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On some special iteration groups

by

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Abstract. Let f be a real function fulfilling the following conditions: (H) f is defined and absolutely monotonic in an interval $[0, a)$, $0 < f(x) < x$ for $x \in (0, a)$, moreover

$$\lim_{x\to 0+} \frac{f(x)}{x} = s, \quad 0 < s < 1.$$

An iteration group $\{f^u\}$ is called *absolutely monotonic* if for every positive u the function $f^u(x)$ is an absolutely monotonic function of x.

The main result of this paper is

THEOREM 2. *Let function f fulfil hypothesis* (H). *Then f has an absolutely monotonic iteration group if and only if*

$$h^{(n)}(0) \leqslant 0 \quad for \quad n = 2, 3, \ldots,$$

where h is an analytic solution of the equation

$$h[f(x)] = f'(x)h(x)$$

such that $h(0) = 0$, $h'(0) = 1$.

In the proof of this theorem we use S. Dubuc's theorem about fractional iteration (Ann. Inst. Grenoble, 21 (1) (1971), pp. 171–251).

A function f is called *absolutely monotonic in an interval* $[0, a)$ if

$$\Delta_h^p f(x) = \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} f(x+ih) \geqslant 0$$

for all $x \in [0, a)$, $h \geqslant 0$ and non-negative integers p, where $0 \leqslant x \leqslant x + ph < a$.

It is obvious that the limit of a sequence of absolutely monotonic functions is absolutely monotonic.

Let f be a real function fulfilling the following conditions:

(H) f is defined and absolutely monotonic in an interval $[0, a)$, $0 < f(x) < x$ for $x \in (0, a)$, moreover,

(1) $$\lim_{x\to 0} \frac{f(x)}{x} = s, \quad 0 < s < 1.$$

An iteration group $\{f^u\}$ of f is called *convex* if for every positive u the function $f^u(x)$ is a convex function of x (cf. [6] and [7]).

An iteration group $\{f^u\}$ of f is called *absolutely monotonic* if for every positive u the function $f^u(x)$ is an absolutely monotonic function of x.

Every absolutely monotonic function f is analytic and

$$f(x) = sx + \sum_{i=2}^{\infty} a_i x^i \quad \text{for} \quad x \in [0, a), \tag{2}$$

where

$$a_i \geqslant 0 \quad \text{for} \quad i = 2, 3, \ldots \tag{3}$$

Let

$$\hat{f}(z) = sz + \sum_{i=2}^{\infty} a_i z^i \tag{4}$$

be the extension of f onto the disc $|z| < a$. It is shown by G. Koenigs [3] that there exist a positive number $r_0 \in (0, a)$ and an analytic function σ for $|z| < r_0$ such that

$$\sigma[\hat{f}(z)] = s\sigma(z) \tag{5}$$

and

$$\sigma'(0) = 1. \tag{6}$$

This function is unique.

Let

$$\hat{h}(z) \overset{\text{df}}{=} \frac{\sigma(z)}{\sigma'(z)}. \tag{7}$$

The function $\hat{h}$ is analytic in a neighbourhood of zero. Let

$$\hat{h}(z) = z + \sum_{i=2}^{\infty} b_i z^i. \tag{8}$$

Serge Dubuc proved in [2] the following theorem:

THEOREM. *Let* (1)-(8) *hold. Then every equation*

$$\hat{\varphi}^m(z) = \hat{f}(z) \tag{9}$$

for $m = 2, 3, \ldots$ *has in a neighbourhood of zero an analytic solution*

$$\hat{\varphi}(z) = \sum_{i=1}^{\infty} c_i z^i \tag{10}$$

such that

$$c_i \geqslant 0 \quad \textit{for} \quad i = 1, 2, \ldots$$

if and only if

$$b_i \leqslant 0 \quad \textit{for} \quad i = 2, 3, \ldots, \tag{C}$$

where b_i *are coefficients in* (8).

Let $m = 2$ and let condition (C) be fulfilled. Then (9) has the form

$$\hat{\varphi}^2(z) = \hat{f}(z). \tag{11}$$

The formal solutions (10) of equation (11) can be found from the formula (cf. [8])

$$a_n = \sum_{i=1}^{n} \sum_{\substack{p_1, \ldots, p_i \in N \\ p_1 + \ldots + p_i = n}} c_i c_{p_1} \ldots c_{p_i} \quad \text{for} \quad n = 1, 2, \ldots \tag{12}$$

