THEOREM 3. *Let the function g fulfil hypothesis* (H_1) *and let conditions* (2)-(8) *be fulfilled for f defined by* (28). *Then g has a completely monotonic iteration group if and only if condition* (C) *is fulfilled (for f defined by* (28)).

EXAMPLE 3. Let

$$g(x) = (2-s)x - x^2 \tag{29}$$

for $x \in (0, 1-s]$. The function g is completely monotonic. Suppose that there exists a completely monotonic function X such that

$$X^2(x) = g(x)\,. \tag{30}$$

Then $\varphi(x) = 1-s-X(1-s-x)$ is an absolutely monotonic solution of (24), where

$$f(x) = 1-s-g(1-s-x) = sx + x^2.$$

But this is impossible (cf. Example 2). Therefore equation (30) with g given by (29) has no completely monotonic solution.

Theorem 3 and Example 3 answer in the negative U. T. Bödewadt's conjecture [1] that for a completely monotonic g the equation

$$\varphi^n(x) = g(x)$$

always has a unique completely monotonic solution for every positive integer n.

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Compact absolute retracts as factors of the Hilbert space

by

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Abstract. It is shown that if X is a compact ANR then $X \times l_2$ is an l_2-manifold.

Let (Y, ϱ) be a metric space and let r be a retraction of Y onto its subspace X. We shall call the retraction *regular* (with respect to ϱ), if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varrho(r(y), y) < \varepsilon$ whenever the ϱ-distance from y to X is less than δ. Our main theorem is:

THEOREM 1. *If $(E, \|\ \|)$ is a normed linear space and $r\colon E \xrightarrow{\text{onto}} X \subset E$ is a retraction which is regular with respect to $\|\ \|$, then $X \times \Sigma_{l_1} E \cong \Sigma_{l_1} E$. If moreover X is complete in the norm $\|\ \|$, then also $X \times \Pi_{l_1} E \cong \Pi_{l_1} E$.*

Here, "$\cong$" means "is homeomorphic to", and $\Sigma_{l_1} E$ and $\Pi_{l_1} E$ denote respectively $\{(t_i) \in E^\infty\colon t_i = 0$ for almost all $i \in N\}$ and $\{(t_i) \in E^\infty\colon \sum \|t_i\| < \infty\}$, both spaces equipped with the norm $|||(t_i)||| = \sum \|t_i\|$. As a corollary we conclude that if X is a compact absolute retract and E is an infinite-dimensional Fréchet space, then $X \times E \cong E$.

The problem whether a given space is a cartesian factor of the Hilbert cube or of a locally convex linear metric space has been studied by several authors (see [0], [11], [14]–[18] and also [5] pp. 266 and 269, [9a] p. 30 and [13] p. 265). The strongest results in this direction were obtained by J. E. West, who proved (among other theorems) that if K is a contractible locally finite-dimensional simplicial complex endowed with its metric topology, then $K \times E^\infty \cong E^\infty$ for every Fréchet space E of sufficiently large density character. The methods used by West in proving this were closely connected with those he developed in [14] for investigating factors of the Hilbert cube; they depend on "approximating" the space $K \times E^\infty$ by sets homeomorphic to E^∞.

D. W. Henderson in his recent paper [8] considered the situation where X is a retract of a finite-dimensional space F, and he succeeded in an explicit writing of a homeomorphism $f\colon X \times \varinjlim F^i \xrightarrow{\text{onto}} \varinjlim F^i$. The symbol $\varinjlim F^i$ denotes here the direct limit of finite powers of F; this space, which as a set can be identified with ΣF, is however not metrizable and thus is not a normed linear space. Our idea of proving Theorem 1 is to define a topology $\mathcal{G}$ on ΣE such that: 1) the "Henderson map" $f\colon X\times(\Sigma E,\mathcal{G})\to(\Sigma E,\mathcal{G})$ remains still a homeomorphism (1), and 2) the space $(\Sigma E,\mathcal{G})$ is homeomorphic to a normed linear space.

