generic extension of a model-class M implies the existence of an extension which is hereditarily ordinal definable over M (thus, the definability of $0^{\#}$ is not an exception).

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Reçu par la Rédaction le 10. 7. 1972

On semi-uniformities

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Abstract. Regular Hausdorff extensions of topological spaces are studied as completions of generalized uniformities.

Key words. Regular extensions, completions, semi-uniform spaces.

Introduction. In this paper we will consider a generalization of uniform space first defined by Morita [2] which plays an important role in the completion and extension theory of topological spaces. Morita's paper is little known and consequently his original ideas are not referred to as often as they should be.

Our point of view is that one of the most important kinds of information a structure on a space X provides, besides a topology for X, is a topological extension of X. The most satisfactory extension theory appears to us to lie in the setting of regular Hausdorff spaces and the fundamental structures are the semi-uniformities presented here.

1. Preliminaries. In [2, I–IV], Morita considers families of open covers of a topological space which satisfy certain uniformity conditions. In this paper, the theory of semi-uniform spaces will be developed independently of a topology. Much of the terminology is that used in uniform space theory, and the reader is refered to [1].

If $\mathcal{S}$ is a family of coverings of a set X, and $\mathfrak{U}_1$ and $\mathfrak{U}_2 \in \mathcal{S}$, then $\mathfrak{U}_1$ is said to *locally star-refine* $\mathfrak{U}_2$ in $\mathcal{S}$ (and written $\mathfrak{U}_1 < s\mathfrak{U}_2$) if for each $A \in \mathfrak{U}_1$ there is a covering $\mathfrak{U}_A \in \mathcal{S}$ and a set $B \in \mathfrak{U}_2$ such that $\operatorname{st}(A, \mathfrak{U}_A) \subset B$.

A family of coverings in which each covering has a local star-refinement is called a *semi-normal* family.

A *semi-uniformity* $\mathcal{S}$ on a set X is a family of coverings of X which satisfies

(i) $\mathcal{S}$ is a filter with respect to local star-refinement and (ii) for distinct points $x, y \in X$, there is a covering in $\mathcal{S}$, no member of which contains both x and y.

The concepts of a base and a subbase for a semi-uniformity are analogous to those for a uniformity, [1].

The set X, together with a semi-uniformity $\mathcal{S}$ on X, will be called a *semi-uniform space* and will be denoted by $(X, \mathcal{S})$, or sometimes merely by X. The members of a semi-uniformity will be called *semi-uniform coverings*.

Since star-refinement implies local star-refinement, a uniformity on X is a semi-uniformity and every uniform space is a semi-uniform space. The converse is not true.

Each semi-uniformity $\mathcal{S}$ on X determines a topology $\tau(\mathcal{S})$ on X as follows: $U \in \tau(\mathcal{S})$ if and only if for each $x \in U$ there is a $\mathfrak{U} \in \mathcal{S}$ such that $\operatorname{st}(x, \mathfrak{U}) \subset U$. Two semi-uniformities on X are said to be *compatible* if they determine the same topology.

The family $\{\operatorname{st}(x, \mathfrak{U}): \mathfrak{U} \in \mathcal{S}\}$ is a neighborhood system at x in $\tau(\mathcal{S})$. If $\operatorname{st}(\operatorname{st}(x, \mathfrak{U}_1), \mathfrak{U}_1) \subset \operatorname{st}(x, \mathfrak{U}_2)$, then $\operatorname{cl}[\operatorname{st}(x, \mathfrak{U}_1)] \subset \operatorname{st}(x, \mathfrak{U}_2)$, where closure is with respect to $\tau(\mathcal{S})$.

THEOREM 1.1. *For each semi-uniformity $\mathcal{S}$, $(X, \tau(\mathcal{S}))$ is a T_3-space.*

Proof. If $x \in X$ and $\mathfrak{U}_2 \in \mathcal{S}$, there is a $\mathfrak{U}_1 \in \mathcal{S}$ such that $\operatorname{st}(\operatorname{st}(x, \mathfrak{U}_1), \mathfrak{U}_1) \subset \operatorname{st}(x, \mathfrak{U}_2)$. The preceding remarks show that $\tau(\mathcal{S})$ is regular. Property (ii) implies that $\tau(\mathcal{S})$ is T_1.

THEOREM 1.2. *Each semi-uniform covering in $\mathcal{S}$ has an open (in $\tau(\mathcal{S})$) semi-uniform refinement.*

Proof. If $\mathfrak{U}_1, \mathfrak{U}_2 \in \mathcal{S}$ and $\mathfrak{U}_2 <^s \mathfrak{U}_1$, then for each $A \in \mathfrak{U}_2$, there is a $\mathfrak{U}_A \in \mathcal{S}$ and a $B \in \mathfrak{U}_1$ such that $\operatorname{st}(A, \mathfrak{U}_A) \subset B$. For each $x \in A$, $\operatorname{st}(x, \mathfrak{U}_A) \subset B$. Thus $A \subset \operatorname{int} B$ and $\mathfrak{U}_2$ refines $\mathfrak{B} = \{\operatorname{int} B: B \in \mathfrak{U}_1\}$, which refines $\mathfrak{U}_1$. Since $\mathcal{S}$ is a filter, $\mathfrak{B} \in \mathcal{S}$.

By observing that for each $A \subset X$ and for each $\mathfrak{U} \in \mathcal{S}$, $\operatorname{cl} A \subset \operatorname{st}(A, \mathfrak{U})$, one can conclude that each covering in $\mathcal{S}$ has a semi-uniform refinement composed of regular open sets (i.e. sets which are the interiors of their closures).

A topological space (X, τ) is said to be *semi-uniformizable* if there is a semi-uniformity $\mathcal{S}$ on X such that $\tau = \tau(\mathcal{S})$. Such a semi-uniformity is said to be *compatible with the topology*.

THEOREM 1.3. *Each T_3 space is semi-uniformizable.*

Proof. The family of all open coverings is a base for a semi-uniformity which is compatible with the original topology.

A semi-uniformity generated by the base of all open coverings of a T_3-topology will be called a *fine semi-uniformity*.

In view of Theorem 1.2, Theorems 1.1 and 1.3 are the same as Theorem 1 of Morita [2, I], and semi-uniform spaces are the regular T-uniformities of Morita.

