]
\[ \leq 2 \sum_{k=1}^{m} \|\|\pi(d_k)||| \]
\[ \leq 2 \sum_{k=1}^{m} \|\|\pi(d_k)||| \]

This easily leads to the conclusion that \( \pi \) is continuous at every point \( \delta \in Z \). Similarly one shows that the map \( \pi : Z \to B \),

\[ \pi(s) = \left( \pi(\|\|\delta e_1\|) \right) \]

is continuous too. Obviously \( \pi \circ \pi = B \circ \pi = \pi \) is the identity.

Let us note that if \( B \) is a Banach space, then the assertion of Lemma 1a follows from more general theorems of Cz. Besaga (3, Proposition 5.3 and § 6); moreover the homeomorphism \( \pi \) we use is very similar to that constructed by S. Mazur in 1929 for \( L_\infty \) spaces. See [3] and the references given there for more information on the subject.

**Remark 1.** Propositions 1 and 1a and their proofs remain valid if we assume only that \( B \) is an additive group and \( \|\| \) is a group norm on \( B \) (i.e. \( \|a\| = 0 \) if \( a = 0 \) and \( \|a + b\| \leq \|a\| + \|b\| \) for \( a, b \in B \)).

**Some applications.** First of all, we have

**Remark 2.** Let \( X \) be a compact absolute retract and let \( (B, \|\|) \) be a normed linear space such that \( X \) can be (topologically) embedded into \( B \). Then \( X \times \sigma B = \sigma B \) and \( X \times \Pi B \cong \Pi B \).

**Proof.** Every retraction onto a compact set is regular. Hence we get

**Theorem 2.** Let \( X \) be a compact absolute retract. Then \( X \times F \cong \sigma F \) in any of the following cases

(a) \( F \) is an infinite-dimensional Fréchet space,

(b) \( F \) is a \( \sigma \)-compact locally convex linear metric space which contains an infinite-dimensional compact convex set,

(c) \( F \) is an infinite-dimensional locally convex linear metric space and both \( X \) and \( F \) are countable unions of finite-dimensional compact sets.

**Proof.** (a) Let us first assume \( F \) is \( l_1 \), the space of summable sequences. Since \( X \) can be topologically embedded into \( l_1 \), we get \( X \times l_1 \cong \Pi l_1 \) and thus \( X \times F \cong F \). If \( F \) is an arbitrary infinite-dimensional
Frohlich space, then, by the theorems of Brelle-Graves (see e.g. [10]),
Corollary 7.3) and of Kude-Anderson [3], X × F ≃ X × I × F ≃ I × F ≃ F.
(b) Let E = (i_λ × E; sup |λ| = ∞; E) be regarded as a subspace
of I_λ. By [12]. p. 126, any space F satisfying the assumptions of (b) is
homeomorphic to E; in particular, X × I × F ≃ X × I × F ≃ F. Hence X × F ≃ X × I × F ≃ F.
(c) Denote E = (i_λ × I; i_λ = 0 for all but finitely many i). Using
Proposition 4.6 of [4] and Theorem 3 of [12] we easily show (in the same
way as in the proof of Proposition 5 of [12]) that F ≃ E ≃ Σ_i E
for every space F satisfying the assumptions of (c). Moreover it follows
from [4] or from [2] that, under the assumptions of (c), X can be
topologically embedded into F. Hence X × F ≃ X × Σ_i E ≃ Σ_i E ≃ F.

COROLLARY 1. Let X be a compact ARN; let F be a locally convex linear
metric space and suppose that one of the assumptions (a)-(c) is satisfied.
Then X × F is homeomorphic to an open subset of F.

Proof. The cone Y = (X × [0,1])|X×I| is a compact AR which can be
embedded into F × I and, by the theorems we quoted in the proof of
Theorem 3, F × I ≃ F. Hence X × I × F ≃ I × F is homeomorphic to
an open subset of F × F.

