C-separated sets and unicoherence

by

R. F. Dickman, Jr. and **L. R. Rubin** (Blacksburg, Va.)

Abstract. We say that a subset A of a topological space X is *C-separated* provided there exist disjoint closed and connected sets H and K of X such that $A \subset (H \cup K)$ and A meets both H and K. A connected space X has *Property C* provided that every separated closed set is C-separated. In this paper we give a conjecture concerning unicoherence for locally connected, connected normal spaces and show it is equivalent to a special case of a conjecture of A. H. Stone. We show that our conjecture holds in spaces which have Property C. As the major result of this paper we show that every locally compact, locally connected, connected Lindelöf Hausdorff space has Property C.

1. Definitions and terminology. Let X be a topological space. By a *continuum of* X we mean a non-empty closed and connected subset of X and by a *region* we mean an open and connected set. Note that a continuum is not necessarily compact. A set $A \subset X$ is *conditionally compact* provided $\bar{A}$ is compact.

A set $A \subset X$ is *C-separated* provided there exist disjoint continua H and K of X such that $A \subset (H \cup K)$ and $A \cap H \neq \emptyset \neq A \cap K$. We say X has *Property C* provided that every separated closed subset of X is C-separated. Recall that X is *unicoherent* if whenever $X = H \cup K$ where H and K are continua, $H \cap K$ is a continuum. Definitions not given herein may be found in [2] or [7].

2. Some conjectures. In [5] A. D. Wallace proved that a Peano continuum S is multicoherent (i.e., not unicoherent) if and only if there exist continua S_1, S_2, and S_3 of S such that $S = S_1 \cup S_2 \cup S_3$ and for $i, j \in \{1, 2, 3\}$, $S_i \cap S_j \neq \emptyset$ while $S_1 \cap S_2 \cap S_3 = \emptyset$. A. H. Stone has made the following conjecture:

(*) Let X be a locally connected, connected normal space and let $n > 2$ be an integer. If X is not unicoherent, X can be expressed as the union of n continua $A_1, A_2, \ldots, A_n$ whose nerve is a closed n-gon (i.e., with indices taken mod n, A_i meets A_{i-1} and A_{i+1} and no others and no three of these sets have a non-empty intersection).

In [4] Stone proved (*) for $n = 3$. He also has established that (*) is true when X is finitely coherent or when X is compact.

We offer the following conjecture:

(**) Let X be a locally connected, connected normal space. Then X is unicoherent if and only if for each disjoint pair $\{A, B\}$ of continua of X there exists a continuum C of X such that C separates A and B in X.

In a private communication A. H. Stone reported his conjecture to us and at the same time he stated that he believed that (*) for $n = 4$ was equivalent to (**).

LEMMA 1. *Let X be a unicoherent, locally connected, connected normal space and let A and B be disjoint continua of X. Then there exists a continuum C of X that separates A and B in X.*

Proof. This follows from a well-known characterization of unicoherence. See [7, pp. 47–49].

THEOREM 1. *Let X be a locally connected, connected normal space. Then (*) for $n = 4$ is equivalent to (**).*

Proof. Suppose that (*) for $n = 4$ holds. By Lemma 1 we need only show that if every pair of disjoint continua of X can be separated by a continuum in X, then X is unicoherent. Suppose to the contrary that X is not unicoherent. Then by (*) there exist continua A_1, A_2, A_3, A_4 such that $X = \bigcup A_i$, $A_i \cap A_j \neq \emptyset$ if and only if $|i(\mathrm{mod}\,4) - j(\mathrm{mod}\,4)| \leqslant 1$, and no three of the A_i's have a non-empty intersection. By our assumption there exists a continuum C such that C separates A_1 and A_3 in X. But then $C \subset A_2 \cup A_4$, $A_2 \cap A_4 = \emptyset$ and C meets each of the continua A_2 and A_4, implying C is not connected. This contradiction implies (**) holds.

