Characterization of generic extensions of models of set theory

by

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Abstract. We prove the following theorem: For a model-class M the following two conditions are equivalent: a) there is an α-C.C. M-complete Boolean algebra $B \in M$ and an M-complete ultrafilter Z on B such that $V = M(Z)$, b) for every function $f \in V$, there exists a function $h \in M$ such that for each x from the domain of f, $f(x) \in h(x)$ and the cardinality of $h(x)$ is smaller than α. Some related problems and results are discussed, e.g. the existence of a complete totally non-homogenous Boolean algebra with prescribed properties is proved.

Among all the methods of constructing an extension of a given model of the theory of sets, Cohen's method of generic extensions is the most powerful one. The main subject of this paper is to characterize those extensions of a model of the theory of sets, which are of the above-mentioned type, i.e. generic ones.

P. Hájek and P. Vopěnka in [22] have proved that every extension (with the class form of the axiom of choice) is "generic", but over a class of forcing conditions. Since in this case one can obtain very poor information, we preserve the notion of generic extension for the case of a set of forcing conditions and we shall be interested in such extensions only.

Historically, one of the first characterizations of generic extensions has been given by P. Hájek and P. Vopěnka in [22], p. 207 (see also the remark preceding our Theorem 3.3). We have found a seemingly weaker condition (already known in literature), which is equivalent to the genericity of an extension. Namely ("Apr" is defined in (1.6)):

THEOREM A. *Let M be a model-class, $\alpha \in \mathrm{Card}^M$. Then the following two conditions are equivalent*:

(i) *V is an α-C.C.-generic extension of M*,

(ii) $\mathrm{Apr}_{M,V}(\alpha)$.

The theorem has two aspects: 1) $\mathrm{Apr}_{M,V}(\alpha)$ implies that V is a generic extension of M (in our opinion, this fact is a little suprising) and 2) this extension is constructed by using an α-C.C. set of conditions.

In § 1, we recall all the definitions and results which are needed in the other parts. § 2 is devoted to the proofs of two, more or less technical, theorems on Booleans algebras. § 3 is the main part of the paper. We suppose that Corollary 3.2 is of its own interest. In § 4 we prove Theorem A and some related results.

The construction of a real $x \subseteq \omega_0$ such that "$L(x)$ is a generic extension of L, $\aleph_3^L$ is collapsed to $\aleph_2^L$ but all the other cardinals are preserved" (owing to J. Silver and R. Solovay [9], p. 103) has turned out to be a good counterexample for our false hypotheses (see the remark after problem 3). Using elementary properties of the "Apr", we have concluded that in $L(x)$ there exists a non-constructible hereditarily ordinal definable set. § 5 is devoted to a slight generalization of this reasoning, which leads to (a sketch of) a proof of the following

THEOREM B. *For every infinite cardinal α, there exists a complete Boolean algebra B such that*

(i) *B possesses a set of generators of the power α,*

(ii) *B satisfies* $(2^\alpha)^{++}$-C.C,

(iii) *B is totally non-homogenous.*

In literature, there are known examples of small totally non-homogenous complete Boolean algebras (none of them allowing a non-identical automorphism) in the constructible universe L, but the constructions use special properties of L and/or ω_0 (the Boolean algebra constructed in K. McAloon [14] is relatively "large"). Anyway, we consider the simple proof of this theorem as more interesting than the theorem itself.

Finally, in § 6 we present some open questions closely related to the subject of this paper.

We consider this paper as a (free) continuation of our previous note [3]. During the summer session of the Prague set theory seminarium at Čingov in June 1971, we discussed the result of note [3] with B. Balcar and P. Vopěnka. The work on this paper has been strongly influenced by that discussion, namely the results presented in § 3 (which may be considered as a direct generalization of the results contained in [3]). I would like to express my thanks to them.

The methods of constructions used in this paper imply interesting results on Boolean algebras, which we shall eventually publish elsewhere.

§ 1. Preliminaries. All the reasonings of this paper are applicable to any sufficiently strong theory of sets (e.g. to the Zermelo-Fraenkel set theory ZFC or to the theory of semisets — see [22]). In order to express our results in a precise way, we shall formulate them all in the Gödel-Bernays set theory GB with the axiom of regularity (see [7]) and with the set form of the axiom of choice AC: "every set can be well-ordered".

We use the standard set-theoretical notations. An ordinal is the set of all smaller ordinals, a cardinal is an initial ordinal. ξ, η, ζ denote ordinals, $\alpha, \beta, \gamma, \delta$ denote cardinals. $\mathcal{P}(x)$ is the set of all subsets of x, ${}^xy$ is the set of all mappings from x into y. V is the universal class: the class of all sets. α^+ is the first cardinal greater than α.

A class M is called a *model-class* iff M is transitive, closed under Gödel's operations, and almost universal, and the axiom of choice holds true in M. In other words, M is a model-class iff $\langle M, \epsilon\rangle$ is a model of ZFC. If we define

$$\mathrm{Cls}_M(X) \equiv X \subseteq M \;\&\; (\forall y)(y \in M \rightarrow y \cap X \in M)\,,$$

we obtain a model of GB. If $\square$ is some concept of the set theory, then $\square^M$ means the corresponding concept constructed in this model. E.g. Card^M is the class of all infinite ordinals which are cardinal numbers in the sense of the model Cls_M, ϵ, i.e. the class of those infinite ordinals which cannot be mapped in a one-to-one way on some smaller ordinal by a map belonging to M.

