a cluster and by hypothesis, not bound. Thus, there is a $\mathfrak{U} \in \mathcal{S}$ such that $\mathfrak{U} < \mathfrak{A}_2 \cap X$. This implies that $\mathfrak{U}^* < \mathrm{cl}\,\mathfrak{A}_2 < \mathfrak{A}_1$ and $\mathfrak{A}_1 \in \mathcal{S}^*$.

Among all semi-uniformities compatible with a given topology there is a finest, namely the fine semi-uniformity. We will define $(X, \mathcal{S})$ to be a *fine space* provided $\mathcal{S}$ is the fine semi-uniformity compatible with $\tau(\mathcal{S})$. A fine space is complete.

THEOREM 5.2. *Every continuous function from a fine space into a semi-uniform space is a mapping.*

THEOREM 5.3. *The fine spaces form a coreflective subcategory of the category of semi-uniform spaces.*

Proof. If $\mathcal{S}$ is a semi-uniformity on X, let $\alpha\mathcal{S}$ denote the compatible fine semi-uniformity. It is easy to see that if $(Y, \mathcal{S}_1)$ is a fine space, then any map $f\colon (Y, \mathcal{S}_1) \to (X, \mathcal{S}_2)$ factors uniquely $f = c \circ f'$ where $f'\colon (Y, \mathcal{S}_1) \to (X, \alpha\mathcal{S}_2)$ and $c\colon (X, \alpha\mathcal{S}_2) \to (X, \mathcal{S}_2)$.

A fine uniform space is a fine semi-uniform space if and only if the semi-uniform topology is paracompact.

A subspace of a fine space will be called *subfine*. For each semi-uniformity $\mathcal{S}$ on X, let $\sigma\mathcal{S}$ denote the finest semi-uniformity on X whose completion is topologically equivalent to that of $\mathcal{S}$.

THEOREM 5.4. *The subfine spaces form a coreflective subcategory of the category of semi-uniform spaces.*

Proof. If $(Y, \mathcal{S}_1)$ is a subfine space and $f\colon (Y, \mathcal{S}_1) \to (X, \mathcal{S}_2)$ is a mapping, then f has a unique extension to a mapping f^* from $(Y^*, \mathcal{S}_1^*)$ to $(X^*, \mathcal{S}_2^*)$. Since completions are unique, $(Y^*, \mathcal{S}_1^*)$ must be a fine space, so $f^* = c \circ h$ where $h\colon (Y^*, \mathcal{S}_1^*) \to (X^*, \sigma\mathcal{S}_2^*)$ and $c\colon (X^*, \sigma\mathcal{S}_2^*) \to (X^*, \mathcal{S}_2^*)$. If $e\colon Y \to Y^*$ is the embedding, then $c \circ (h \circ e) = f^* \circ e = f$, and $h \circ e\colon (Y, \mathcal{S}_1) \to (X, \sigma\mathcal{S}_2)$ and $c\colon (X, \sigma\mathcal{S}_2) \to (X, \mathcal{S}_2)$.

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On the extension of continuous functions

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Abstract. The principal result is the following. Let X be a dense subspace of Z. Let f be a continuous function from X to a complete semi-uniform space (Y, C). Then f can be continuously extended to Z iff, for $\tau = \{T\} \in C$, $\{\mathrm{Int}_Z \mathrm{cl}_Z f^{-1}(T)\}$ covers Z. In particular, let X be a dense subspace of Z and f a continuous function from X to a regular T_1 space Y. Then f can be continuously extended to Z iff, for $\tau = \{T\}$ an open covering of Y, $\{\mathrm{Int}_Z \mathrm{cl}_Z f^{-1}(T)\}$ is an open covering of Z. Well-known applications will also be discussed.

1. Introduction. This paper contains a theorem which gives very specific circumstances under which a continuous function, whose image is regular T_1, can be continuously extended from a dense subspace onto an entire space. Before this theorem can be given, however, it is necessary to mention some recent results in structural theory.

2. Semi-uniform spaces. In [11] E. F. Steiner and A. K. Steiner have introduced the concept of semi-uniformities. Here, only those points essential to this paper will be discussed. A semi-uniform space consists of a pair X, C where X is a space and C is a collection of coverings of X satisfying:

(i) If $\tau \in C$ then there exists $\tau' \in C$ such that: for each $T' \in \tau'$ there exist $T \in \tau$ and $\tau'' \in C$ where $\mathrm{St}(T', \tau'') \subset T$. (If α is a covering of a set Y and $A \subset Y$, then $\mathrm{St}(A, \alpha) = \bigcup \{T \in \alpha\colon T \cap A \neq \emptyset\}$. If A consists of a single point x, then $\mathrm{St}(x, \alpha) = \mathrm{St}(A, \alpha)$). τ' is said to *semi-star refine* τ.