Now suppose (**) is true and X is not unicoherent. Then by (**) there exist disjoint continua A' and B' in X such that no continuum of X separates A' and B' in X. Let A_0 and B_0 be disjoint continua such that A' is interior to A_0 and B' is interior to B_0. Now if $X \setminus A_0$ is connected, let $A = A_0$; if $X \setminus A_0$ is not connected let U be the component of $X \setminus A_0$ that contains B_0 and in this case let $A = X \setminus U$. In either case A' is interior to A, $A \cap B_0 = \emptyset$ and $X \setminus A$ is connected. In a similar fashion we can find a continuum B such that B' is interior to B, $A \cap B = \emptyset$ and $X \setminus B$ is connected. Let $T = X \setminus (A \cup B)$ and note that $\overline{T}$ is not connected. For otherwise $\overline{T}$ would be a continuum separating A' and B'. Thus $\overline{T} = H \cup K$ where H and K are disjoint non-empty closed sets. Let P denote the union of the components of T which lie in H and let Q denote the union of the components of T which lie in K. Observe that since neither A nor B separates X, and X is locally connected, then the closure of every component of T meets both A and B. Thus $Z = A \cup Q \cup B$ and $Y = A \cup P \cup B$ are continua. Furthermore note that Y is locally connected (in the relative topology of Y) at every point of $\overline{P}$. Let R be any open subset of X containing A such that $\overline{R} \cap B = \emptyset$ and let F denote the frontier of R relative to Y. Let L be the component of $Y \setminus F$ containing A and let M be the component of $Y \setminus \overline{L}$ containing B. Finally let $D_1 = Y \setminus M$ and $D_2 = \overline{M}$. Then D_1 and D_2 are continua, $A \subset \mathrm{Int}_Y D_1$, $B \subset \mathrm{Int}_Y D_2$, $Y = D_1 \cup D_2$ and $D_1 \cap D_2 \subset F \subset Y \setminus (A \cup B)$. Repeat the above construction for Z and obtain continua D_3 and D_4 such that $A \subset \mathrm{Int}_Z D_4$, $B \subset \mathrm{Int}_Z D_3$, $Z = D_3 \cup D_4$ and $D_3 \cap D_4 \subset Z \setminus (A \cup B)$. Then $X = D_1 \cup D_2 \cup D_3 \cup D_4$ is the desired representation of X.

LEMMA 2. *If X is a connected space, X has Property C, and every disjoint pair of continua of X can be separated by a connected subset of X, then X is unicoherent.*

Proof. Suppose $X = H \cup K$ where H and K are continua and $H \cap K$ is separated. Since $H \cap K$ is also closed and X has Property C, there exist disjoint continua C and D such that $H \cap K \subset (C \cup D)$ and $C \cap (H \cap K) \neq \emptyset \neq D \cap (H \cap K)$. By hypothesis there exists a connected subset M of X such that M separates C and D in X. But this implies M meets each of the separated sets $H \setminus (C \cup D)$ and $K \setminus (C \cup D)$, and since M is contained in their union, M is not connected. This contradiction proves the Lemma.

The next theorem is obtained from Lemmas 1 and 2.

THEOREM 2. *If X is a locally connected, connected normal space which has Property C, then X is unicoherent if and only if every pair of disjoint continua in X can be separated by a connected subset of X.*

3. C-separated sets and Property C.

In this section we establish some basic properties of C-separated sets.

LEMMA 3. *Let A and B be subsets of X, suppose $A \subset B$ and that every component of B contains a component of A. Then if B is C-separated so also is A.*

The proof follows easily from the definitions involved.

LEMMA 4. *Let X be locally connected, connected and normal. If A is a separated closed subset of X and A has only finitely many components then A is C-separated.*

Proof. Let $Q_1, \ldots, Q_n$ be the components of A and let $V_1, \ldots, V_n$ be a collection of regions such that for $i, j \in \{1, \ldots, n\}$, $Q_i \subset V_i$ and $\overline{V}_i \cap \overline{V}_j = \emptyset$ if $i \neq j$. Let $\{W_\alpha\}$, $\alpha \in \Gamma$, be a covering of $X \setminus A$ by regions no one of which intersects more than one $\overline{V}_i$. Let $K = \{V_i\}_{i=1}^{n} \cup \{W_\alpha\}_{\alpha \in \Gamma}$ and let S be a simple chain [7, p. 33] of elements of K from Q_1 to Q_2 and let S_1 be the subchain of S such that S_1 contains V_1 and exactly one V_i different from V_1.

Let Q' denote the closure of the union of the links of S_1. Then $B = \bigcup \{Q_i: Q_i \cap Q' = \emptyset\} \cup Q'$ is a separated set with $n-1$ components and every component of B contains a component of A. The result now follows by induction.