If M_1, M_2 are model-classes, $M_1 \subseteq M_2$, we say that M_2 is an *extension of* M_1. If x is a set and M_2 is an extension of M_1, we write $M_2 = M_1(x)$ iff M_2 is the smallest model-class containing both M_1 and x, i.e. $M_1 \subseteq M_2$ and $x \in M_2$. Compare A. Lévy [12], where also the existence of $M(x)$ has been proved for some classes M, e.g. for $M = L$ — the class of all constructible sets.

We suppose that the reader is familiar with the absoluteness of some concepts (see e.g. [8]) and we shall use the corresponding results without any commentary.

Let $\langle P, \leqslant\rangle \in M$ be a partially ordered set. A set $G \subseteq P$, $G \neq P$ is said to be *M-generic over P* iff

a) $x \in G$, $x \leqslant y \in P \rightarrow y \in G$,

b) $x, y \in G \rightarrow (\exists z)(z \in G \;\&\; z \leqslant x \;\&\; z \leqslant y)$,

c) if $A \subseteq P$, $A \in M$, A is dense in P (i.e. $(\forall x)(x \in P,\ x \neq 0 \rightarrow (\exists y)(y \leqslant x \;\&\; y \in A \;\&\; y \neq 0))$), then $A \cap G \neq \emptyset$.

It is well known that if M is a model-class, then

(1.1) if G is an M-generic set, then there exists a model class N such that $N = M(G)$.

For the proof see e.g. [4], [8], [10].

DEFINITION 1.1. Let M_2 be an extension of M_1, M_2 is said to be a *generic extension of* M_1 iff there exists a partially ordered set $\langle P, \leqslant\rangle \in M_1$ and an M_1-generic set $G \subseteq P$ such that $M_2 = M_1(G)$.

If $\mathrm{card}^{M_1}(P) < \alpha$, we say that M_2 is an *α-generic extension.*

If $\langle P, \leqslant\rangle$ satisfies the α-chain condition in M_1 (P is α-C.C., i.e. for every set $X \subseteq P$, $X \in M_1$ of pairwise incompatible elements, $\mathrm{card}^{M_1}(X) < \alpha$), we say that M_2 is an *α-C.C.-generic extension.*

If $B \in M$ is an M-complete Boolean algebra (i.e. a Boolean algebra such that $\bigvee X$ exists for each $X \subseteq B$, $X \in M$), then an M-generic set G over B is actually an M-complete ultrafilter on B. From well-known results it follows that

(1.2) M_2 is a generic extension of M_1 iff there exist an M_1-complete Boolean algebra $B \in M_1$ and an M_1-complete ultrafilter G on B such that $M_2 = M_1(G)$. The "α-generic" and "α-C.C.-generic" extensions correspond to the case of a Boolean algebra with a dense subset (in M_1) of M_1-cardinality less than α and an α-C.C.-Boolean algebra, respectively.

We shall need the following fact (see [2], [17], [22], for the notion of "distributivity" see also [18]):

(1.3) Let M_2 be a generic extension of M_1. Then $\mathcal{P}(\alpha) \cap M_1 = \mathcal{P}(\alpha) \cap M_2$ if and only if the Boolean algebra in (1.2) can be chosen to be $(\alpha, 2)$-distributive (in M_1).

Now we recall some notions and results from [22]. Let M be a model-class, $x, y \subseteq M$. We say that x is *M-dependent on y* $(\mathrm{Dep}_M(x, y))$ iff there exists a relation $r \in M$ such that $x = r''y = \{u\colon (\exists v)(v \in y \ \& \ \langle vu\rangle \in r)\}$. A set $z \subseteq M$ is an *M-support* iff for every $x, y \subseteq M$ which are M-dependent on z also $x - y$ is M-dependent on z. A set $z \subseteq M_1$, $z \in M_2$, is a *total M_1-support for M_2* iff $(\forall x)\big(x \subseteq M_1 \ \& \ x \in M_2 \to \mathrm{Dep}_{M_1}(x, z)\big)$.

(1.4) If z is an M-support, then there exists a generic extension N of M such that $N = M(z)$ and z is a total M-support for N. Moreover, if $z \subseteq a \in M$, $\mathrm{card}^M(a) < \alpha$, then N is an α-generic extension of M.

The first part of (1.4) is proved in [22], p. 222. The second part follows from [1].

B. Balcar and P. Vopěnka have proved a theorem (see [21]) which essentially simplifies our considerations:

(1.5) Let M, N be two model-classes. If M and N have the same sets of ordinals, then $M = N$.