(ii) If $\tau, \tau' \in C$ then there exists $\tau'' \in C$ such that τ'' refines both τ and τ'.

(iii) If $\tau' \in C$ refines a covering τ, then $\tau \in C$.

(iv) For each $x \in X$, $\{\mathrm{St}(x, \tau)\colon \tau \in C\}$ is a base of neighborhoods for x.

(v) For $x, y \in X$ where $x \neq y$, there exists $\tau \in C$ such that $y \notin \mathrm{St}(x, \tau)$.

If the pair X, C satisfies (i) through (v), then C is said to be a *semi-uniformity* on X. This is denoted by (X, C), and (X, C) is said to be a *semi-uniform space*. Note that, if (i) is replaced by: for $\tau \in C$ there exists $\tau' \in C$ such that, for each $T' \in \tau'$ there exists $T \in \tau$ where $\mathrm{St}(T', \tau') \subset T$, then C is a uniformity on X (see [9]).

Each semi-uniform space (X, C) has a unique completion denoted by $(\hat{X}, \hat{C})$. If $X = \hat{X}$ then (X, C) is said to be a *complete* semi-uniform space. It can be deduced from [1] that: if Q is a filter on X containing a member of each covering in C, and if (X, C) is complete, then $\bigcap_{A \in Q} \mathrm{cl}_X A \neq \emptyset$.

If (X, C) and (Y, C') are semi-uniform spaces and f is a function from X to Y, then f is said to be *semi-uniformly continuous* if, for each $\tau \in C'$, $\{f^{-1}[T]:\ T \in \tau\} \in C$. Semi-uniformly continuous functions are always continuous. If f has an inverse and both f and f^{-1} are semi-uniformly continuous, then f is said to be an *isomorphism*. An isomorphism is, in particular, a homeomorphism.

The following facts (found in [11]) make semi-uniformities quite interesting:

(1) A space is regular T_1 iff it is semi-uniformizable. In fact, if X is a regular T_1 space, then the collection of all coverings of X which are refined by open coverings of X is a complete semi-uniformity on X.

(2) If (Y, C) is a complete semi-uniform space and X is dense subspace of Y, then the completion of (X, C'), where C' is the trace of C on X (i.e. $C' = \{\tau:\ \tau = \{T \cap X\}$ where $\{T\} \in C\}$), is isomorphic to (Y, C) under an isomorphism which is the identity on X.

(3) If (X, C) is a semi-uniform space, (Y, C') is a complete semi-uniform space, and f is a semi-uniformly continuous function from (X, C) to (Y, C'), then f can be semi-uniformly continuously extended to $(\hat{X}, \hat{C})$.

The following lemma will also be needed.

LEMMA 2.1. *Let (X, C) be a semi-uniform space. Then (X, C) has the following properties:*

(i) *If $\tau \in C$ then $\{\mathrm{Int}_X T:\ T \in \tau\} \in C$. Hence, each covering in C can be refined by an open covering in C.*

(ii) *If $C' = \big\{\{\mathrm{cl}_X \mathrm{Int}_X T:\ T \in \tau\}:\ \tau \in C\big\}$ then $C' \subset C$, and each covering in C can be refined by a covering in C'.*

C' is therefore a *base* for C (the members of C' are commonly known as *regularly closed* coverings).

Proof. The proof of these statements are easy consequences of the axioms of a semi-uniformity. The possession of these properties by a uniform space are well-known.

Let (X, C) be a semi-uniform space and $(\hat{X}, \hat{C})$ be its semi-uniform completion. It can be deduced from [1] that C is simply the trace of $\hat{C}$ on X. For $\tau \in C$ let $\hat{\tau} \in \hat{C}$ be such that τ is the trace of $\hat{\tau}$ on X. From (ii) of the previous lemma, there exists $\hat{\tau}' \in C$ which refines $\hat{\tau}$ and which is a regularly closed covering of $\hat{X}$. It is easy to show that $\hat{\tau}'$ refines $\{\mathrm{cl}_{\hat{X}} T:\ T \in \tau\}$. Hence,

(a) for $\tau \in C$, $\{\mathrm{cl}_{\hat{X}} T:\ T \in \tau\} \in \hat{C}$.