From (12) we have

$$a_1 = c_1^2, \quad a_n = c_n(c_1^n + c_1) + w_n(c_1, \ldots, c_{n-1}) \quad \text{for} \quad n = 2, 3, \ldots, \tag{13}$$

where w_n is a polynomial with non-negative coefficients. According to (13) a formal solution of (11) such that $c_1 \geqslant 0$ is unique. Moreover, since $c_i \geqslant 0$, we have

$$c_n \leqslant \frac{a_n}{c_1^n + c_1} \leqslant \frac{a_n}{c_1} \quad \text{for} \quad n = 2, 3, \ldots,$$

whence

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} \leqslant \limsup_{n\to\infty} \sqrt[n]{|a_n|}. \tag{14}$$

Inequality (14) implies that

$$R_{\hat{f}} \leqslant R_{\hat{\varphi}}, \tag{15}$$

where $R_{\hat{f}}$ denotes the radius of convergence of series (4) and $R_{\hat{\varphi}}$ denotes that of series (10).

Let

$$\varphi(x) = \sum_{i=1}^{\infty} c_i x^i \quad \text{for} \quad x \in [0, R_{\hat{\varphi}}). \tag{16}$$

The function φ is absolutely monotonic and φ is a solution of the equation

$$\varphi^2(x) = f(x) \tag{17}$$

in $[0, r_1)$ for a certain $r_1 \in (0, a)$.

Suppose that there exists a point $x_0 \in [0, r_1)$ such that $\varphi(x_0) > x_0$. The function φ is increasing; hence

$$f(x_0) = \varphi^2(x_0) \geqslant \varphi(x_0) > x_0,$$

and this contradicts (1). If $b \in (0, a)$ is such that

$$\varphi([0, b]) \subset [0, b], \tag{18}$$

then the superposition $\varphi^2(x) = \varphi[\varphi(x)]$ exists in $[0, b]$. Since series (16) formally satisfies equation (17), the function φ is an actual solution of (17) in $[0, b]$.

Let

$$c = \sup\{x \in [0, a): \varphi(x) < x\}.$$

Suppose that $c < a$; then $\varphi(c) = c$ and condition (18) is fulfilled for $b = c$. Hence φ fulfils (17) in $[0, c]$ and

$$f(c) = \varphi[\varphi(c)] = \varphi(c) = c,$$

which contradicts hypothesis (H). Thus we have shown that the inequality

$$\varphi(x) < x$$

is fulfilled in $[0, a)$ and φ is an actual solution of (17) in $[0, a)$.

We have proved the following

THEOREM 1. *Let the function f fulfil hypothesis* (H) *and let conditions* (2), (3) *and* (C) *be fulfilled. Then equation* (17) *has an absolutely monotonic solution in* $[0, a)$ *such that*

$$0 < \varphi'(0) < 1 \quad \text{and} \quad 0 < \varphi(x) < x \quad \text{in } (0, a).$$

Since f is convex, $0 < f(x) < x$, $f'(x) > 0$ and $\lim_{x\to 0+} f'(x) = s \in (0, 1)$ in $(0, a)$, we have (cf. [5])

COROLLARY 1. *If the hypotheses of Theorem* 1 *are fulfilled, then $\varphi = f^{1/2}$, where $\{f^u\}$ is the principal iteration group of f (concerning definitions cf.* [4], *Chapter* IX).

COROLLARY 2. *Suppose that the hypotheses of Theorem* 1 *are fulfilled. Then, for non-negative integer k, the equation*

$$\varphi^{2^k}(x) = f(x)$$

has an absolutely monotonic solution φ in $[0, a)$ such that $0 < \varphi'(0) < 1$ and $0 < \varphi(x) < x$ in $(0, a)$.

We shall prove the following

THEOREM 2. *Let the function f fulfil hypothesis* (H) *and let conditions* (2)-(8) *be fulfilled. Then f has an absolutely monotonic iteration group if and only if condition* (C) *is fulfilled.*

Proof. For every $u \geqslant 0$ there exists a sequence $p_0, p_1, \ldots$, where $p_k = 0$ or 1 for $k = 1, 2, \ldots$, such that

$$u = \sum_{k=0}^{\infty} p_k 2^{-k}.$$

Moreover, a superposition of absolutely monotonic functions is absolutely monotonic. Therefore, according to Corollaries 1 and 2, the function $f^{u_n}(x)$, where f^{u_n} is the member of the principal iteration group of f with $u_n = \sum_{k=0}^{n} p_k 2^{-k}$, is an absolutely monotonic function of x. According to the definition of a continuous iteration group, $f^u(x)$ is a continuous function of u. Therefore

$$\lim_{n\to\infty} f^{u_n}(x) = f^u(x).$$

Since the limit of a sequence of absolutely monotonic functions is absolutely monotonic, the function $f^u(x)$ is an absolutely monotonic function of x.