Let M_2 be an extension of M_1, $\alpha, \beta, \gamma \in \mathrm{Card}^{M_1}$. The various conditions for the "genericity" of this extension will be formulated by using the following properties of this extension (see [22]):

$$\mathrm{Bd}_{M_1,M_2}(\alpha) \equiv (\forall x \subseteq M_1, x \in M_2)(\exists y \in M_2)(\exists a \in M_1)(y \subseteq a \ \& \ \& \ \mathrm{card}^{M_1}(a) < \alpha \ \& \bigcup y = x),$$

(1.6)
$$\mathrm{Apr}_{M_1,M_2}(\alpha, \beta, \gamma) \equiv (\forall f \in {}^{\alpha}\beta, f \in M_2)\big(\exists g \in {}^{\alpha}\mathcal{P}(\beta)\big)\Big(g \in M_1 \ \& \ \& (\forall \xi \in \alpha)\big(f(\xi) \in g(\xi) \ \& \ \mathrm{card}^{M_1}(g(\xi)) < \gamma\big)\Big),$$

$$\mathrm{Apr}_{M_1,M_2}(\gamma) \equiv (\forall \alpha)(\forall \beta)\mathrm{Apr}_{M_1,M_2}(\alpha, \beta, \gamma).$$

It is well known that an α-generic (α-C.C.-generic) extension M_2 of M_1 possesses the property $\mathrm{Bd}_{M_1,M_2}(\alpha)$ $\big(\mathrm{Apr}_{M_1,M_2}(\alpha)\big)$ (see e.g. [22]).

Let us remark that

(1.7) if $\alpha \leqslant \beta$, $\beta \in \mathrm{Card}^{M_1}$, then each of the conditions "M_2 is an α-generic extension of M_1", "M_2 is an α-C.C.-generic extension of M_1", $\mathrm{Bd}_{M_1,M_2}(\alpha)$, $\mathrm{Apr}_{M_1,M_2}(\beta, \beta, \alpha)$ implies that $\beta \in \mathrm{Card}^{M_2}$.

If M is a model-class, then $\mathrm{HDf}(M)$ denotes the class of all hereditarily M-definable sets. This class is defined similarly to the class $\mathrm{HDf} = \mathrm{HDf}(\emptyset)$ of hereditarily definable sets (= hereditarily ordinal definable in [14]), but allowing parameters from M (see [22], p. 186).

It is easy to see (compare [22], p. 186) that $\mathrm{HDf}(M)$ is a model-class. By a slight modification of the proofs on pages 304 and 320 in [22] one can easily show that

(1.8) If $V = M(x)$, $x \subseteq \alpha$, then V is an β-C.C.-generic extension of $\mathrm{HDf}(M)$, where $\beta = (2^{\alpha})^{+}$.

(1.9) If $V = M(z)$, where z is an M-complete ultrafilter on an M-complete Boolean algebra $B \in M$, then $\mathrm{HDf}(M) = M(z \cap B_{\mathrm{rig}})$, where $B_{\mathrm{rig}} = \{u \in B\colon$ for every $f \in M$, an automorphism of B, $f(u) = u\}$.

Let us recall that a complete Boolean algebra B is said to be *totally non-homogeneous* if there is no partition $\{u_t\colon t \in T\}$ of B such that $B/u_t = \{x \in B,\ x \leqslant u_t\}$ is homogeneous for every $t \in T$.

We shall need the following simple fact:

(1.10) If B_{rig} is not completely distributive then B is totally non-homogeneous.

§ 2. Two theorems on Boolean algebras.

THEOREM 2.1. *Let B be a complete α-C.C. Boolean algebra. Assume that B is not atomic (and thus not completely distributive). Let β be the first cardinal such that B is not (β, α)-distributive. Then $\beta \leqslant \alpha$.*

Remark. The theorem for $\alpha = \aleph_1$ has been (implicitly) proved by H. Gaifman in [6]. In fact, H. Gaifman proves that the Souslin hypothesis implies that every complete $\aleph_1$-C.C., $(\aleph_0, 2)$-distributive Boolean algebra is atomic. From an $\aleph_1$-C.C., $\aleph_0$-distributive, $\aleph_1$-non-distributive complete Boolean algebra, it is easy to construct a Souslin tree (and vice versa). Thus, one needs Theorem 2.1 to pass from "not atomic" to "$\aleph_1$-non-distributive".

In our case, the theorem plays an important role in the proof of Theorem A.

Proof of the theorem. Let B be a complete α-C.C. Boolean algebra. Assume that B is not atomic and that β is the first cardinal such that B is not (β, α)-distributive. Evidently, β is a regular cardinal. Assume $\beta > \alpha$.

Since B is not (β, α)-distributive, there exists a system $\{a_{\xi,\eta} \in B: \xi \in \beta \,\&\, \eta \in \alpha\}$ such that (see [18])

(2.1) for every $\xi \in \beta$, $A_\xi = \{a_{\xi,\eta}: \eta \in \alpha\}$ is a partition, i.e. $\bigvee_{\eta\in\alpha} a_{\xi,\eta} = 1$ and $a_{\xi,\eta_1} \wedge a_{\xi,\eta_2} = 0$ for $\eta_1 \neq \eta_2$,

(2.2) for every $\xi_1 < \xi_2 < \beta$, A_{ξ_2} is a refinement of A_{ξ_1}, i.e. $(\forall\eta_1)(\exists\eta_2)(a_{\xi_2,\eta_1} \leqslant a_{\xi_1,\eta_2})$,

(2.3) the set $\{A_\xi: \xi \in \beta\}$ of partitions has no common refinement, i.e. $u = \bigvee_{\varphi\in\beta_\alpha} \bigwedge_{\xi\in\beta} a_{\xi,\varphi(\xi)} \neq 1$.