The proof that $X\times\Sigma_{l_1}E\cong\Sigma_{l_1}E$. We shall fix here a normed linear space $(E,\|\ \|)$ and a retraction $r\colon E\xrightarrow{\text{onto}}X\subset E$ which is regular in the metric induced by $\|\ \|$. Define

$$w(\lambda)=\sup\{\|r(x+t)-x\|\colon x\in X \text{ and } \|t\|\leqslant\lambda\},\qquad \lambda\in[0,\infty).$$

Because of the regularity of r, w is a non-decreasing function with $\lim_{\lambda\to0}w(\lambda)=0$; in particular w is bounded by $\frac12$ on an interval $[0,\delta)\subset[0,1]$. If we set

$$K=\operatorname{conv}\big(\{(\lambda,\mu)\in[0,\delta)\times[0,\tfrac12]\colon \mu\leqslant w(\lambda)\}\cup\{1\}\times[0,1]\big)$$

and

$$u(\lambda)=\begin{cases}\sup\{\mu\colon(\lambda,\mu)\in K\} & \text{when } \lambda\in[0,1],\\ \frac12+\frac12\lambda & \text{when } \lambda\in(1,\infty),\end{cases}$$

then u will satisfy the conditions

(0) $$u(\lambda)\in[\lambda,1]\text{ if }\lambda\in[0,1]\quad\text{and}\quad u(\lambda)\leqslant\lambda\text{ if }\lambda\geqslant1,$$

(1) $$\|r(x+t)-x\|\leqslant u(\|t\|)\quad\text{for all } x\in X \text{ and } t\in E \text{ with } \|t\|<\delta.$$

Setting finally $u^1=u$, $u^{i+1}=u\circ u^i$ and $v=\sum_{i\geqslant1}2^{-i}u^i$, we get a homeomorphism v of $[0,\infty)$ onto itself such that

(2) $$v\circ u=\sum_{i\geqslant2}2^{-i+1}u^i\leqslant 2v$$

and, because $v(0)=0$ and v is a concave function,

(3) $$v\Big(\sum_{i\geqslant1}\lambda_i\Big)\leqslant\sum_{i\geqslant1}v(\lambda_i)\quad\text{and}\quad v(2\lambda)\leqslant2v(\lambda),\qquad \lambda,\lambda_1,\lambda_2\ldots\in[0,\infty).$$

Now let us define Henderson's [8] map $f\colon X\times\Sigma E\to\Sigma E$; here ΣE denotes the set $\{(t_i)\in E^\infty\colon t_i=0 \text{ for almost all } i\}$ and (at the moment) no topology on ΣE or on X is considered. We agree that, from now till the end of the next section, $t=(t_i)$ and $s=(s_i)$ will stand for the points lying in E^∞. Set

(4) $$f(x,t)=\big(x+t_1,\,x+t_2-r(x+t_1),\,x+t_3-r(x+t_2),\,\ldots\big).$$

It is easy to see that f has the map

(5) $$s\mapsto\big(g_\infty(s),\,(g_i(s)-g_\infty(s))\big)\in X\times\Sigma E,\qquad s\in\Sigma E,$$

as inverse, where in the definition above

(6) $$g_1(s)=s_1,\qquad g_{i+1}(s)=s_{i+1}+rg_i(s)\quad\text{and}\quad g_\infty(s)=\lim_{i\to\infty}g_i(s).$$

PROPOSITION 1. *The formula*

(7) $$\varrho(t)=\sum_{i\geqslant1}2^iv(\|t_i\|)\quad\textit{and}\quad d(s,t)=\varrho(s-t),\qquad s,t\in\Sigma E,$$

defines a metric d on ΣE, and both f and g are continuous when considered as maps between $X\times(\Sigma E,d)$ and $(\Sigma E,d)$.

Proof. The triangle inequality follows immediately from (3). Given $t\in\Sigma E$ let us now denote $m(t)=\inf\{j\colon t_i=0 \text{ if } i\geqslant j\}$. We have

(8) $$\|x+s_i-r(x+s_{i-1})\|\leqslant\|s_i\|+u(\|s_{i-1}\|)$$

whenever $x\in X$ and $s_{i-1},s_i\in E$ satisfy $\max(\|s_{i-1}\|,\|s_i\|)<\delta$. Hence $\sum\|t_i-s_i\|<\delta$ implies