3. Main theorem and applications.

THEOREM 3.1. *Let X be a dense subspace of a space Z. Let f be a continuous function from X into a complete semi-uniform space (Y, C). f can be continuously extended to Z iff the following holds: for each $\tau \in C$, $\{\mathrm{Int}_Z \mathrm{cl}_Z f^{-1}[T]:\ T \in \tau\}$ is an open covering of Z.*

In particular, since Y is complete relative to the semi-uniformity generated by all open coverings, f can be continuously extended to Z iff, for each open covering τ of Y, $\{\mathrm{Int}_Z \mathrm{cl}_Z f^{-1}[T]:\ T \in \tau\}$ covers Z.

Proof. The proof of the necessity is trivial. Assume then that X, Y, (Y, C), and f are as in the hypothesis. Let $t \in Z$. Define $D_t = \{V \cap X:\ V$ is open in Z and $t \in V\}$. Define $E_t = \{A \subset Y:\ f[U] \subset A$ for some $U \in D_t\}$. Clearly, E_t has the f.i.p. If $\tau \in C$ is a closed covering of Y, then there exists $T \in \tau$ such that $t \in K = \mathrm{Int}_Z \mathrm{cl}_Z f^{-1}[T]$. Since $f^{-1}[T]$ is closed in X and $K \cap X \in D_t$, it follows that $f[K \cap X] \subset T \in K_t$. Since, by Lemma 2.1, C has a base of closed coverings, a member of each covering in C is in E_t. Since E_t has the f.i.p. and Y is complete, $S = \bigcap_{A \in E_t} \mathrm{cl}_Y A \neq \emptyset$. If $y_1 \neq y_2$ belong to Y, then there exists a closed covering $\tau \in C$ such that $y_1 \in \mathrm{St}(y_1, \tau) \subset Y - \{y_2\}$. If $T \in \tau$ is in E_t, then T cannot contain both y_1 and y_2. It follows that S consists of a singleton. Let $S = \{y_t\}$. Extend f by defining $\hat{f}$ from Z to Y so that $\hat{f}(t) = y_t$. Clearly, $\hat{f}$ restricted to X is f. It remains to show that $\hat{f}$ is continuous. For $t \in Z$, assume that $t \in \mathrm{cl}_Z A$ for $A \subset Z$. Suppose that $y_t \notin \mathrm{cl}_Y \hat{f}[A]$. Then there exists a closed covering $\tau \in C$ such that $y_t \in \mathrm{St}(y_t, \tau) \subset Y - \mathrm{cl}_Y \hat{f}[A]$. There exists $T \in \tau$ such that $t \in \mathrm{Int}_Z \mathrm{cl}_Z f^{-1}[T]$. Since $t \in \mathrm{cl}_Z A$, there exists $t_1 \in A$ such that $t_1 \in \mathrm{Int}_Z \mathrm{cl}_Z f^{-1}[T]$. Thus, $\hat{f}(t_1) = y_{t_1} \in T \cap \hat{f}[A]$ and $y_t \in T$. This is in contradiction to $\mathrm{St}(y_t, \tau) \subset Y - \mathrm{cl}_Y \hat{f}[A]$. Thus, $\hat{f}(t) \in \mathrm{cl}_Y f[A]$. Hence, $\hat{f}$ is continuous.

For X a dense subspace of a space Z, Y a regular T_1 space, and f a continuous function from X to Y, necessary and sufficient conditions for continuously extending f to Z are given in many general topology texts (e.q. [1]). The conditions in these texts, however, are less specific than the condition given in the previous theorem. The necessity for the regularity of the image space is indicated in [2]. Frolík [7] states a theorem analogous to Theorem 3.1. Rather than the image being a complete semi-uniform space as in Theorem 3.1, he demands that the image be complete relative to another type of structure.

The previous theorem has an alternate which is easier to apply in some cases.

THEOREM 3.2. *Let X be a dense subspace of a space Z. Let f be a continuous function from X into a complete semi-uniform space (Y, C). f can be continuously extended to Z iff the following holds: if $\tau \in C$, then $\{\mathrm{cl}_Z f^{-1}[Y-T]: T \in \tau\}$ has empty intersection.*

In particular, f can be continuously extended to Z iff, for each collection of closed sets $\{A_a\}$ in Y with empty intersection, $\{\mathrm{cl}_Z f^{-1}[A_a]\}$ has empty intersection.