We may suppose that $u = 0$ (if not, consider $B|-u$).

If $A \subseteq B$ is a partition, $a \in B$, $a \neq 0$, we say that A *divides* a iff there are $x_1, x_2 \in A$, $x_1 \neq x_2$, $x_1 \wedge a \neq 0$, $x_2 \wedge a \neq 0$.

By (2.3) and $u = 0$, for every $a \in B$, $a \neq 0$, there is an $\xi \in \beta$ such that A_ξ divides a.

We define a sequence of ordinals $\{\delta_\xi: \xi \in \beta\}$ by transfinite induction. We set $\delta_0 = 0$ and $\delta_\lambda = \lim_{\xi<\lambda} \delta_\xi$ for the limit λ. If δ_ξ is already defined, let τ_η be the first ordinal ζ such that A_ζ divides $a_{\delta_\xi,\eta}$. Let $\delta_{\xi+1} = \sup\{\tau_\eta: \eta \in \alpha\}$. Since β is regular, δ_ξ can be defined for every $\xi \in \beta$. By definition, $A_{\delta_{\xi+1}}$ divides each element of A_{δ_ξ}. Let $a \in A_{\delta_\alpha}$, $a \neq 0$. For every $\xi < \alpha$ there exists an ordinal $\psi(\xi) < \alpha$ such that $a \leqslant a_{\delta_\xi,\psi(\xi)}$. Since A_{δ_ξ} divides every $v \in A_{\delta_\eta}$, $\eta < \xi$, the sequence $\{a_{\delta_\xi,\psi(\xi)}: \xi \in \alpha\}$ is strictly decreasing — a contradiction with the α-chain condition, q.e.d.

The second theorem will be used in the construction of a suitable ultrafilter on the free $<\alpha$-complete Boolean algebra $\mathfrak{A}_{<\alpha,\beta}$ with β generators (see L. Rieger [16] and also R. Sikorski [18]; $\mathfrak{A}_{\alpha,\beta} = \mathfrak{A}_{<\alpha^+,\beta}$).

If M is a model-class, then every mapping $f \in M$ of generators of $\mathfrak{A}^M_{\alpha,\beta}$ into an M-α-complete Boolean algebra $B \in M$ can be extended to an M-α-complete homomorphism. R. Solovay has shown (see [19]) that, roughly speaking, in the case of $\alpha = \beta = \aleph_0$ the same holds true without the assumptions "$f \in M$ and $B \in M$". For mappings in the two-element Boolean algebra $\{0, 1\}$ a similar result follows from a theorem in [22] (see pp. 210–221). Following the ideas of [22], we extend the above-mentioned results.

THEOREM 2.2. *Let M be a model-class, $\alpha \in$ Card, $\beta \in \mathrm{Card}^M$, α being regular. Let $C = \mathfrak{A}^M_{<\alpha,\beta}$ be the free $<\alpha$-complete Boolean algebra with β generators constructed inside M. Let $C^\# = \mathfrak{A}_{<\alpha,\beta}$ (i.e. the same algebra constructed in the whole universe V). Then the identical mapping of generators of C onto the set of generators of $C^\#$ can be extended to a one-to-one M-$<\alpha$-complete homomorphism $\#: C \to C^\#$, i.e. if $X \subseteq C$, $X \in M$, $\mathrm{card}^M(X) < \alpha$, then $\#(\bigvee X) = \bigvee_{x\in X} \#(x)$.*

Proof. The proof is rather complicated. We outline its idea (compare [22], pp. 211–214). Let us recall the construction of the algebra $\mathfrak{A}_{<\alpha,\beta}$. For every $\xi < \alpha$, we define a set H_ξ of Boolean polynomials: H_0 is the set of generators, $H_\lambda = \{-x: x \in \bigcup_{\xi<\lambda} H_\xi\} \cup \{\bigvee X: X \subseteq \bigcup_{\xi<\lambda} H_\xi \,\&\, \mathrm{card}(X) < \alpha\}$. The canonical congruence relation "$\sim$" on $H = \bigcup_{\xi<\alpha} H_\xi$ is defined by induction as in [22], pp. 211–212. Thus $x \sim_0 y$ iff 1) $x = y$ or 2) $x = 0$ $(0 = \bigvee\emptyset)$ and $y = -\bigvee\{u, -u\}$ for some $u \in H$ or 3) $x = \bigvee X$ and $y = \bigvee Y$, where each $u \in Y$ is such that $u = \bigvee U_u$ and $X = \bigcup_{u\in Y} U_u$. Let $x \sim_{<\xi} y$ denote $(\exists\eta \in \xi) x \sim_\eta y$ and let $X \sim\sim_{<\xi} Y$ denote

$$(\forall u \in X)(\exists v \in Y)(u \sim_{<\xi} v) \,\&\, (\forall u \in Y)(\exists v \in X)(u \sim_{<\xi} v)\,.$$