2. The category of semi-uniform spaces. A function f from a semi-uniform space $(X, \mathcal{S})$ to a semi-uniform space $(Y, \mathcal{R})$ is called a *mapping* if for every covering $\mathfrak{B} \in \mathcal{R}$, $f^{-1}(\mathfrak{B}) = \{f^{-1}(A): A \in \mathfrak{B}\}$ is a covering in $\mathcal{S}$.

THEOREM 2.1. *Every mapping is continuous.*

The composition of mappings is a mapping; thus, the collection of semi-uniform spaces and mappings forms the *category of semi-uniform spaces*, which has the uniform spaces as a full subcategory.

As in uniform spaces, subspaces, free sums, direct products and quotients exist, and are defined analogously. The definitions will be restated here for the purpose of unity and completeness.

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are semi-uniformities on X, $\mathcal{S}_1$ is said to be *coarser than* $\mathcal{S}_2$ (and $\mathcal{S}_2$ *finer than* $\mathcal{S}_1$) if $\mathcal{S}_1 \subset \mathcal{S}_2$.

A function e from a semi-uniform space X into a semi-uniform space Y is an *embedding* if the semi-uniformity on X is the coarsest one making e a mapping. A subset $A \subset Y$ is a *subspace* if the identity $i: A \to Y$ is an embedding (i.e. the structure on A is the relativized structure of Y).

If $\{X_\lambda: \lambda \in \Lambda\}$ is a family of semi-uniform spaces, the *sum* Σ is defined to be the ordered pairs (x, λ), where $\lambda \in \Lambda$ and $x \in X_\lambda$. The cannonical injections $i_\lambda: X_\lambda \to \Sigma$ are defined as $i_\lambda(x) = (x, \lambda)$, and the semi-uniformity of Σ is all coverings whose inverse image under i_λ is a semi-uniform covering for each $\lambda \in \Lambda$. If Y is any semi-uniform space and $f: \Sigma \to Y$ and $g: \Sigma \to Y$ are two different mappings, then there is a $\lambda \in \Lambda$ such that $f \circ i_\lambda \neq g \circ i_\lambda$. For any family $\{f_\lambda\}$ of mappings $f_\lambda: X_\lambda \to Y$, there is a mapping $f: \Sigma \to Y$ such that $f \circ i_\lambda = f_\lambda$.

The *product* Π of a family $\{X_\lambda: \lambda \in \Lambda\}$ of semi-uniform spaces is the Cartesian product set with the coarsest semi-uniformity making each projection $p_\lambda: \Pi \to X_\lambda$ a mapping. The existence of such a semi-uniformity is guaranteed by the following theorem.

THEOREM 2.2. *Let $\{f_\lambda: \lambda \in \Lambda\}$ be a family of functions on a set X into various semi-uniform spaces $(Y_\lambda, \mathcal{S}_\lambda)$ which separates points. Then there is a coarsest semi-uniformity on X making each f_λ a mapping.*

Proof. For each $\lambda \in \Lambda$, $\{f_\lambda^{-1}(\mathfrak{U}): \mathfrak{U} \in \mathcal{S}_\lambda\}$ is a semi-normal family. Since $\{f_\lambda\}$ separates points, the family of finite intersections of covers in $\{f_\lambda^{-1}(\mathfrak{U}): \mathfrak{U} \in \mathcal{S}_\lambda, \lambda \in \Lambda\}$ is a base for the desired semi-uniformity.

If Y is any semi-uniform space and $f: Y \to \Pi$ and $g: Y \to \Pi$ are different mappings, then there is a $\lambda \in \Lambda$ such that $p_\lambda \circ f \neq p_\lambda \circ g$. Also, for every family $\{f_\lambda\}$ of mappings $f_\lambda: Y \to X_\lambda$, there exists a mapping $f: Y \to \Pi$ such that $p_\lambda \circ f = f_\lambda$; namely, $f(y)_\lambda = f_\lambda(y)$.

A *quotient map* $q: X \to Q$ is an onto mapping such that whenever $q = g \circ f$ and $g: Q' \to Q$ is one-to-one and onto, then g is an isomorphism. Every map $f: X \to Y$ has the form $g \circ q$ where $q: X \to Q$ is a quotient map and $g: Q \to Y$ is one-to-one. The space Q is simply the set $f[X]$ with the finest semi-uniformity making f a mapping, q agrees with f, and g is the identity.

THEOREM 2.3. *The semi-uniform topology of a subspace, sum, or product is the subspace, sum, or product topology, respectively.*

As is the case for uniform quotients, the quotient topology need not agree with the semi-uniform topology on quotient spaces.

EXAMPLE. Let R denote the real numbers with the usual uniformity, let $X = (0,1) \cup (1,2) \subset R$ and $Y = (0,1] \subset R$, each with the subspace semi-uniformity. The function $f\colon X \to Y$ defined as $f(x) = x$, $0 < x < 1$ and $f(x) = 1$ for $1 < x < 2$ is a mapping. The singleton $\{1\}$ is open in the quotient topology on Y, but not in the topology of the quotient semi-uniformity, since every semi-uniform covering of X contains a set which intersects both $(0,1)$ and $(1,2)$.

3. Complete spaces and completions. A family $\mathcal{F}$ of subsets of a semi-uniform space $(X, \mathcal{S})$ is said to be *bound* if every covering in $\mathcal{S}$ contains some element which intersects each member of $\mathcal{F}$. $\mathcal{F}$ is called a *cluster* if it contains at least one member of each semi-uniform covering and is maximal with respect to being bound.

In [2, I], Morita defines an equivalence class of Cauchy families as follows: A family $\mathcal{C}$ of subsets of a semi-uniform space $(X, \mathcal{S})$ is a Cauchy family if $\mathcal{C}$ has the finite intersection property and for any covering $\mathfrak{U}_1 \in \mathcal{S}$ there is a set $F \in \mathcal{C}$ and a covering $\mathfrak{U}_2 \in \mathcal{S}$ such that $\mathrm{st}(F, \mathfrak{U}_2) \subset U \in \mathfrak{U}_1$. Cauchy families $\mathcal{C}$ and $\mathcal{C}'$ are equivalent if for any $F \in \mathcal{C}$ and any $\mathfrak{U}_1 \in \mathcal{S}$ there is any $F' \in \mathcal{C}'$ and $\mathfrak{U}_2 \in \mathcal{S}$ such that $\mathrm{st}(F', \mathfrak{U}_2) \subset \mathrm{st}(F, \mathfrak{U}_1)$.