On the other hand, if f has an absolutely monotonic iteration group, then for every positive integer m there exists an absolutely monotonic solution of the equation

$$\varphi^m(x) = f(x)$$

and, according to Theorem 2, condition (C) is fulfilled.

COROLLARY 3. *If f fulfils hypothesis* (H) *and condition* (C) *is fulfilled, then f has a convex iteration group in* $(0, a)$.

Let hypothesis (H) be fulfilled. According to (5), (7) and in view of the relation

$$\sigma'[\hat{f}(z)]\hat{f}'(z) = s\sigma'(z)$$

we obtain

$$\hat{h}[\hat{f}(z)] = \hat{f}'(z)\hat{h}(z). \tag{19}$$

Moreover, $\hat{h}$ is of the form (8). From (8) and (19) we have

$$\psi[f(z)] = g(z)\psi(z) \tag{20}$$

for

$$\psi(z) = 1 + \sum_{i=2}^{\infty} b_i z^{i-1}$$

and

$$g(z) = \begin{cases} \dfrac{z\hat{f}'(z)}{\hat{f}(z)} & \text{for } z \neq 0, \\ 1 & \text{for } z = 0. \end{cases}$$

The function ψ is a solution of the linear homogeneous equation (20) in a neighbourhood of zero and

$$\lim_{z\to 0} \psi(z) = 1.$$

It can be proved similarly to Theorem 5.2 in [4] that

$$\psi(z) = \lim_{n\to\infty} \frac{\hat{f}^n(z)}{z \prod_{i=0}^{n-1} \hat{f}'[\hat{f}^i(z)]},$$

whence

$$\hat{h}(z) = \lim_{n\to\infty} \frac{\hat{f}^n(z)}{[\hat{f}^n(z)]'}. \tag{21}$$

EXAMPLE 1. Let

$$f(x) = \frac{1}{2}(x+x^2+\ldots) = \frac{1}{2}\cdot\frac{x}{1-x} \quad \text{for} \quad x \in [0, \tfrac{1}{2});$$

then

$$\hat{f}(z) = \frac{1}{2}(z+z^2+\ldots) = \frac{1}{2}\cdot\frac{z}{1-z} \quad \text{for} \quad |z| < \frac{1}{2}.$$

We have

$$\hat{f}^n(z) = \frac{1}{2^n}\cdot\frac{z}{1-(2-2^{1-n})z} \quad \text{and} \quad (\hat{f}^n)'(z) = \frac{1}{2^n}\cdot\frac{1}{[1-(2-2^{1-n})z]^2}.$$

According to (21)

$$\hat{h}(z) = \lim_{n\to\infty}\frac{\hat{f}^n(z)}{(\hat{f}^n)'(z)} = \lim_{n\to\infty}[1-(2-2^{1-n})z]z = z-2z^2.$$

We have shown that condition (C) is fulfilled. Therefore f has an absolutely monotonic iteration group in $[0, \frac{1}{2})$.

A formal solution of (19) can be found from the equations (cf. [8])

$$sb_2+2a_2 = a_2+b_2s^2,$$
$$a_3+2a_2b_2s+b_3s^3 = sb_3+2a_2b_2+3a_3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

whence we have

$$b_2 = \frac{a_2}{s(s-1)},$$

$$b_3 = \frac{2a_2b_2(1-s)+2a_3}{s(s^2-1)} = 2\frac{a_3s-a_2^2}{s^2(s^2-1)}.$$

If

$$a_3s-a_2^2 < 0, \tag{22}$$

then

$$b_3 > 0. \tag{23}$$

Inequality (23) is incompatible with (C). If condition (22) is fulfilled, then the function f cannot have an absolutely monotonic iteration group.

EXAMPLE 2. Let $f(x) = sx+x^2$, $0 < s < 1$. Since the function f fulfils (22), it does not have an absolutely monotonic iteration group.

We shall show more, viz. that equation

$$\varphi^2(x) = f(x) \tag{24}$$

has no absolutely monotonic solution. Suppose that equation (24) has a solution in the form

$$\varphi(x) = \sum_{i=1}^{\infty} c_i x^i.$$

Then (cf. [8])

$$sx+x^2 = \varphi^2(x) = c_1x+c_2(c_1+c_1^2)x^2+(c_1c_3+2c_1c_2^2+c_3c_1^3)x^3+\ldots \tag{25}$$

From (25) we obtain either

$$c_1 = -\sqrt{s} \tag{26}$$

or

$$c_1 = \sqrt{s}, \quad c_2 = \frac{1}{s+\sqrt{s}} \quad \text{and} \quad c_3 = \frac{-2}{(1+s)(s+\sqrt{s})^2}. \tag{27}$$

Formulas (26) and (27) show that φ is not absolutely monotonic.