We define: $x \sim_\xi y$ iff one of the following conditions is satisfied:

$$x \sim_{<\xi} y\,,\quad y \sim_{<\xi} x\,,\quad -x \sim_{<\xi} -y\,,$$
$$(\exists u)(x \sim_{<\xi} u \,\&\, u \sim_{<\xi} y)\,,$$
$$(\exists X, Y)(X \sim\sim_{<\xi} Y \,\&\, x = \bigvee X \,\&\, y = \bigvee Y)\,,$$
$$(\exists u, v)(u \vee -v \sim_{<\xi} -0 \,\&\, x = u \vee v \,\&\, y = u)\,,$$
$$(\exists u, v)(u \vee v \sim_{<\xi} u \,\&\, x = u \vee -v \,\&\, y = -0)\,.$$

We let $x \sim y$ iff $(\exists\xi < \alpha)(x \sim_\xi y)$. Then the Boolean algebra $\mathfrak{A}_{<\alpha,\beta}$ is the set of the sets of congruent polynomials $H/\!\sim\, = \{[x]: x \in H\}$. It is easy to show that e.g. $\bigvee_{\eta\in\lambda} [x_\eta] = [\bigvee_{\eta\in\lambda} x_\eta]$ for $\lambda < \alpha$.

Evidently H^M (i.e. the set of corresponding polynomials constructed in M) is a subset of H. Also, for any $x, y \in H^M$, x is congruent to y iff x is congruent to y in M. Thus, the identity mapping of H^M into H induces the homomorphism $\#: C \to C^\#$. q.e.d.

§ 3. How to find a support. The strongest information contained in Theorem A is the fact that $\mathrm{Apr}_{M,V}(\alpha)$ implies the existence of a set of conditions and of an M-generic subset of that set with suitable properties. The problem is essentially simplified by (1.4). In this part we present two methods for obtaining an M-generic set (see the proofs of Theorems 3.1 and 3.3). The first method is a generalization of that presented in [3], the second one is due to P. Hájek and P. Vopěnka [22].

The proof of Theorem A heavily depends on

THEOREM 3.1. *Let M be a model-class, $x \subseteq \beta$, $\alpha \in \mathrm{Card}^M$, α being regular. Let $\delta = \mathrm{card}^M(\bigcup_{\varepsilon<\alpha} {}^{\varepsilon}\beta \cap M)$, $\gamma = \mathrm{card}^M(\mathfrak{P}(\delta) \cap M)$. If $\mathrm{Apr}_{M,V}(\gamma, \delta, \alpha)$, then there exists a model-class N such that*

a) *N is an α-C.C.-generic extension of M,*

b) *$N = M(x)$.*

Proof. By (1.7), $\alpha, \beta, \delta, \gamma$ are cardinal numbers (in V). We set $D = \mathfrak{A}^M_{<\alpha,\beta}$, $C = \mathfrak{A}^M_{\gamma,\beta}$, $D^{\#} = \mathfrak{A}_{<\alpha,\beta}$, $C^{\#} = \mathfrak{A}_{\gamma,\beta}$. Let $\{g_\xi\colon \xi \in \beta\}$ be the set of generators for all C, D, $C^{\#}$, $D^{\#}$. Evidently D is a (regular) subalgebra of C, $D^{\#}$ is a (regular) subalgebra of $C^{\#}$. By Theorem 2.2, there exists an M-γ-complete one-to-one homomorphism $\#\colon C \to C^{\#}$. Let π be the γ-complete homomorphism of $C^{\#}$ in $\{0,1\}$ induced by $\pi(g_\xi) = 1 \equiv \xi \in x$. We define

$$u \in Z \equiv u \in C \;\&\; \pi(\#(u)) = 1\,.$$

Evidently, Z is an M-γ-complete ultrafilter on C.

Let $\mathcal{F} = \{X_\xi\colon \xi \in \gamma\}$ be an enumeration (in M) of those subsets X of D for which $\bigvee^D X = 1$ and $X \in M$. Since $\mathrm{card}^M(X) \leqslant \delta < \gamma$, the union $\bigvee^C X$ does exist (it need not be equal to 1!) for any $X \in \mathcal{F}$. Let $X'_\xi = X_\xi \cup \{-\bigvee^C X_\xi\}$. By a suitable enumeration in M, we have $X'_\xi = \{u_{\xi,\eta}\colon \eta \in \delta\}$. Since $\bigvee^C X'_\xi = 1$ and Z is an M-γ-complete ultrafilter, there is an $\eta \in \delta$ such that $u_{\xi,\eta} \in Z$. We set $f(\xi)$ equal to the first ordinal η for which $u_{\xi,\eta} \in Z$. By $\mathrm{Apr}_{M,V}(\gamma, \delta, \alpha)$ there exists a function $h \in M$, $h\colon \gamma \to \mathfrak{P}(\delta)$ such that $(\forall \xi \in \gamma)\big(f(\xi) \in h(\xi) \;\&\; \mathrm{card}^M(h(\xi)) < \alpha\big)$. The intersection $u = \bigwedge^C_{\xi \in \gamma} \bigvee^C_{\eta \in h(\xi)} u_{\xi,\eta}$ belongs to Z.

Now we show that $B = \{u \wedge v\colon v \in D\}$ is an α-C.C. (in M) M-complete Boolean algebra and $M(Z \cap B) = M(x)$.