LEMMA 3.1. *Each cluster contains a unique equivalence class of Cauchy families.*

Proof. Let $\mathcal{F}$ be a cluster and define $\mathcal{G}$ to be the family of all members $F \in \mathcal{F}$ such that there is a $A \in \mathcal{F}$ and a $\mathfrak{U} \in \mathcal{S}$ and $\mathrm{st}(A, \mathfrak{U}) \subset F$. If $F_1, \ldots, F_n \in \mathcal{G}$, then $\mathrm{st}(A_i, \mathfrak{U}_i) \subset F_i$ for $A_i \in \mathcal{F}$ and $\mathfrak{U}_i \in \mathcal{S}$. Let $\mathfrak{U} \in \mathcal{S}$ be a common refinement of the $\mathfrak{U}_i$. Since $\{A_i\}$ is bound, $\emptyset \neq \bigcap \mathrm{st}(A_i, \mathfrak{U}) \subset \bigcap F_i$. Thus $\mathcal{G}$ has the finite intersection property. It is easy to verify that $\mathcal{G}$ contains at least one member of each semi-uniform cover, and is a Cauchy family. Suppose $\mathcal{C}$ is a Cauchy family equivalent to $\mathcal{G}$ and $C \in \mathcal{C}$. For each $\mathfrak{U} \in \mathcal{S}$, there is a $\mathfrak{U}_1 \in \mathcal{S}$ such that $\mathfrak{U}_1 <^s \mathfrak{U}$. Let $A \in \mathfrak{U}_1 \cap \mathcal{G}$. Then there is a $U \in \mathfrak{U}$ and $\mathfrak{U}_A \in \mathcal{S}$ such that $\mathrm{st}(A, \mathfrak{U}_A) \subset U$. Since $F \cap \mathrm{st}(A, \mathfrak{U}_A) \neq \emptyset$ for each $F \in \mathcal{F}$, U intersects each member of $\mathcal{F}$. By the equivalence of $\mathcal{C}$ to $\mathcal{G}$, there is a $C' \in \mathcal{C}$ and a $\mathfrak{U}' \in \mathcal{S}$ such that $\mathrm{st}(C', \mathfrak{U}') \subset \mathrm{st}(A, \mathfrak{U}_A) \subset U$. Since $C \cap C' \neq \emptyset$, it follows that $U \cap C \neq \emptyset$. Thus $\mathcal{F} \cup \{C\}$ is bound, which implies that $C \in \mathcal{F}$.

If $\mathcal{C}$ and $\mathcal{C}'$ are Cauchy families and $\mathcal{C} \cup \mathcal{C}'$ is bound, then $\mathcal{C} \sim \mathcal{C}'$. Let $C \in \mathcal{C}$ and $\mathfrak{U} \in \mathcal{S}$. There is a $C' \in \mathcal{C}'$ and $\mathfrak{U}' \in \mathcal{S}$ such that $\mathrm{st}(C', \mathfrak{U}') \subset U$ for some $U \in \mathfrak{U}$. Since $\{C, C'\}$ is bound, $\emptyset \neq C \cap \mathrm{st}(C', \mathfrak{U}') \subset C \cup U$. Thus $\mathrm{st}(C', \mathfrak{U}') \subset U \subset \mathrm{st}(C, \mathfrak{U})$ and the equivalence is shown. As a cluster is a bound family, it can contain only one equivalence class.

LEMMA 3.2. *Each equivalence class of Cauchy families is a cluster.*

Proof. Let $\mathcal{K}$ be an equivalence class of Cauchy families. Since each Cauchy family is contained in a Cauchy ultrafilter, $\mathcal{K}$ contains at least one member of each semi-uniform cover.

Let $\mathfrak{U} \in \mathcal{S}$ and let $\mathfrak{U}_1 \in \mathcal{S}$ such that $\mathfrak{U}_1 <^s \mathfrak{U}$. If $A \in \mathfrak{U}_1 \cap \mathcal{K}$, there is a $U \in \mathfrak{U}$ and $\mathfrak{U}_A \in \mathcal{S}$ such that $\mathrm{st}(A, \mathfrak{U}_A) \subset U$. If $E \in \mathcal{K}$, then E and A are in equivalent Cauchy families $\mathcal{C}$ and $\mathcal{C}'$, respectively. Thus there is a $B \in \mathcal{C}$ and a $\mathfrak{U}_1 \in \mathcal{S}$ such that $\mathrm{st}(B, \mathfrak{U}_1) \subset \mathrm{st}(A, \mathfrak{U}_A) \subset U$. Since $E \cap B \neq \emptyset$, $E \cap U \neq \emptyset$. It follows that $\mathcal{K}$ is a bound family.

Suppose $\mathcal{K} \cup \{A\}$ is bound. If $\mathcal{G}$ is the family of all $E \in \mathcal{K}$ such that $\mathrm{st}(F, \mathfrak{U}) \subset E$ for some $F \in \mathcal{K}$, $\mathfrak{U} \in \mathcal{S}$, then $\mathcal{G} \cup \{A\}$ is a Cauchy family. To see this, it suffices to show that $\mathcal{G} \cup \{A\}$ has the finite intersection property. If $E_1, \ldots, E_n \in \mathcal{G}$, then as in Lemma 1, there is a $\mathfrak{U} \in \mathcal{S}$ and $F_1, \ldots, F_n \in \mathcal{K}$ such that $\mathrm{st}(F_i, \mathfrak{U}) \subset E_i$ for each i. Since $\{F_1, \ldots, F_n, A\}$ is bound, $\emptyset \neq A \cap \bigcap \mathrm{st}(F_i, \mathfrak{U}) \subset A \cap \bigcap E_i$.

From Lemma 3.1 it follows that the equivalence class containing $\mathcal{G} \cup \{A\}$ and $\mathcal{K}$ are contained in the same cluster and are thus identical. Therefore $A \in \mathcal{K}$ and $\mathcal{K}$ is maximal with respect to being bound.

From Lemmas 3.1 and 3.2 it follows that clusters and equivalence classes of Cauchy families are identical.

A cluster $\mathcal{F}$ is said to *converge to* $x \in X$ if $\mathrm{st}(x, \mathfrak{U}) \in \mathcal{F}$ for each $\mathfrak{U} \in \mathcal{S}$ (or, what is equivalent, if $\bigcap \{\bar{F}\colon F \in \mathcal{F}\} = \{x\}$). Distinct clusters cannot converge to the same point. The family $\mathcal{F}_x = \{A \subset X\colon x \in \bar{A}\}$ is a cluster converging to x. A space $(X, \mathcal{S})$ is said to be *complete* if each cluster converges.