Proof. The proof of the necessity is trivial. For the sufficiency, let X, Z, (Y, C), and f be as in the hypothesis. Let $\tau \in C$. Since

$$\{\mathrm{cl}_Z f^{-1}[Y-T]: T \in \tau\}$$

has empty intersection, it follows that

$$\{Z-\mathrm{cl}_Z f^{-1}[Y-T]: T \in \tau\}$$

is an open covering of Z. For each $T \in \tau$,

$$Z-\mathrm{cl}_Z f^{-1}[Y-T] \subset \mathrm{Int}_Z \mathrm{cl}_Z f^{-1}[T].$$

Thus, $\{\mathrm{Int}_Z \mathrm{cl}_Z f^{-1}[T]: T \in \tau\}$ is an open covering of X. Hence, the condition of Theorem 3.1 is satisfied.

Several applications of these two theorems will now be mentioned.

APPLICATION 3.1. *Let (X, C) and (Y, C') be semi-uniform spaces. Let (Y, C') be complete and f be a semi-uniformly continuous function from X to Y. Let $(\hat{X}, \hat{C})$ denote the semi-uniform completion of (X, C). Then f can be continuously extended to $\hat{X}$* (see [11]).

Proof. If $\tau \in C'$, then $\tau' = \{f^{-1}[T]: T \in \tau\} \in C$. By (a), $\{\mathrm{cl}_{\hat{X}} T: T \in \tau'\} \in \hat{C}$. By (i) of Lemma 2.1, $\{\mathrm{cl}_{\hat{X}} T: T \in \tau'\}$ can be refined by an open covering of $\hat{X}$. Theorem 3.1 then gives the result.

APPLICATION 3.2. *Let X be a complete metric space. Then X is a G_δ subset of each of its Hausdorff extensions* (see [3]).

Proof. Let X be as in the hypothesis. Let Y be a Hausdorff extension of X. Let C be the complete uniformity on X generated by the metric. Then C has a countable base D. Define $S = \bigcap_{\tau \in D} \bigcup_{T \in \tau} \mathrm{Int}_Y \mathrm{cl}_Y T$. By Theorem 3.1, the identity function from X to X has a continuous extension from S to X. Since S is Hausdorff and X is a dense subspace of S, $X = S$. It follows that X is a G_δ subset of Y.

In [12] Taĭmanov has given an extension theorem for continuous functions into compact spaces. General extension theorems concerning Wallman compactifications (see [6] for a general reference) and the Smirnov compactification of a proximity space (see [10]) are corollaries of this result. Consequently, they will not be mentioned separately.

If the image space Y is compact, the word "finite" may be placed in the appropriate positions in the previous two theorems. The following is a stronger result.

APPLICATION 3.3. *Let X be a dense subspace of a space Z. Let Y be a compact Hausdorff space and f be a continuous function from X to Y. f can be continuously extended to Z iff the following holds: if $\mathrm{cl}_Y A \cap \mathrm{cl}_Y B = \emptyset$ for subsets A and B of Y, then $\mathrm{cl}_Z f^{-1}[A] \cap \mathrm{cl}_Z f^{-1}[B] = \emptyset$* (see [12]).

Proof. The proof of the necessity is easy. Assume then that X, Z, f, and Y are as in the hypothesis. Let $\{A_a\}$ be an arbitrary collection of closed sets in Y. Let t belong to the intersection of $\{\mathrm{cl}_Z f^{-1}[A_a]\}$. By Theorem 3.2, it suffices to show that $\{A_a\}$ has non-empty intersection. Let θ be the family of all open sets in Z containing t. Define $\theta' = \{0 \cap X: 0 \in \theta\}$. Define $\theta'' = \{f[0]: 0 \in \theta'\}$. Since θ' has the f.i.p., θ'' does also. Define