(9) $$\begin{aligned}d\big(f(x,t),f(y,s)\big)&-2v(\|x-y+t_1-s_1\|)-\\&-\sum_{i=2}^{m(t)}2^iv(\|x+t_i-r(x+t_{i-1})-y-s_i-r(y+s_{i-1})\|)\\&\leqslant\sum_{i>m(t)}2^iv(\|s_i\|+u(\|s_{i-1}\|))\\&\leqslant\sum_{i>m(t)}2^iv(\|s_i\|)+2\sum_{i\geqslant m(t)}2^ivu(\|s_i\|)\\&\leqslant5\sum_{i\geqslant m(t)}2^iv(\|s_i\|)\leqslant5d(s,t),\end{aligned}$$

because of the inequalities (2) and (3). Since moreover

(10) $$\varrho(t-s)\geqslant\sum_{i\geqslant1}u(\|t_i-s_i\|)\geqslant\sum_{i\geqslant1}\|t_i-s_i\|\quad\text{if}\quad\varrho(t-s)\leqslant1,$$

we infer, for every fixed $(x,t)\in X\times\Sigma E$, that $d\big(f(x,t),f(y,s)\big)\leqslant\tau_{x,t}(y,s)$ where $\tau_{x,t}$ is a continuous function of (y,s) and $\tau_{x,t}(x,t)=0$. Hence f is continuous at the (arbitrarily given) point $(x,t)\in X\times\Sigma E$.

(1) We shall write $f\colon X\times(Y,\mathcal{G})\to(Z,\mathcal{G}_1)$ (respectively $f\colon X\times(Y,\varrho)\to(Z,d)$) if we want to stress the topologies (resp. the metrics) in which the spaces in question are treated. The space X will be always treated in the topology induced by $\|\ \|$.

Now let us consider the function g. If $s \in \Sigma E$ and $i_0 \in N$ fulfil $\|s_i\| < \delta$ for all $i \geqslant i_0$, then

$$(11)\qquad \|rg_i(s)-rg_{i-1}(s)\| = \|r(s_i+rg_{i-1}(s))-rg_{i-1}(s)\| \leqslant u(\|s_i\|)\,, \qquad i \geqslant i_0\,,$$

$$(12)\qquad \|g_i(s)-rg_i(s)\| \leqslant \|s_i+rg_{i-1}(s)-rg_i(s)\| \leqslant \|s_i\|+u(\|s_i\|) \leqslant 2u(\|s_i\|)\,, \qquad i \geqslant i_0\,,$$

$$(13)\qquad \|g_\infty(s)-g_i(s)\| = \|rg_{i+m(s)}(s)-g_i(s)\| \leqslant \sum_{j=i+1}^{m(s)+i} \|rg_j(s)-rg_{j-1}(s)\| + \|rg_i(s)-g_i(s)\| \leqslant \sum_{j=i+1}^{m(s)+i} u(\|s_j\|)+2u(\|s_i\|) \leqslant 2\sum_{j\geqslant i} u(\|s_j\|)\,, \qquad i \geqslant i_0\,.$$

Therefore, given another point $t \in \Sigma E$ we get for $i \geqslant \max(i_0, m(t))$:

$$(14)\qquad \|g_i(s)-g_i(t)\| = \|s_i+rg_{i-1}(s)-rg_{m(t)}(t)\| \leqslant \|s_i\| + \sum_{j=m(t)+1}^{i-1} \|rg_j(s)-rg_{j-1}(s)\| + \|rg_{m(t)}(s)-rg_{m(t)}(t)\| \leqslant \|rg_{m(t)}(s)-rg_{m(t)}(t)\| + \sum_{j\geqslant m(t)} u(\|s_j\|) \leqslant \|rg_{m(t)}(s)-rg_{m(t)}(t)\| + d(s,t)\,.$$

By (14) and (10),

$$d(t,s) < \min(\delta, \varepsilon/2) \quad \text{and} \quad \|rg_{m(t)}(s)-rg_{m(t)}(t)\| < \varepsilon/2$$

together imply $\|g_i(s)-g_i(t)\| < \varepsilon$ for all $i \geqslant m(t)$; this shows that the functions g_i are equicontinuous at (every) point $t \in \Sigma E$. Therefore $g_\infty = \lim_{i\to\infty} g_i$ is a continuous function. Denoting by $\hat{d}$ the product metric on $X \times \Sigma E$ (the sum of the norm $\|\ \|$ on X and of the metric d on ΣE), we further get for all $s, t \in \Sigma E$ with $d(s,t) < \delta$ (use (10), (13), (3) and (2)):