THEOREM 3.1. *Every T_3 space has a compatible semi-uniformity which is complete.*

The fine semi-uniformity defined in Theorem 1.3 is complete, as was noted by Morita [2, I, Theorem 8].

THEOREM 3.2. *A closed subspace of a complete space is complete.*

Proof. A cluster in the subspace is contained in a cluster in the large space, which must converge to a point in the subspace.

THEOREM 3.3. *The product of complete spaces is complete.*

Proof. If $\mathcal{F}$ is a cluster in a product ΠX_a of complete spaces, then $p_a(\mathcal{F})$ is a cluster in X_a and must converge to $x_a \in X_a$. It is easy to see that $\mathcal{F}$ converges to $x = (x_a)$.

Remark. The relation between clusters and Cauchy filters, Lemmas 3.1 and 3.2, shows that a semi-uniform space is complete if and only if each Cauchy filter converges. Thus the concepts of completeness agree in uniform spaces and semi-uniform spaces.

A semi-uniform space Y is a *completion* of a space X if Y is complete and contains a dense subspace isomorphic to X.

The existence and uniqueness of completions were established by Morita [2, I, II]; we will simply outline their construction.

THEOREM 3.4. *Each semi-uniform space $(X, \mathcal{S})$ has a unique completion.*

Let X^* be the set of all clusters of X. For each $F \subset X$, define $F^* = \{\mathcal{F} \in X^*: F \in \mathcal{F}\}$, and for each $\mathfrak{U} \in \mathcal{S}$, let $\mathfrak{U}^* = \{U^*: U \in \mathfrak{U}\}$. The collection $\{\mathfrak{U}^*: \mathfrak{U} \in \mathcal{S}\}$ is a base for a semi-uniformity $\mathcal{S}^*$ on X^* which is complete. The embedding $e(x) = \mathcal{F}_x$ for $x \in X$ is an isomorphism and $e[X]$ is a dense subspace of X^*. For each $A \subset X$, A^* is the closure of A in X^* with respect to $\tau(\mathcal{S}^*)$.

The above construction of the completion differs slightly from Morita since he did not assume $\tau(\mathcal{S})$ to be T_1, and thus he defined X^* to be $X \cup \{\text{nonconvergent equivalence classes}\}$.

The uniqueness of the completion follows from the following theorem.

THEOREM 3.5. *If f is a mapping on a subspace X of a semi-uniform space into a complete semi-uniform space Y, then f may be extended uniquely to a mapping on the closure of X.*

Proof. Since X is dense in $\mathrm{cl}X$ and Y is T_2, f has at most one continuous extension.

For each $x \in \mathrm{cl}X$, the collection $\mathcal{F}_x = \{A \subset X: x \in \mathrm{cl}A\}$ is a cluster in X. Since f is a mapping, $f[\mathcal{F}_x]$ is contained in a unique cluster in Y, which converges to a point, $F(x)$.

Thus F: $\mathrm{cl}X \to Y$ is defined and coincides with f on X. If $A, B \subset Y$ and $\mathrm{cl}_Y A \subset B$, then $\mathrm{cl}f^{-1}(A) \subset F^{-1}(B)$. Thus if $\mathfrak{U}_1, \mathfrak{U}_2$ are semi-uniform coverings of Y and $\mathfrak{U}_1 < s\mathfrak{U}_2$, then $\mathrm{cl}f^{-1}(\mathfrak{U}_1) < F^{-1}(\mathfrak{U}_2)$ and F is a mapping on $\mathrm{cl}X$.

COROLLARY. *A mapping f: $X \to Y$ has an extension to a mapping f^*: $X^* \to Y^*$* [2, II, Theorem 3].

The completion functor is thus a reflector and has the following preservation properties.

THEOREM 3.6.

(a) *If X is a uniform space, then X^* is also.*

(b) *If $X \subset Y$, then $X^* \subset Y^*$.*

(c) $(\Pi X_a)^* = \Pi X_a^*$ (*= means isomorphic*).

(d) $(\Sigma X_a)^* = \Sigma X_a^*$.

(e) *If q is a quotient, then q^* is also.*

Proof. (a) If $\mathfrak{U}_1$ star-refines $\mathfrak{U}_2$ and $\mathfrak{U}_2$ star-refines $\mathfrak{U}_3$, then $\mathfrak{U}_1^*$ star-refines $\mathfrak{U}_3^*$.

(b) Since $e[X] \subset e[Y] \subset Y^*$, and since $\mathrm{cl}\,e[X]$ is complete, the uniqueness of the completion implies $X^* = \mathrm{cl}\,e[X] \subset Y^*$.

(c) ΠX_a is a dense semi-uniform subspace of both ΠX_a^* and $(\Pi X_a)^*$. As these two spaces are complete, they must be isomorphic.

The proof of (d) is identical to the proof of (c).

(e) If q^*: $X^* \to Y^*$ factors into $g \circ h$ where h: $X^* \to A$ and g: $A \to Y^*$ with g one-to-one and onto, then q factors into $g' \circ h'$, where h': $X \to g^{-1}[Y]$ and g': $g^{-1}[Y] \to Y$. Clearly g' is one-to-one and onto, and is thus an isomorphism. Since $g^{-1}[Y]$ is a dense subspace of A, $g^{-1}[Y]$ is a dense subspace of A^*. It follows that A^* is isomorphic to Y^* and hence that g is an isomorphism.

THEOREM 3.7. *For each T_3 extension Y of a topological space X, there is a compatible semi-uniformity $\mathcal{S}$ on X, such that the completion of $(X, \mathcal{S})$ is topologically equivalent to Y.*

Proof. Y is complete with respect to the semi-uniformity generated by the base consisting of all open covers of Y. The trace of this semi-uniformity on X (i.e. the subspace semi-uniformity) is the desired one.

4. Properties of the induced topologies. In this section we are concerned with the problem of determining the topological properties of X and X^* with respect to structural properties of the semi-uniformity. Throughout, all topological spaces are assumed to be T_3.