If $Y \in M$ is a set of pairwise disjoint elements of B, then there is a set $X \in M$, $X \subseteq D$ such that $Y \subseteq \{u \wedge v\colon v \in X\}$. By the definition of $\mathcal{F}$, there is an ordinal $\xi \in \gamma$ such that $X = X_\xi$. Evidently $Y \subseteq \{u \wedge u_{\xi,\eta}\colon \eta \in h(\xi)\}$. Thus, B is α-C.C. in M and also M-complete.

It is easy to see that $Z \cap B$ is an M-complete ultrafilter on B (note that $u \in Z$!). Hence the model-class $M(Z \cap B)$ does exist. Since $x = \{\xi\colon g_\xi \in Z\} = \{\xi\colon u \wedge g_\xi \in Z \cap B\}$, we have $x \in M(Z \cap B)$.

If N is an extension of M, $x \in N$, then we can define an M-complete ultrafilter Z' in N in the same way as we have defined Z in V. Evidently, we obtain $Z' \subseteq Z$ and, since Z' is an ultrafilter, $Z' = Z$. Thus also $Z \cap B \in N$. q.e.d.

As a direct consequence of the theorem we obtain the following generalization of our theorem proved in [3]:

COROLLARY 3.2. *Let $V = M(x)$, $x \subseteq \beta$, $\alpha \in \mathrm{Card}^M$, α being regular. Let δ, γ be the same as in the theorem. Then the following conditions are equivalent:*

(i) *V is an α-C.C.-generic extension of M,*

(ii) $\mathrm{Apr}_{M,V}(\alpha)$,

(iii) $\mathrm{Apr}_{M,V}(\gamma, \delta, \alpha)$.

For $\alpha = \beta^+$ and $M = L$, this theorem follows from our theorem in [3], as J. L. Krivine has remarked.

The proof of Theorem 3.1 is a slight modification of the proofs in [3]. In fact, one can show, in the same way as in [3], that $Z \cap D$ is an M-support and then prove a theorem analogous to Proposition 3 in [3].

In [22] p. 207, P. Hájek and P. Vopěnka have proved that V is a generic extension of M iff $\mathrm{Bd}_{M,V}(\equiv (\exists\alpha)\mathrm{Bd}_{M,V}(\alpha))$. This result may be refined in an almost trivial way. We state it explicitly, hoping that it may be useful in a comparison with Theorem A.

THEOREM 3.3. *Let M be a model-class, $\alpha \in \mathrm{Card}^M$. Then V is an α-generic extension of M if and only if $\mathrm{Bd}_{M,V}(\alpha)$.*

Proof. Let f be a one-to-one mapping of $\mathfrak{P}(\alpha)$ onto some cardinal β. Let $g = \{<\xi\eta>\colon \eta \in f^{-1}(\xi)\}$. Using $\mathrm{Bd}_{M,V}(\alpha)$ (namely the fact that every set M-depends on a subset of α), one can easily show that g is a total M-support for V. Again by $\mathrm{Bd}_{M,V}(\alpha)$, there are $z \subseteq \alpha \in M$, $\mathrm{card}^M(\alpha) < \alpha$ such that $\mathrm{Dep}_M(g, z)$. Thus z is also a total M-support for V. The theorem follows by (1.4). q.e.d.

§ 4. Proof of Theorem A. Theorem A follows directly from Corollary 3.2 and the following

THEOREM 4.1. *Let M be a model-class. If $\mathrm{Apr}_{M,V}(\alpha)$, then $V = M(\mathfrak{P}(\alpha))$.*

Proof. Assume $V \neq M(\mathfrak{P}(\alpha))$, i.e. there is a model-class N such that $\mathfrak{P}(\alpha) \in N$, $M \subseteq N$ and $N \neq V$. By (1.5), there exists a set $x \subseteq N$, $x \notin N$. Evidently $\mathrm{Apr}_{M,V}(\alpha)$ implies $\mathrm{Apr}_{N,V}(\alpha)$. Thus, by Theorem 3.1, there exists a model-class $N' = N(x)$ such that N' is an α-C.C.-generic extension of N. Since $\mathfrak{P}(\alpha) \subseteq N \subseteq N'$, by (1.3) there exist an α-C.C. $(\alpha, 2)$-distributive N-complete Boolean algebra $B \in N$ and an N-complete ultrafilter Z on B such that $N' = N(Z)$. By Theorem 2.1, B is atomic. Hence $N' = N$ — a contradiction. q.e.d.

As the strongest consequence of Theorem A, we consider the fact that $\mathrm{Apr}_{M,V}(\alpha)$ implies $V = M(x)$ for some set $x \subseteq M$. There is a natural problem: find some estimate for the power of the set x. More precisely, let $\mathrm{Apr}_{M,V}(\alpha)$, $V = M(x)$, $x \subseteq \beta$. Give an estimate for the cardinal β.

Of course, there is no estimate of β inside the model-class M. (For any β, one can construct an α^+-C.C.-generic extension in which $2^\alpha > \beta$ and $V \neq L(x)$ for any $x \subseteq \beta$.) From Theorem 4.1 it follows that $\beta \leqslant \mathrm{card}(\mathfrak{P}(\alpha))$. Using an idea of K. Kunen (see K. Kunen [11], p. 89) we give a better estimate for β in terms of the weak power of α.