$$\mathfrak{U} = \{E \subset Y: 0 \subset E \text{ for } 0 \in \theta''\}.$$

Since Y is compact, there exists $y \in Y$ such that $y \in \bigcap_{A \in \mathfrak{U}} \mathrm{cl}_Y A$. It suffices to show that y belongs to each A_a in $\{A_a\}$. If, for some a, $y \notin A_a$, then there exists an open set V in Y such that $y \in V \subset \mathrm{cl}_Y V \subset Y-A_a$. Since $[\mathrm{cl}]_Y V] \cap A_a = \emptyset$, it follows from the hypothesis that $[\mathrm{cl}_Z f^{-1}[\mathrm{cl}_Y V]] \cap \cap \mathrm{cl}_Z f^{-1}[A_a] = \emptyset$. Since $t \in \mathrm{cl}_Z f^{-1}[A_a]$, $t \in Z-\mathrm{cl}_Z f^{-1}[\mathrm{cl}_Y V]$. This implies that $D = [[Z-\mathrm{cl}_Z f^{-1}[\mathrm{cl}_Y V]] \cap X]$ belongs to θ' and that y belongs to $\mathrm{cl}_Y f[D]$. But this is contradictory to the easily proven fact that $\mathrm{cl}_Y f[D] \subset Y-V$. Hence, $y \in A_a$.

If the image space Y in Theorem 3.2 is Lindelöf, then the conditions in the theorem can clearly be restricted to a countable collection of closed sets. A stronger result by Engelking [5] states that this same restriction suffices when Y is only realcompact. There is, in fact, a slightly stronger result using only countable collections of zero-sets.

APPLICATION 3.4. *Let X be a dense subspace of a space W. Let f be a continuous function from X into a realcompact space Y. f can be continuously extended to W iff the following holds: if $\{Z_i\}$ is a countable collection of zero-sets in Y with empty intersection, then $\{\mathrm{cl}_W f^{-1}[Z_i]\}$ has empty intersection* (see [5]).

Proof. Since Y is realcompact, Y is complete relative to a uniformity with a base of countable cozero-set coverings. This is shown in Chapter 15 of [8]. Since every uniformity is a semi-uniformity, Theorem 3.2 gives the result.

A particular consequence of the next application of Theorem 3.2 is that the Tychonoff preimage of a realcompact space under a perfect mapping (continuous, closed, onto, and the preimage of a singleton is compact) is realcompact.

Application 3.5. *Let X be a Tychonoff space and f be a continuous function from X onto a realcompact space Y. Assume that if Z is a zero-set in X, then $f[Z]$ is closed in Y. Also, assume that, if $y \in Y$ then $f^{-1}(y)$ is Lindelöf. Then X is realcompact* (see [4]).

Proof. Let θ be an arbitrary zero-set ultrafilter on X which is closed under countable intersections and which has empty intersection. Define $\theta' = \{f[A]: A \in \theta\}$. If $y \in \bigcap_{A' \in \theta'} A'$, then, since $f^{-1}(y)$ is Lindelöf, since θ is closed under countable intersections, and since θ has empty intersection, there exists $Z \in \theta$ such that $f^{-1}(y) \cap Z = \emptyset$. But $y \in f[Z]$ implies that this is impossible. Thus, it must follow that $\bigcap_{A' \in \theta'} A' = \emptyset$. If νX is the Hewitt realcompactification of X, then f has a continuous extension to νX. Then, by Theorem 3.2, $\bigcap_{A' \in \theta'} \mathrm{cl}_{\nu X} f^{-1}[A'] = \emptyset$. But, since $\emptyset \neq \bigcap_{A \in \theta} \mathrm{cl}_{\nu X} A \subset \bigcap_{A' \in \theta'} \mathrm{cl}_{\nu X} f^{-1}[A']$, this is impossible. It follows that θ must have empty intersection. Hence, X is realcompact.

The following theorem is interesting because it essentially combines a condition analogous to that given in Theorem 3.1 and a condition given by Taĭmanov in [12]. The theorem will not be proven here, but the approach is similar to that used in proving Taĭmanov's result.

Theorem 3.3. *Let X be a dense subspace of a space Z. Let (Y, C) be a complete semi-uniform space. Let f be a continuous function from X to Y. f can be continuously extended to Z iff the following conditions hold:*

(i) *If $\tau \in C$, then $\{\mathrm{cl}_Z f^{-1}[T]: T \in \tau\}$ is a covering of Z (compare this with the stronger condition in Theorem* 3.1).

(ii) *If $\mathrm{cl}_Y A \cap \mathrm{cl}_Y B = \emptyset$ for subsets A and B of Y, then $\mathrm{cl}_Z f^{-1}[A] \cap \mathrm{cl}_Z f^{-1}[B] = \emptyset$.*

Let Y be a compact Hausdorff space and let C be the complete semi-uniformity on Y generated by all open coverings. Since Y is compact, (i) holds in the previous theorem. In this case, the previous theorem reduces to the condition used in Application 3.3.

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