If (X, τ) is a topological space and $\mathcal{C}$ is a family of coverings of X generated by the finite open covers, then an extension of X may be obtained (in an identical manner as X^*) whose topology is compact [2, II, Theorem 4]. However, $\mathcal{C}$ may not be a semi-uniformity and the topology of the extension need not be T_3. It is known that the family of all finite open covers of a topological space (X, τ) generates a compatible uniformity if and only if τ is normal. The same result is true for semi-uniform spaces.

THEOREM 4.1. *The family of all finite open covers of a topological space (X, τ) is a base for a compatible semi-uniformity if and only if τ is normal.*

Proof. If τ is normal, the finite open covers generate a compatible uniformity. Conversely, suppose the finite open covers are a base for a compatible semi-uniformity. Let A and B be disjoint closed subsets of X. Then $\{X-A, X-B\}$ is a semi-uniform covering and has a finite, open local-star refinement $\mathfrak{U} = \{U_1, \ldots, U_n\}$. For each U_i, $\mathrm{cl}\mathfrak{U}_i \subset X-A$ or $\mathrm{cl}\,U_i \subset X-B$. If $V = \bigcup\{U_i: \mathrm{cl}\,U_i \subset X-B\}$, then $A \subset V \subset \mathrm{cl}V \subset X-B$, i.e. A and B are separated by disjoint open sets $V, X-\mathrm{cl}V$, respectively.

The next few theorems are concerned with the consequences of a semi-uniformity having a base of finite covers.

THEOREM 4.2. *If the finite coverings in a semi-uniformity* $\mathcal{S}$ *form a base for* $\mathcal{S}$, *then* $\mathcal{S}$ *is a precompact uniformity.*

Proof. It suffices to show that the finite coverings form a base for a uniformity. Let $\mathfrak{U}$ be a finite covering locally-star-refined by a finite covering $\mathfrak{U}' = \{A_1, A_2, \ldots, A_n\}$. For each A_i, there is a finite covering $\mathfrak{U}_i \in \mathcal{S}$ and a $U_i \in \mathfrak{U}$ such that $\mathrm{st}(A_i, \mathfrak{U}_i) \subset U_i$. There is a finite covering $\mathfrak{U}_0 \in \mathcal{S}$ which refines $\mathfrak{U}'$ and $\mathfrak{U}_i$, $1 \leqslant i \leqslant n$. Clearly $\mathfrak{U}_0$ star-refines $\mathfrak{U}$.

THEOREM 4.3. *A compact topological space* (X, τ) *has only one compatible semi-uniformity.*

Proof. There exists at least one compatible semi-uniformity, Theorem 1.3. Suppose $\mathcal{S}$ is a compatible semi-uniformity on X, and let $\mathfrak{A}$ be any open cover of X. For each $x \in X$, there is a $\mathfrak{U}_x \in \mathcal{S}$ and an $A_x \in \mathfrak{A}$ such that $\mathrm{st}\big(\mathrm{st}(x, \mathfrak{U}_x), \mathfrak{U}_x\big) \subset A_x$. Since $\tau = \tau(\mathcal{S})$ is compact,

$$X \subset \bigcup \{\mathrm{st}(x_i, \mathfrak{U}_{x_i}) \mid 1 \leqslant i \leqslant k\}.$$

If $\mathfrak{U} \in \mathcal{S}$ refines each $\mathfrak{U}_{x_i}$, then $\mathfrak{U}$ refines $\mathfrak{A}$ and $\mathfrak{A} \in \mathcal{S}$.

THEOREM 4.4. *A semi-uniformity* $\mathcal{S}$ *on* X *has a base of finite covers if and only if the completion* $(X^*, \mathcal{S}^*)$ *is a compact topological space.*

Proof. The sufficiency follows from Theorem 4.2. If $\tau(\mathcal{S}^*)$ is compact, the family of all finite open covers of $\big(X^*, \tau(\mathcal{S}^*)\big)$ is a base for $\mathcal{S}^*$. Thus $\mathcal{S}$ has a base of finite covers.

As a partial generalization cf compact extensions we have the following

THEOREM 4.5. *If* $\mathcal{S}$ *is a complete semi-uniformity on* X *with a base of countable covers, then every ultrafilter of closed sets with the countable intersection property converges.*

Proof. Let $\mathcal{F}$ be an ultrafilter of closed sets having the countable intersection property and let $\mathfrak{U}$ be any countable covering in $\mathcal{S}$. Since $X = \bigcup \{U_i\colon U_i \in \mathfrak{U}\}$, and $\mathcal{F}$ has the countable intersection property, some member $U_j \in \mathfrak{U}$ intersects each set in $\mathcal{F}$. Thus $\mathrm{cl}\, U_j \in \mathcal{F}$. Each covering in $\mathcal{S}$ is refined by a countable closed covering in $\mathcal{S}$, and thus $\mathcal{F}$ can be extended to a cluster. This cluster, and hence $\mathcal{F}$, must converge since $\mathcal{S}$ is complete.

COROLLARY A. *If* $\mathcal{S}$ *is a complete semi-uniformity on* X *with a base of countable covers and* $\big(X, \tau(\mathcal{S})\big)$ *is normal, then* $\big(X, \tau(\mathcal{S})\big)$ *is realcompact.*

Proof. It suffices to point out that if $\big(X, \tau(\mathcal{S})\big)$ is normal, then every covering in $\mathcal{S}$ can be refined by a zero-set covering in $\mathcal{S}$.

COROLLARY B. *If* $\mathcal{S}$ *is a semi-uniformity on* X *with a base of countable coverings and* $\big(X^*, \tau(\mathcal{S}^*)\big)$ *is normal, then* $\big(X^*, \tau(\mathcal{S}^*)\big)$ *is a realcompactification of* $\big(X, \tau(\mathcal{S})\big)$.

Remark. The converse of Theorem 4.5 (and the corollaries) is not true. For example, let X be the discrete space of cardinal c and let $\mathcal{S}$ be the fine semi-uniformity on X. Then $(X, \mathcal{S})$ is complete, every ultrafilter of closed sets with the countable intersection property converges (since $\big(X, \tau(\mathcal{S})\big)$ is realcompact), but the covering of singletons is in $\mathcal{S}$ and has no countable refinement.

This example shows that it is not necessary for each covering in $\mathcal{S}$ to have a countable semi-uniform refinement for $\tau(\mathcal{S}^*)$ to be realcompact. However, we have the following.