We recall the definition of a partition relation (see Erdös-Hajnal-Rado [5]). $\gamma \to (\alpha)^2_\beta$ means that for every function f defined on $[\gamma]^2$

$=\{x\subseteq\gamma\colon \operatorname{card}(x)=2\}$ with values in the cardinal β, there exist a set $Y\subset\gamma$ and an ordinal $\xi\in\beta$ such that $\operatorname{card}(Y)=\alpha$ and $f(x)=\xi$ for each $x\in[Y]^2$.

THEOREM 4.2. *Let M be a model-class, and let $\mathrm{Apr}_{M,V}(\alpha)$ hold true. Let $\beta=\sum_{\delta<\alpha}2^\delta$. If $\gamma^+\to(\beta^+)^2_\alpha$ holds true in M, then $\operatorname{card}(\mathcal{S}(\alpha))\leqslant\gamma$.*

Proof. Let us suppose that $\operatorname{card}(\mathcal{S}(\alpha))>\gamma$, i.e. there is a set $\{y_\xi\colon \xi\in\gamma^+\}$ of distinct subsets of α. For $\{\xi_1,\xi_2\}\in[\gamma^+]^2$ we set $f(\xi_1,\xi_2)$ equal to the least ordinal $\eta\in\alpha$ for which $\eta\cap y_{\xi_1}\neq\eta\cap y_{\xi_2}$. By $\mathrm{Apr}_{M,V}(\alpha)$, there is a function $g\in M$, $g\colon[\gamma^+]^2\to\alpha$ such that $f(\xi_1,\xi_2)\leqslant g(\xi_1,\xi_2)$ for each $\{\xi_1,\xi_2\}\in[\gamma^+]^2$. Since $g\in M$ and $\gamma^+\to(\beta^+)^2_\alpha$ holds true in M, there exist $Y\subseteq\gamma^+$ and an ordinal $\eta\in\alpha$ such that $\operatorname{card}^M(Y)=\operatorname{card}^V(Y)=\beta^+$ and $g(\xi_1,\xi_2)=\eta$ for $\{\xi_1,\xi_2\}\in[Y]^2$. Thus $\{y_\xi\cap\eta\colon \xi\in Y\}$ are distinct subsets of $\eta<\alpha$ — contradiction. q.e.d.

For example, if the generalized continuum hypothesis (GCH) holds true in M, then $\beta^{++}\to(\beta^+)^2_\alpha$ also holds true in M (see [5], p. 130). Thus

COROLLARY 4.3. *If M is a model-class,* GCH *holds true in M and $\mathrm{Apr}_{M,V}(\alpha)$, then $\operatorname{card}(\mathcal{S}(\alpha))\leqslant(\sum_{\delta<\alpha}2^\delta)^+$.*

§ 5. Proof of Theorem B.

In this part we outline the proof of Theorem B. The reader familiar with R. Jensen's and R. Solovay's [9] paper can easily fill the details.

The main idea of the proof is contained in the following simple observation:

Let M be a model-class, $V=M(x)$, $x\subseteq\alpha$, $\alpha\in\mathrm{Card}^M$. If

(5.1) $$\neg\mathrm{Apr}_{M,V}\big((2^\alpha)^+\big),\quad\text{then}\quad \mathrm{HDf}(M)\neq M.$$

In fact, by (1.8), V is an $(2^\alpha)^+$-C.C.-generic extension of $\mathrm{HDf}(M)$; thus $\mathrm{Apr}_{\mathrm{HDf}(M),V}\big((2^\alpha)^+\big)$. Since $\neg\mathrm{Apr}_{M,V}\big((2^\alpha)^+\big)$, we obtain $\mathrm{HDf}(M)\neq M$.

Now we shall slightly modify a construction due to J. Silver and R. Solovay (see [9], pp. 103–104). Let $\beta_1=2^\alpha$, $\beta_2=\beta_1^+$. Let C_0 be the complete Boolean algebra of regular open subsets of the topological space ${}^{\beta_1}\beta_2$ with $\leqslant\alpha$-product topology. In the Boolean extension ${}^{C_0}V$ there exists a set $A_0\subseteq\beta_1$ such that ${}^{C_0}V=V(A_0)$. Moreover, $\neg\mathrm{Apr}_{V,{}^{C_0}V}(\beta_2)$. Now we use "almost disjoint sets" method (inside the model ${}^{C_0}V$) to construct a Boolean algebra $C_1\in{}^{C_0}V$ such that ${}^{C_1}({}^{C_0}V)=V(A_1)$ for a set $A_1\subseteq\alpha$. It is well known (see e.g. [8]) that there exists a complete Boolean algebra $B\in V$ such that ${}^BV$ is isomorphic to ${}^{C_1}({}^{C_0}V)$. Hence, ${}^BV=V(A)$, where $A\subseteq\alpha$ and $\neg\mathrm{Apr}_{V,{}^BV}(\beta_2)$.

Evidently B possesses a set of generators of power α. By a simple computation (see [9]), B is β_2^+-C.C. By (5.1), $\mathrm{HDf}(V)\neq V$ in ${}^BV$. Since by (1.9) $\mathrm{HDf}(V)={}^{B_{\mathrm{rig}}}V$ in ${}^BV$, we conclude that B_{rig} is not completely distributive. The theorem follows by (1.10).

§ 6. Some open problems.