It is clear that if X × I × F ≃ F for a locally convex linear metric space,
then X is (homeomorphic to a retract of F and hence an absolute
retract (see [7] or [6]). Similarly, if X × F is homeomorphic to an open
subset of F and F is a locally convex linear metric space, then X × F is AR(N).

Theorem 2 and Corollary 1 give us characterizations of absolute retract,
respective absolute neighborhood retract, in the class of compact spaces.

Setting F = Σ_i E we infer from Corollary 1 that X × Σ_i E is homeo-
monic to an open subset of Σ_i E in any case where X is a compact ARN
which is a countable union of finite-dimensional compact sets. Let us
recall that every open subset of Σ_i E has a structure of a countable
metric simplicial complex; for the proof of this and other properties of
Σ_i E, manifolds see [9]. In a sense, the (c)-part of Theorem 2 and of
Corollary 1 is a "metric version" of the results of Henderson [8].

Our second application is

THEOREM 2. Let X be a closed convex subset of a normed linear space
(E, | |). Then X × Σ_i E ≃ Σ_i E; if moreover X is complete in the norm | |,
then also X × Σ_i E ≃ Σ_i E.

Proof. By the theorem of Dugundji, there is a retraction (τ) ε E into X
such that |x − τ(x)| = 4 inf {∥x − s∥: s ∈ X} for all x ∈ E. Then τ is regular and our
assertion follows from Theorem 1.

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Аналог теоремы Куратовского-Дугундзю

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Abstract. The paper is dedicated to the problem of extension of mappings in the category of metrizable uniform spaces and uniformly continuous mappings. Unless otherwise mentioned, all spaces and maps belong to this category.

Theorem 1. For a complete space $Y$, the following three conditions are equivalent:

1) $Y \leftrightarrow LC^n$, where $n \geq 0$;
2) for any closed subset $A$ of $X$, where $X\setminus A$ is precompact and relative dimension $\text{red}(X\setminus A) \leq n + 1$, every mapping $f: A \rightarrow Y$ has an extension $f_0: U \rightarrow Y$ for some uniform neighborhood $U$ of $A$;
3) for every closed embedding $i: X \rightarrow Z$ where $Z\setminus AX$ is precompact and $\text{red}(Z\setminus AX) \leq n + 1$, there exists retraction $r_0: U \rightarrow Z$ from some uniform neighborhood $U$ of $iX$.

Theorem 4. If $A$ is closed in $X$ and $\text{red}(X\setminus A) = 0$, there exists retraction $r: X \rightarrow A$ under these very conditions every mapping $f: A \rightarrow Y$ has an extension $f_0: X \rightarrow Y$ for arbitrary uniform spaces $Y$ (not necessarily metrizable).

Various variations of Theorem 1 are also proved. Examples showing the essentiality of the precompactness of $X\setminus A$ and the completeness of $Y$ are given: Theorem 1 is not true if even one of these conditions is removed.

Настоящая работа посвящена вопросу о продолжении равномерно-
непрерывных отображений

$\xymatrix{ A \ar[r]^f & Y \ar[d]^f \ar@{-->}[d] \ar@{-->}[u] \ar[r]_{\bar{f}} \ar@{-->}[r] & X }$

для метризуемых пространств, где $\bar{f}$ — равномерное вложение, $f$ — за-
данное отображение, $\bar{f}$ — искомое продолжение. Как известно, для не-
прерывных отображений подобная задача решена при следующих (доста-
точно широких) известных условиях Куратовского [7]: 1) $A$ — замкнуто в $X$, 2) $\text{dim}(X\setminus A) \leq n + 1$, 3) $Y \subseteq LC^n$. В изучаемом здесь случае (ка-
терогория метризуемых равномерных пространства с равномерно-
непрерывными отображениями) ситуация значительно сложнее, чем в топологическом случае (категория метризуемых топологических пространств с непрерывными отображениями), где верна классическая теорема Куратовского-Дугунд-
джи [6], [7].