A function g is called *completely monotonic in* $(0, a]$ if $f \in C^\infty((0, a])$ and $(-1)^{k-1}f^{(k)}(x) \geq 0$ for $x \in (0, a]$ and $k = 1, 2, \ldots$ (cf. [9]).

Let a function g fulfil the following conditions:

(H_1) g is completely monotonic in $(0, a]$; moreover, $x < g(x) < a$ in $(0, a)$ and

$$\lim_{x\to a-}\frac{a-g(x)}{a-x} = s, \quad 0 < s < 1.$$

An iteration group $\{g^u\}$ is called *completely monotonic* if for every positive u the function $g^u(x)$ is a completely monotonic function of x.

A function g fulfils hypothesis (H_1) if and only if the function

$$f(x) = a-g(a-x) \tag{28}$$

fulfils hypothesis (H). Moreover, the function g has a completely monotonic iteration group if and only if the function f defined by (28) has an absolutely monotonic iteration group. The formula

$$g^u(x) = a-f^u(a-x)$$

gives the relation between those iteration groups.

THEOREM 3. *Let the function g fulfil hypothesis* (H_1) *and let conditions* (2)-(8) *be fulfilled for f defined by* (28). *Then g has a completely monotonic iteration group if and only if condition* (C) *is fulfilled* (*for f defined by* (28)).

EXAMPLE 3. Let

$$g(x) = (2-s)x - x^2 \tag{29}$$

for $x \in (0, 1-s]$. The function g is completely monotonic. Suppose that there exists a completely monotonic function X such that

$$X^2(x) = g(x) . \tag{30}$$

Then $\varphi(x) = 1-s-X(1-s-x)$ is an absolutely monotonic solution of (24), where

$$f(x) = 1-s-g(1-s-x) = sx + x^2.$$

But this is impossible (cf. Example 2). Therefore equation (30) with g given by (29) has no completely monotonic solution.

Theorem 3 and Example 3 answer in the negative U. T. Bödewadt's conjecture [1] that for a completely monotonic g the equation

$$\varphi^n(x) = g(x)$$

always has a unique completely monotonic solution for every positive integer n.

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Compact absolute retracts as factors of the Hilbert space

by

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Abstract. It is shown that if X is a compact ANR then $X \times l_2$ is an l_2-manifold.

Let (Y, ϱ) be a metric space and let r be a retraction of Y onto its subspace X. We shall call the retraction *regular* (with respect to ϱ), if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varrho(r(y), y) < \varepsilon$ whenever the ϱ-distance from y to X is less than δ. Our main theorem is:

THEOREM 1. *If* $(E, \| \|)$ *is a normed linear space and* $r\colon E \xrightarrow{\text{onto}} X \subset E$ *is a retraction which is regular with respect to* $\| \|$, *then* $X \times \Sigma_{l_1} E \cong \Sigma_{l_1} E$. *If moreover X is complete in the norm* $\| \|$, *then also* $X \times \Pi_{l_1} E \cong \Pi_{l_1} E$.

Here, "$\cong$" means "is homeomorphic to", and $\Sigma_{l_1} E$ and $\Pi_{l_1} E$ denote respectively $\{(t_i) \in E^\infty\colon t_i = 0$ for almost all $i \in N\}$ and $\{(t_i) \in E^\infty\colon \sum \|t_i\| < \infty\}$, both spaces equipped with the norm $|||(t_i)||| = \sum \|t_i\|$. As a corollary we conclude that if X is a compact absolute retract and E is an infinite-dimensional Fréchet space, then $X \times E \cong E$.

The problem whether a given space is a cartesian factor of the Hilbert cube or of a locally convex linear metric space has been studied by several authors (see [0], [11], [14]–[18] and also [5] pp. 266 and 269, [9a] p. 30 and [13] p. 265). The strongest results in this direction were obtained by J. E. West, who proved (among other theorems) that if K is a contractible locally finite-dimensional simplicial complex endowed with its metric topology, then $K \times E^\infty \cong E^\infty$ for every Fréchet space E of sufficiently large density character. The methods used by West in proving this were closely connected with those he developed in [14] for investigating factors of the Hilbert cube; they depend on "approximating" the space $K \times E^\infty$ by sets homeomorphic to E^∞.

D. W. Henderson in his recent paper [8] considered the situation where X is a retract of a finite-dimensional space F, and he succeeded in an explicit writing of a homeomorphism $f\colon X \times \varinjlim F^i \xrightarrow{\text{onto}} \varinjlim F^i$. The symbol $\varinjlim F^i$ denotes here the direct limit of finite powers of F; this