$$\hat{d}(g(s), g(t)) - \|g_\infty(s)-g_\infty(t)\| - \sum_{i\leqslant m(t)} 2^i v(\|g_i(s)-g_\infty(s)-g_i(t)+g_\infty(t)\|) = \sum_{i>m(t)} 2^i v(\|g_i(s)-g_\infty(s)\|) \leqslant \sum_{i>m(t)} 2^i v\Big(2\sum_{j\geqslant i} u(\|s_j\|)\Big) \leqslant 4\sum_{i>m(t)} 2^i \sum_{j\geqslant i} v(\|s_j\|) = 4\sum_{j>m(t)} 2^j v(\|s_j\|)(1+2^{-1}+\ldots+2^{-j+m(t)}) \leqslant 8d(s,t)\,.$$

Hence g is continuous at every point $t \in \Sigma E$, which completes our proof.

Now, the first assertion of Theorem 1 follows from the lemma below:

LEMMA 1. *Let $(E, \|\ \|)$ be a normed linear space, let $v\colon [0,\infty) \xrightarrow{\text{onto}} [0,\infty)$ be a homeomorphism and let d be the metric on ΣE defined by* (7). *Then the formula*

$$(15)\qquad A(t) = \Big(2^i v(\|t_i\|)\frac{t_i}{\|t_i\|}\Big)\,, \quad \textit{where } \frac{0}{\|0\|} = 0$$

defines a homeomorphism of $(\Sigma E, d)$ onto $(\Sigma E, |||\ |||)$ (recall that $|||t||| = \sum_{i\geqslant 1}\|t_i\|$, $t \in \Sigma E$).

Proof. We have

$$|||A(s)-A(t)||| - \sum_{i\leqslant m(t)} 2^i \Big\| v(\|s_i\|)\frac{s_i}{\|s_i\|} - v(\|t_i\|)\frac{t_i}{\|t_i\|}\Big\| = \sum_{i>m(t)} 2^i v(\|s_i\|) \leqslant d(s,t)\,.$$

Since the map $s \mapsto \sum_{i=1}^{m(t)} 2^i \|v(\|s_i\|)s_i\|s_i\|^{-1} - v(\|t_i\|)t_i\|t_i\|^{-1}\|$ is continuous, A must be continuous at the given point $t \in \Sigma E$. Moreover

$$(16)\qquad A^{-1}(s) = \Big(v^{-1}(2^{-i}\|s_i\|)\frac{s_i}{\|s_i\|}\Big)$$

and

$$d(A^{-1}(s), A^{-1}(t)) - \sum_{i=1}^{m(t)} 2^i v(\|v^{-1}(2^{-i}\|s_i\|)s_i\|s_i\|^{-1} - v^{-1}(2^{-i}\|t_i\|)t_i\|t_i\|^{-1}\|) = \sum_{i>m(t)} 2^i v(v^{-1}(2^{-i}\|s_i\|)) \leqslant |||s-t|||\,.$$

Hence A^{-1} is also a continuous map.

Proof of the second part of the theorem. Under our previous notation, let us denote by Z the set $\{t = (t_i) \in E^\infty\colon \sum 2^i v(\|t_i\|) < \infty\}$. Because of (3), Z is a linear space; we shall consider it under the metric $\bar{d}(s,t) = \bar{\varrho}(s-t)$, where $\bar{\varrho}(s) = \sum_{i\geqslant 1} 2^i v(\|s_i\|)$. Observe that

$$(10\text{a})\qquad \bar{d}(s,t) \geqslant \sum_{i\geqslant 1} u(\|s_i-t_i\|) \geqslant \sum_{i\geqslant 1}\|s_i-t_i\| \quad \text{if} \quad \bar{d}(s,t) < 1\,.$$

Let $\bar{f}\colon X \times Z \to E^\infty$ be the natural extension of the map f, defined by the same formula (4).