THEOREM 4.6. *Each realcompact extension of a topological space* (X, τ) *is topologically equivalent to* $\big(X^*, \tau(\mathcal{S}^*)\big)$, *where* $\mathcal{S}$ *is a compatible semi-uniformity on* X *possessing a base of countable covers.*

Proof. If $\mathcal{R}$ denotes the usual uniformity on the reals, then for each continuous real-valued function f on a realcompact space Y, $f^{-1}(\mathcal{R})$ is a normal family with a countable base. The uniformity on Y generated by $\{f^{-1}(\mathcal{R})\}$ has a countable base, is compatible with the topology of Y, and is complete. The trace of this uniformity on a dense subspace X has the required properties.

We turn now to the question of when $\tau(\mathcal{S})$ and $\tau(\mathcal{S}^*)$ are completely regular. Clearly $\tau(\mathcal{S})$ is completely regular if and only if $\mathcal{S}$ is compatible to a uniformity. An answer of this kind says nothing about the structure of $\mathcal{S}$ itself. If $\mathcal{S}$ contains a compatible uniformity, $\tau(\mathcal{S})$ is completely regular. We do not know if the converse holds, except in the following case.

THEOREM 4.7. *If* $\big(X, \tau(\mathcal{S})\big)$ *is completely regular and locally compact, then* $\mathcal{S}$ *contains a compatible uniformity.*

Proof. The family of finite open covers of sets whose complements or closures are compact is a base for a compatible precompact uniformity on X. To show that this uniformity is contained in $\mathcal{S}$, it suffices to prove that if $X = V \cup U$ where $V, U \in \tau(\mathcal{S})$ and $\overline{V}$ and $X - U$ are compact, then $\{V, U\} \in \mathcal{S}$. If $x \in X - U$, then there is a $\mathfrak{U}_x \in \mathcal{S}$ such that $\mathrm{st}\big(\mathrm{st}(x, \mathfrak{U}_x), \mathfrak{U}_x\big) \subset V$. Since $X - U$ is compact, there are a finite number of x_i such that $\{\mathrm{st}(x_i, \mathfrak{U}_{x_i})\colon 1 \leqslant i \leqslant n\}$ covers $X - U$. There is a $\mathfrak{U} \in \mathcal{S}$ which refines $\mathfrak{U}_{x_i}$ for each i. Then, since $\mathrm{st}(X - U, \mathfrak{U}) \subset V$, $\mathfrak{U}$ refines $\{V, U\}$, and $\{V, U\} \in \mathcal{S}$.

The condition that $\mathcal{S}$ contain a compatible uniformity is not sufficient to guarantee complete regularity of $\tau(\mathcal{S}^*)$ as the following example shows.

Let (Y, τ) be a T_3 space which is not completely regular and which has a dense, locally compact subspace X. See 18G of [3] for such a space. If $\mathcal{S}$ is the fine semi-uniformity on Y compatible with τ, then $\mathcal{S}_X$, the subspace semi-uniformity on X, contains a compatible uniformity by Theorem 4.7. However, since $\mathcal{S}_X^* = \mathcal{S}$, $\tau(\mathcal{S}_X^*) = \tau$ is not completely regular.

A cover $\mathfrak{U}$ of X is *star-finite* if for each $A \in \mathfrak{U}$, the card $\{U \in \mathfrak{U}: U \cap A \neq \emptyset\} = |\mathrm{st}(A, \mathfrak{U})|$ is finite. A family $\mathcal{D}$ of covers is said to be a *mutually star-finite family* if for $\mathfrak{U}_1, \mathfrak{U}_2 \in \mathcal{D}$, $|\mathrm{st}(A, \mathfrak{U}_2)|$ is finite for each $A \in \mathfrak{U}_1$. Each cover in a mutually star-finite family is star-finite.

THEOREM 4.8. *If $\mathcal{B}$ is a mutually star-finite base for a semi-uniformity $\mathcal{S}$ on X, then $\tau(\mathcal{S})$ is completely regular.*

Proof. We will show, that for each element $\mathfrak{U} \in \mathcal{B}$, there is a sequence $\mathfrak{U} = \mathfrak{U}_1', \mathfrak{U}_2', \mathfrak{U}_3', \ldots$ of open covers of X such that $\mathfrak{U}_{i+1}' <^* \mathfrak{U}_i'$ (i.e. $\mathfrak{U}_{i+1}'$ star-refines $\mathfrak{U}_i'$). The semi-uniformity generated by these normal sequences is a uniformity equivalent to $\mathcal{S}$.

Without loss of generality, we may assume that $\mathcal{B}$ consists of open coverings.

Let $\mathfrak{U}_1, \mathfrak{U}_2 \in \mathcal{B}$ such that $\mathfrak{U}_2 < s\mathfrak{U}_1$. For each $A \in \mathfrak{U}_2$ there is a covering $\mathfrak{U}_A$ and $U_A \in \mathfrak{U}_1$ such that $\mathrm{st}(A, \mathfrak{U}_A) \subset U_A$. Choose $\mathfrak{U}_A' \in \mathcal{B}$ such that $\mathfrak{U}_A'$ refines $\mathfrak{U}_B$ for each $B \in \mathfrak{U}_2$ for which $B \cap \mathrm{st}(A, \mathfrak{U}_2) \neq \emptyset$, and having the property that if $C \in \mathfrak{U}_A'$ and $C \cap A \neq \emptyset$, then $C \subset B$ for some $B \in \mathfrak{U}_2$. Since only finitely many $B \in \mathfrak{U}_2$ intersect $\mathrm{st}(A, \mathfrak{U}_2)$, there is at least one covering in $\mathcal{B}$ which refines each $\mathfrak{U}_B$ and $\mathfrak{U}_2$.

Define $\mathfrak{U}_2' = \{V: V \in \mathfrak{U}_A' \text{ and } V \cap A \neq \emptyset, A \in \mathfrak{U}_2\}$. $\mathfrak{U}_2'$ is an open cover of X, is star-finite, and $\mathfrak{U}_2' <^* \mathfrak{U}_1$. For each $A \in \mathfrak{U}_2$, there is a $\mathfrak{U}_A'' \in \mathcal{B}$ such that $\mathfrak{U}_A'' <^s \mathfrak{U}_A'$. Define $\mathfrak{U}_3 = \{V: V \in \mathfrak{U}_A'' \text{ and } V \cap A \neq \emptyset, A \in \mathfrak{U}_2\}$. $\mathfrak{U}_3$ is a star-finite open cover of X, $\mathfrak{U}_3$ locally star-refines $\mathfrak{U}_2'$ with respect to $\mathcal{B}$, and $\{\mathfrak{U}_3, \mathfrak{U}_2'\}$ is mutually star-finite.