The simplest example of an extension which is not a generic one is the Easton type model. However, in that model $V\neq L(x)$ for any set x and the "genericity" is destroyed in an uninteresting way. We are interested in small extensions only, i.e. in such an extension N of M that $N=M(x)$ for some set x.

Of course the first problem connected with the subject of this paper is

PROBLEM 1. Assume $V=L(x)$, $x\subseteq\omega_0$. If V is not a generic extension of L, what can we say about x?

The existence of non-generic small extensions follows from the axioms of large cardinals, e.g. $L(0^\#)$ is not a generic extension of L.

PROBLEM 2. Let $V=L(x)$, $x\subseteq\omega_1$, V not being a generic extension of L. Does a set $y\subseteq\omega_0$ exist such that $L(y)$ is not a generic extension of L?

For all known small extensions which are not generic, we can find a set $y\subseteq\omega_0$ with this property, namely $0^\#$.

Let $V=L(x)$, $x\subseteq\omega_0$, V being a generic extension of L. Then there exists a cardinal α such that $\mathrm{Bd}_{L,V}(\alpha)$. Evidently, no estimate for the cardinal α can be given in L (counter-example: collapse a sufficiently big cardinal). Therefore

PROBLEM 3. Give (if possible) an estimate for α in V.

Generalizing J. Silver's and R. Solovay's construction in [9], one can see that generally $\alpha\geqslant\aleph_{\omega_0}$ (there is a model of ZFC in which $V=L(x)$, $x\subseteq\omega_0$ and $\neg\mathrm{Apr}_{L,V}(\aleph_n)$, $n\in\omega_0$; thus neither $\mathrm{Bd}(\aleph_{\omega_0})$).

As direct consequences of our results we obtain $\mathrm{Bd}(\alpha)\to\mathrm{Apr}(\alpha)$ and $\mathrm{Apr}(\alpha)\to\mathrm{Bd}\big((2^\alpha)^+\big)$. The proofs of both implications are based on the properties of generic extensions. Thus

PROBLEM 4. Prove those implications directly.

We do not know whether the estimate given in Corollary 4.3 is the best possible. The simplest related question may be formulated as

PROBLEM 5. Let $\mathrm{Apr}_{L,V}(\omega_1)$, $2^{\aleph_0}=\aleph_2$. Does $2^{\aleph_1}=\aleph_2$ hold true?

(Let us remark that by Corollary 4.3 we have $2^{\aleph_1}\leqslant\aleph_3$).

The axiom of the simultaneous collapse ASC says that

$$(\forall\alpha)(\exists\beta)(\beta\geqslant\alpha\ \&\ \beta\in\mathrm{Card}^L\ \&\ \beta\notin\mathrm{Card}).$$

If V is a generic extension of L, then $\neg$ASC. In every known small extension which is not a generic one, the axiom ASC holds true. Thus

PROBLEM 6. Let $V=L(x)$. If V is not a generic extension of L, does ASC hold true?

Finally, let us remark that, by (1.8), if $M_2=M_1(x)$, then M_2 is a generic extension of $\mathrm{HDf}(M_1)$. Hence, the existence of a small non-generic extension of a model-class M implies the existence of an extension which is hereditarily ordinal definable over M (thus, the definability of $0^{\#}$ is not an exception).

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On semi-uniformities

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Abstract. Regular Hausdorff extensions of topological spaces are studied as completions of generalized uniformities.

Key words. Regular extensions, completions, semi-uniform spaces.

Introduction. In this paper we will consider a generalization of uniform space first defined by Morita [2] which plays an important role in the completion and extension theory of topological spaces. Morita's paper is little known and consequently his original ideas are not referred to as often as they should be.

Our point of view is that one of the most important kinds of information a structure on a space X provides, besides a topology for X, is a topological extension of X. The most satisfactory extension theory appears to us to lie in the setting of regular Hausdorff spaces and the fundamental structures are the semi-uniformities presented here.

1. Preliminaries. In [2, I–IV], Morita considers families of open covers of a topological space which satisfy certain uniformity conditions. In this paper, the theory of semi-uniform spaces will be developed independently of a topology. Much of the terminology is that used in uniform space theory, and the reader is refered to [1].

If $\mathcal{S}$ is a family of coverings of a set X, and $\mathfrak{U}_1$ and $\mathfrak{U}_2 \in \mathcal{S}$, then $\mathfrak{U}_1$ is said to *locally star-refine* $\mathfrak{U}_2$ in $\mathcal{S}$ (and written $\mathfrak{U}_1 < s\mathfrak{U}_2$) if for each $A \in \mathfrak{U}_1$ there is a covering $\mathfrak{U}_A \in \mathcal{S}$ and a set $B \in \mathfrak{U}_2$ such that $\mathrm{st}(A, \mathfrak{U}_A) \subset B$.

A family of coverings in which each covering has a local star-refinement is called a *semi-normal* family.

A *semi-uniformity* $\mathcal{S}$ on a set X is a family of coverings of X which satisfies

(i) $\mathcal{S}$ is a filter with respect to local star-refinement and (ii) for distinct points $x, y \in X$, there is a covering in $\mathcal{S}$, no member of which contains both x and y.

The concepts of a base and a subbase for a semi-uniformity are analogous to those for a uniformity, [1].