PROPOSITION 1a. *The map $\bar{f}$ has its range contained in Z. If moreover X is complete in the norm $\|\ \|$, then $\bar{f}$ is a homeomorphism of $X \times (Z, \bar{d})$ onto $(Z, \bar{d})$.*

Proof. Given points $(x,t), (y,s) \in X \times Z$ we have for every integer m which is so large that $i \geqslant m$ implies $\|t_i\|+\|s_i\| < \delta$:

(9a) $$\bar d(\bar f(y,s),\bar f(x,t)) - 2v(\|x-y+t_1-s_1\|) - \sum_{i=2}^{m} 2^i v(\|x+t_i-r(x+t_{i-1})-y-s_i+r(y+s_{i-1})\|)$$
$$\leqslant \sum_{i>m} 2^i v(\|x+t_i-r(x+t_{i-1})\|+\|y+s_i-r(x+s_{i-1})\|)$$
$$\leqslant \sum_{i>m} 2^i v(\|t_i\|+u(\|t_{i-1}\|)+\|s_i\|+u(\|s_{i-1}\|))$$
$$\leqslant 5\sum_{i\geqslant m} 2^i v(\|t_i\|)+5\sum_{i\geqslant m} 2^i v(\|s_i\|).$$

Seeting here $x=y$, $s=(0,0,\ldots)$ and m an integer which is large enough we infer that $\operatorname{image}(\bar f)\subset Z$. If further $(x,t)\in X\times Z$ and $\varepsilon\in(0,\delta)$ are fixed, and m is an integer with $\sum_{i\geqslant m} 2^i v(\|t_i\|)<\varepsilon/30$, then $\bar d(s,t)<\varepsilon/30$ implies

$$\sum_{i\geqslant m} 2^i v(\|s_i\|)\leqslant \sum_{i\geqslant m} 2^i v(\|t_i\|+\|t_i-s_i\|)<\varepsilon/15$$

and, consequently,

$$\bar d(\bar f(y,s),\bar f(x,t))$$
$$\leqslant 2v(\|x-y+t_1-s_1\|)+\sum_{i=2}^{m} 2^i v(\|x+t_i-r(x+t_{i-1})-y-s_i-r(y+s_{i-1})\|)+\varepsilon/2.$$

Thus $\bar f$ is continuous at the point $(x,t)\in X\times Z$.

Now assume X to be $\|\ \|$-complete and for each $i\geqslant 1$ denote by $\bar g_i\colon Z\to E$ the natural extension onto Z of the map g_i defined by (6). Given $s\in Z$, $(r\bar g_i(s))$ is by (11) and (10a) a Cauchy sequence; define $\bar g_\infty(s)=\lim_{i\to\infty} r\bar g_i(s)\in X$. By (12) also the sequence $(\bar g_i(s))$ converges to $\bar g_\infty(s)$. Arguing as in the proof of Proposition 1 (the necessary changes, similar to those given above, are left to the reader) one shows that all the $\bar g_i$'s are equicontinuous and that the map

$$s\overset{\bar g}{\mapsto}\big(\bar g_\infty(s),(\bar g_i(s)-\bar g_\infty(s))\big)\in X\times E^\infty,\qquad s\in Z,$$

has its range contained in $X\times Z$ and is continuous when considered as a map of $(Z,\bar d)$ into $X\times(Z,\bar d)$. Since further ΣE is $\bar d$-dense in Z, $\bar g$ and $\bar f$ must be inverse to each other.

To finish the proof of Theorem 1 it remains to demonstrate

LEMMA 1a. *Under the notation of Proposition* 1a *the spaces* $(Z,\bar d)$ *and* $(\Pi_{l_1}E,|||\ |||)$ *are homeomorphic.*

Proof. Let us consider the map $A\colon Z\to E^\infty$ defined by (15). Then $|||A(s)|||=\bar\varrho(s)<\infty$ for all $s\in Z$, and therefore $\operatorname{image}(A)\subset\Pi_{l_1}E$. Moreover for all $s,t\in Z$ and $m\geqslant 1$ we have

$$|||A(s)-A(t)|||-\sum_{i\leqslant m} 2^i\big\|v(\|s_i\|)s_i\|s_i\|^{-1}-v(\|t_i\|)t_i\|t_i\|^{-1}\big\|$$
$$\leqslant\sum_{i>m} 2^i(v(\|t_i\|)+v(\|s_i\|))$$
$$\leqslant\sum_{i>m} 2^i v(\|t_i\|)+\sum_{i>m} 2^i(v(\|t_i\|)+v(\|s_i-t_i\|))$$
$$\leqslant 2\sum_{i>m} 2^i v(\|t_i\|)+\bar d(s,t).$$

This easily leads to the conclusion that A is continuous at every point $t\in Z$. Similarly one shows that the map $B\colon\Pi_{l_1}E\to Z$,

(16a) $$B(s)=\Big(v^{-1}(2^{-i}\|s_i\|)\frac{s_i}{\|s_i\|}\Big)$$

is continuous too. Obviously $A\circ B=B\circ A=$ the identity.