For each $A \in \mathfrak{U}_3$, coverings $\mathfrak{U}_A \in \mathcal{B}$ and sets $U_A \in \mathfrak{U}_2'$ may be chosen as before. Since $\mathfrak{U}_3$ refines $\mathfrak{U}_2$, each $A \in \mathfrak{U}_3$ is contained in some $A_0 \in \mathfrak{U}_2$. Let $\mathfrak{U}_A' \in \mathcal{B}$ refine $\mathfrak{U}_B$ for all B in $\mathfrak{U}_3$ for which $B \cap \mathrm{st}(A, \mathfrak{U}_3) \neq \emptyset$, and let $\mathfrak{U}_A'$ refine $\mathfrak{U}_{A_0}''$. Then if $C \in \mathfrak{U}_A'$ and $C \cap A \neq \emptyset$, there is a $B \in \mathfrak{U}_3$ containing C.

The star-finite open covering $\mathfrak{U}_3'$ is derived from $\mathfrak{U}_3$ as $\mathfrak{U}_2'$ was from $\mathfrak{U}_2$ and $\mathfrak{U}_3' <^* \mathfrak{U}_2'$. The procedure may be continued inductively.

COROLLARY. *If for each $\alpha \in \Omega$, $\mathcal{D}_\alpha$ is a mutually star-finite family which forms a filter base with respect to local-star refinement, and the family of all finite intersections from $\{\mathcal{D}_\alpha\}$ is a base for a semi-uniformity $\mathcal{S}$, then $\tau(\mathcal{S})$ is completely regular.*

The usual uniformity $\mathcal{R}$ on the reals has a base satisfying the hypothesis of Theorem 4.8, so each realcompact space has a compatible complete semi-uniformity satisfying the conditions of the corollary.

If $\mathcal{S}$ has a subbase $\{\mathcal{D}_\alpha\}$ as in the corollary, then so does $\mathcal{S}^*$, and $\tau(\mathcal{S}^*)$ is completely regular.

THEOREM 4.9. *If $\mathcal{B}$ is a mutually star-finite subfamily of $\mathcal{S}$, then $\mathcal{B}^* = \{\mathfrak{U}^*: \mathfrak{U} \in \mathcal{B}\}$ is a mutually star-finite subfamily of $\mathcal{S}^*$.*

Unfortunately, not all completely regular spaces have a compatible complete semi-uniformity with a subbase of star-finite covers. For example, each star-finite open cover of a countably compact space must be finite, so a countably compact, non compact space has no compatible, complete semi-uniformity as above.

Morita [2, IV] pointed out that if a semi-uniformity $\mathcal{S}$ on X has a countable base, then $\tau(\mathcal{S})$ is metrizable. A semi-uniformity with a countable base need not be a uniformity, as the following example shows.

Let X be the real numbers. For each pair of integers $n \geqslant 1$, $i \geqslant 1$ let $U_i(n)$ be an interval of length $1-2^{-n}$ with center $n+i$. Let $\mathfrak{U}_i = \{U_i(n): n = 1, 2, \ldots\} \cup \{\text{all intervals of length } 2^{-i}\}$. Then $\{\mathfrak{U}_n\}$ is a base for a semi-uniformity on X which is not a uniformity.

THEOREM 4.10. *For each semi-uniformity with a countable base, there is a compatible finer uniformity with a countable base.*

Proof. If $\{\mathfrak{U}_n\}$ is a countable base for a semi-uniformity $\mathcal{S}$, we can assume $\mathfrak{U}_{n+1}$ refines $\mathfrak{U}_n$. For each $y \in X$, and each integer n, let $W_n(y) = \mathrm{st}(y, \mathfrak{U}_n)$. Then $\{W_n(y): n = 1, 2, \ldots\}$ is a nested neighborhood base at y having the property: (Q) for any n, there is a $m(n, y) > n$ such that $W_m(x) \cap W_m(y) \neq \emptyset$ implies that $W_m(x) \subset W_n(y)$.

Let $y \in X$. For each n there is a $p(n) > n$ and a $U \in \mathfrak{U}_n$ such that $W_p(y) \subset U$. Let $k(n) = \max\big(p(n), m(n)\big)$. Define $V_1(y) = W_1(y)$ and $V_j(y) = W_{k^j(1)}(y)$, where $k^j(1) = k\big(k^{j-1}(1)\big)$ for $j > 1$. Clearly $k^j(1) \geqslant j+1$ and $V_j(y) \subset W_j(y)$.

For each y we have another nested neighborhood base $\{V_n(y)\}$ satisfying (Q). Let $\mathfrak{A}_n = \{V_n(y): y \in X\}$. The covers $\mathfrak{A}_n$ are open, $\mathfrak{A}_n$ refines $\mathfrak{U}_n$ and if $V_{n+1}(x) \cap V_{n+1}(y) \neq \emptyset$, then either $V_{n+1}(y) \subset V_n(x)$ or $V_{n+1}(x) \subset V_n(y)$.

If $\mathfrak{C}_n = \{\mathrm{st}(x, \mathfrak{A}_n): x \in X\}$, then $\mathfrak{C}_{n+2} <^* \mathfrak{C}_n$ and $\mathfrak{C}_{n+1} < \mathfrak{A}_n < \mathfrak{U}_n$. Thus $\{\mathfrak{C}_n\}$ is a countable base for a uniformity which is finer than $\mathcal{S}$. Since each cover $\mathfrak{C}_n$ is open in $\tau(\mathcal{S})$, this is a compatible uniformity.

5. Fine and subfine spaces. In general, distinct semi-uniformities on a set X can give topologically equivalent extensions. However, among those giving equivalent topological extensions, there is a finest, namely the trace on X of the fine semi-uniformity compatible with the topology of the extension. This semi-uniformity can be characterized as follows.

THEOREM 5.1. *$\mathcal{S}^*$ is the fine uniformity on $\big(X^*, \tau(\mathcal{S}^*)\big)$ if and only if every bound family in $(X, \mathcal{S})$ is contained in a cluster.*

Proof. Let $\mathcal{S}^*$ be the fine uniformity on $\big(X^*, \tau(\mathcal{S}^*)\big)$ and $\mathcal{F}$ a family of subsets of X which is not in a cluster. Then $\{X^* - F^*: F \in \mathcal{F}\}$ is an open cover of X^* and is thus refined by $\mathfrak{U}^*$ for some $\mathfrak{U} \in \mathcal{S}$. Thus no member of $\mathfrak{U}$ intersects each $F \in \mathcal{F}$ and $\mathcal{F}$ is not bound.