Let us note that if E is a Banach space, then the assertion of Lemma 1a follows from more general theorems of Cz. Bessaga ([3], Proposition 5.2 and § 6); moreover the homeomorphism A we use is very similar to that constructed by S. Mazur in 1929 for L_p-spaces. See [3] and the references given there for more information on the subject.

Remark 1. Propositions 1 and 1a and their proofs remain valid if we assume only that E is an additive group and $\|\ \|$ is a group norm on E (i.e. $\|a\|=0$ iff $a=0$ and $\|a-b\|\leqslant\|a\|+\|b\|$ for $a,b\in E$).

Some applications. First of all, we have:

Remark 2. Let X be a compact absolute retract and let $(E,\|\ \|)$ be a normed linear space such that X can be (topologically) embedded into E. Then $X\times\Sigma_{l_1}E\cong\Sigma_{l_1}E$ and $X\times\Pi_{l_1}E\cong\Pi_{l_1}E$.

Proof. Every retraction onto a compact set is regular. Hence we get

THEOREM 2. *Let X be a compact absolute retract. Then $X\times F\cong F$ in any of the following cases*

(a) *F is an infinite-dimensional Fréchet space,*

(b) *F is a σ-compact locally convex linear metric space which contains an infinite-dimensional compact convex set,*

(c) *F is an infinite-dimensional locally convex linear metric space and both X and F are countable unions of finite-dimensional compact sets.*

Proof. (a) Let us first assume $F=l_1$, the space of summable sequences. Since X can be topologically embedded into l_1, we get $X\times\Pi_{l_1}l_1\cong\Pi_{l_1}l_1$ and thus $X\times l_1\cong l_1$. If now F is an arbitrary infinite-dimensional

Fréchet space, then, by the theorems of Bartle-Graves (see e.g. [10], Corollary 7.3) and of Kadec-Anderson [1], $X\times F\cong X\times l_1\times F\cong l_1\times F\cong F$.

(b) Set $E=\{(\lambda_i)\in l_1\colon \sup i^2|\lambda_i|<\infty\}$; E is regarded as a subspace of l_1. By [12] p. 126, any space F satisfying the assumptions of (b) is homeomorphic to E; in particular $\Sigma_{l_1}E\cong E\cong F$. Hence $X\times F\cong X\times \Sigma_{l_1}E\cong \Sigma_{l_1}E\cong F$.

(c) Denote $E=\{(\lambda_i)\in l_1\colon \lambda_i=0$ for all but finitely many $i\}$. Using Proposition 4.6 of [4] and Theorem 3 of [12] one easily shows (in the same way as in the proof of Proposition 5 of [12]) that $F\cong E\cong \Sigma_{l_1}E$ for every space F satisfying the assumptions of (c). Moreover it follows from [4] or from [2] that, under the assumptions of (c), X can be topologically embedded into E. Hence $X\times F=X\times\Sigma_{l_1}E\cong\Sigma_{l_1}E\cong F$.

COROLLARY 1. *Let X be a compact* ANR, *let F be a locally convex linear metric space and suppose that one of the assumptions* (a)-(c) *is satisfied. Then $X\times F$ is homeomorphic to an open subset of F.*

Proof. The cone $Y=(X\times[0,1])_{/X\times\{1\}}$ is a compact AR which can be embedded into $F\times R$, and, by the theorems we quoted in the proof of Theorem 2, $F\times R\cong F$. Hence $X\times(0,1)\times F\cong X\times F$ is homeomorphic to an open subset of $Y\times F\cong F$.

It is clear that if $X\times F\cong F$ for a locally convex linear metric space, then X is (homeomorphic to) a retract of F and hence is an absolute retract (see [7] or [6]). Similarly, if $X\times F$ is homeomorphic to an open subset of F and F is a locally convex linear metric space, then $X\in \mathrm{ANR}(\mathfrak{M})$. Thus Theorem 2 and Corollary 1 give us characterizations of absolute retracts (resp. absolute neighbourhood retracts) in the class of compact spaces.

Setting $F=l_1^f\underset{\mathrm{df}}{=}\Sigma_{l_1}R$ we infer from Corollary 1 that $X\times l_1^f$ is homeomorphic to an open subset of l_1^f in any case where X is a compact ANR which is a countable union of finite-dimensional compact sets. Let us recall that every open subset of l_1^f has a structure of a countable metric simplicial complex; for the proof of this and other properties of l_1^f-manifolds see [9]. In a sense, the (c)-part of Theorem 2 and of Corollary 1 is a "metric version" of the results of Henderson [8].

Our second application is

THEOREM 3. *Let X be a closed convex subset of a normed linear space $(E,\|\ \|)$. Then $X\times\Sigma_{l_1}E\cong\Sigma_{l_1}E$; if moreover X is complete in the norm $\|\ \|$, then also $X\times\Pi_{l_1}E\cong\Pi_{l_1}E$.*

Proof. By the theorem of Dugundji, there is a retraction (²) $r\colon E\xrightarrow{\text{onto}}X$ such that $\|r(t)-t\|\leqslant 4\inf\{\|t-x\|\colon x\in X\}$ for all $t\in E$. Then r is regular and our assertion follows from Theorem 1.

(²) The required retraction is defined in [7], p. 359 (one has to set there $X=L$ and $f=\mathrm{id}_A$ to get the retraction onto a convex subset A of the normed linear space L).

In particular we infer that if X is a closed subset of the space $l_1(A)$, where A is an infinite set, then $X\times l_1(A)\cong l_1(A)$.

Added in proof. The author has recently shown that every (complete) $\mathrm{AR}(\mathfrak{M})$-space can be embedded into a complete normed linear space as its regular retract; the proof will appear in [19]. This implies that $X\times H$ is homeomorphic to H whenever $X\in\mathrm{AR}(\mathfrak{M})$ is complete metrizable and H is a Hilbert space of density character not less than that of X.

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Аналог теоремы Куратовского-Дугунджи

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Abstract. The paper is dedicated to the problem of extension of mappings in the category of metrizable uniform spaces and uniformly continuous mappings. Unless otherwise mentioned, all spaces and maps belong to this category.

THEOREM 1. *For a complete space Y, the following three conditions are equivalent:*

i) $Y \in LC^n$, *where* $n \geqslant 0$;

ii) *for any closed subset A of X, where $X \setminus A$ is precompact and relative dimension $r\Delta d(X \setminus A) \leqslant n+1$, every mapping $f\colon A \to Y$ has an extension $f_U\colon U \to Y$ for some uniform neighborhood U of A;*

iii) *for every closed embedding $i\colon Y \to Z$ where $Z \setminus iY$ is precompact and $r\Delta d(Z \setminus iY) \leqslant n+1$, there exists retraction $r_U\colon U \to iY$ from some uniform neighborhood U of iY.*

THEOREM 4. *If A is closed in X and $r\Delta d(X \setminus A) = 0$, there exists retraction $r\colon X \to A$; under these very conditions every mapping $f\colon A \to Y$ has an extension $F\colon X \to Y$ for arbitrary uniform spaces Y (not necessarily metrizable).*

Various variations of Theorem 1 are also proved. Examples showing the essentiality of the precompactness of $X \setminus A$ and the completeness of Y are given: Theorem 1 is not true if even one of these conditions is removed.

Настоящая работа посвящена вопросу о продолжении равномерно-непрерывных отображений

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle \tilde{f}} & \\ X & & \end{array}$$

для метризуемых пространств, где i — равномерное вложение, f — заданное отображение, а $\tilde{f}$ — искомое продолжение. Как известно, для непрерывных отображений подобная задача решена при следующих (достаточно широких) известных условиях Куратовского [7]: 1) A — замкнуто в X, 2) $\dim(X \setminus A) \leqslant n+1$, 3) $Y \in C^n \cap LC^n$. В изучаемом здесь случае (категория метризуемых равномерных пространств с равномерно-непрерывными отображениями) ситуация значительно сложнее, чем в топологическом случае (категория метризуемых топологических пространств с непрерывными отображениями), где верна классическая теорема Куратовского-Дугунджи [5], [7].