Conversely, let $\mathfrak{A}_1, \mathfrak{A}_2$ be open covers of X^* such that $\mathrm{cl}\mathfrak{A}_2 < \mathfrak{A}_1$. Since $\bigcap \{X^* - A: A \in \mathfrak{A}_2\} = \emptyset$, $\{X - A: A \in \mathfrak{A}_2\}$ is not contained in a cluster and by hypothesis, not bound. Thus, there is a $\mathfrak{U} \in \mathcal{S}$ such that $\mathfrak{U} < \mathfrak{A}_2 \cap X$. This implies that $\mathfrak{U}^* < \mathrm{cl}\,\mathfrak{A}_2 < \mathfrak{A}_1$ and $\mathfrak{A}_1 \in \mathcal{S}^*$.

Among all semi-uniformities compatible with a given topology there is a finest, namely the fine semi-uniformity. We will define $(X, \mathcal{S})$ to be a *fine space* provided $\mathcal{S}$ is the fine semi-uniformity compatible with $\tau(\mathcal{S})$. A fine space is complete.

THEOREM 5.2. *Every continuous function from a fine space into a semi-uniform space is a mapping.*

THEOREM 5.3. *The fine spaces form a coreflective subcategory of the category of semi-uniform spaces.*

Proof. If $\mathcal{S}$ is a semi-uniformity on X, let $\alpha\mathcal{S}$ denote the compatible fine semi-uniformity. It is easy to see that if $(Y, \mathcal{S}_1)$ is a fine space, then any map $f\colon (Y, \mathcal{S}_1) \to (X, \mathcal{S}_2)$ factors uniquely $f = c \circ f'$ where $f'\colon (Y, \mathcal{S}_1) \to (X, \alpha\mathcal{S}_2)$ and $c\colon (X, \alpha\mathcal{S}_2) \to (X, \mathcal{S}_2)$.

A fine uniform space is a fine semi-uniform space if and only if the semi-uniform topology is paracompact.

A subspace of a fine space will be called *subfine*. For each semi-uniformity $\mathcal{S}$ on X, let $\sigma\mathcal{S}$ denote the finest semi-uniformity on X whose completion is topologically equivalent to that of $\mathcal{S}$.

THEOREM 5.4. *The subfine spaces form a coreflective subcategory of the category of semi-uniform spaces.*

Proof. If $(Y, \mathcal{S}_1)$ is a subfine space and $f\colon (Y, \mathcal{S}_1) \to (X, \mathcal{S}_2)$ is a mapping, then f has a unique extension to a mapping f^* from $(Y^*, \mathcal{S}_1^*)$ to $(X^*, \mathcal{S}_2^*)$. Since completions are unique, $(Y^*, \mathcal{S}_1^*)$ must be a fine space, so $f^* = c \circ h$ where $h\colon (Y^*, \mathcal{S}_1^*) \to (X^*, \sigma\mathcal{S}_2^*)$ and $c\colon (X^*, \sigma\mathcal{S}_2^*) \to (X^*, \mathcal{S}_2^*)$. If $e\colon Y \to Y^*$ is the embedding, then $c \circ (h \circ e) = f^* \circ e = f$, and $h \circ e\colon (Y, \mathcal{S}_1) \to (X, \sigma\mathcal{S}_2)$ and $c\colon (X, \sigma\mathcal{S}_2) \to (X, \mathcal{S}_2)$.

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Reçu par la Rédaction le 7. 9. 1972

On the extension of continuous functions

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Abstract. The principal result is the following. Let X be a dense subspace of Z. Let f be a continuous function from X to a complete semi-uniform space (Y, C). Then f can be continuously extended to Z iff, for $\tau = \{T\} \in C$, $\{\mathrm{Int}_Z \mathrm{cl}_Z f^{-1}(T)\}$ covers Z. In particular, let X be a dense subspace of Z and f a continuous function from X to a regular T_1 space Y. Then f can be continuously extended to Z iff, for $\tau = \{T\}$ an open covering of Y, $\{\mathrm{Int}_Z \mathrm{cl}_Z f^{-1}(T)\}$ is an open covering of Z. Well-known applications will also be discussed.

1. Introduction. This paper contains a theorem which gives very specific circumstances under which a continuous function, whose image is regular T_1, can be continuously extended from a dense subspace onto an entire space. Before this theorem can be given, however, it is necessary to mention some recent results in structural theory.

2. Semi-uniform spaces. In [11] E. F. Steiner and A. K. Steiner have introduced the concept of semi-uniformities. Here, only those points essential to this paper will be discussed. A semi-uniform space consists of a pair X, C where X is a space and C is a collection of coverings of X satisfying:

(i) If $\tau \in C$ then there exists $\tau' \in C$ such that: for each $T' \in \tau'$ there exist $T \in \tau$ and $\tau'' \in C$ where $\mathrm{St}(T', \tau'') \subset T$. (If a is a covering of a set Y and $A \subset Y$, then $\mathrm{St}(A, a) = \bigcup \{T \in a\colon T \cap A \neq \emptyset\}$. If A consists of a single point x, then $\mathrm{St}(x, a) = \mathrm{St}(A, a)$). τ' is said to *semi-star refine* τ.

(ii) If $\tau, \tau' \in C$ then there exists $\tau'' \in C$ such that τ'' refines both τ and τ'.

(iii) If $\tau' \in C$ refines a covering τ, then $\tau \in C$.

(iv) For each $x \in X$, $\{\mathrm{St}(x, \tau)\colon \tau \in C\}$ is a base of neighborhoods for x.

(v) For $x, y \in X$ where $x \neq y$, there exists $\tau \in C$ such that $y \notin \mathrm{St}(x, \tau)$.

If the pair X, C satisfies (i) through (v), then C is said to be a *semi-uniformity* on X. This is denoted by (X, C), and (X, C) is said to be a *semi-uniform space*. Note that, if (i) is replaced by: for $\tau \in C$ there exists $\tau' \in C$ such that, for each $T' \in \tau'$ there exists $T \in \tau$ where $\mathrm{St}(T', \tau') \subset T$, then C is a uniformity on X (see [9]).