Lemma 5. *Let X be a locally compact, locally connected Hausdorff space and suppose A is a compact subset of a region V of X. Then there is a continuum L of X with $A \subset L \subset V$.*

The proof of this lemma is routine.

Lemma 6. *Let A be a closed, disconnected subset of a locally connected, connected Hausdorff space X. Then if A has a component Q having a compact neighborhood, A is C-separated.*

Proof. Let V be a conditionally compact region of X such that $Q \subset V$, $\mathrm{Fr} V \cap A = \emptyset$, and $A \not\subset V$. Choose V so that $\overline{V}$ has a compact neighborhood. Let W be a conditionally compact neighborhood of $\overline{V}$ such that $\mathrm{Fr} W \cap A = \emptyset$ and $W \cap A = V \cap A$. Let S denote the set of complementary domains of $\overline{V}$. If $U \in S$ and $U \cap A \neq \emptyset$, then $U \cap \cap \mathrm{Fr} W \neq \emptyset$. Since S is a pairwise disjoint set and S is an open cover of the compact set $\mathrm{Fr} W$, there is a finite set, say $\{S_1, S_2, \ldots, S_n\}$, of elements of S that covers $\mathrm{Fr} W$. Thus $A \subset (\bigcup S_i) \cup V$. Applying Lemma 5, there is a continuum L of X with $V \cap A \subset L \subset V$. Let $B = L \cup \overline{S}_1 \cup \overline{S}_2 \cup \ldots \cup \overline{S}_n$. Then B has at least two components, one of which is L; and L and at least one other component of B intersects A. In fact, $A \subset B$. By Lemmas 3 and 4, A is C-separated since B has only finitely many components.

Definition. A continuous function f of X onto a space Y is *non-alternating* provided that for every $y \in Y$, if $X \setminus f^{-1}(y) = H \cup K$ is a separation, then $f(H) \cap f(K) = \emptyset$ [6, p. 127]. When $Y = [0, 1]$, a continuous surjection $f: X \to Y$ is non-alternating provided for every $y \in (0, 1)$, $X \setminus f^{-1}(y)$ has exactly two components [1, Lemma 1].

Lemma 7. *Let X be a locally connected, connected normal space. Then X has Property C if and only if for every pair of disjoint non-empty closed sets A and B of X there exists a non-alternating map $f: X \to [0, 1]$ such that $f(A \cup B) = \{0, 1\}$.*

This is Lemma 4 and Remark 5 of [1].

4. C-separation in locally compact Lindelöf spaces. In this section we prove the major result of this paper.

Suppose U is an open subset of a space X; $C, D, Q_1, \ldots, Q_n$ are continua in X that meet U; and B is a closed subset of X contained in U such that $B \cup C \cup D \cup Q_1 \cup \ldots \cup Q_n = C_1 \cup D_1$ where C_1 and D_1 are disjoint continua, $C \subset C_1$ and $D \subset D_1$. Then we say (C_1, D_1) is a *division* of $\bigcup Q_i$ in U relative to (C, D).

Lemma 8. *Let X be a locally connected regular space, U be a region of X, and $\{C, D, Q_1, \ldots, Q_n\}$ be a pairwise disjoint set of continua of X all intersecting U. Then there exists a division (C_1, D_1) of $\bigcup Q_i$ in U relative to (C, D).*

Proof. Let E be a covering of U by regions such that for each $V \in E$, $\overline{V} \subset U$ and $\overline{V}$ meets at most one element of $\{C, D, Q_1, \ldots, Q_n\}$. Let $\{V_1, \ldots, V_r\}$ be a simple chain of elements of E from Q_1 to $C \cup D$. Choose s so that $\overline{V}_s$ is the first element of $\{\overline{V}_1, \ldots, \overline{V}_r\}$ that intersects $C \cup D$, say $\overline{V}_s$ intersects C. Then no element of $\{\overline{V}_1, \ldots, \overline{V}_s\}$ intersects D. Let $\{Q'_1, \ldots, Q'_m\}$ be the members of $\{Q_1, \ldots, Q_n\}$ which intersect $\overline{V}_1 \cup \overline{V}_2 \cup \ldots \cup \overline{V}_s$. Then let $C' = C \cup \overline{V}_1 \cup \ldots \cup \overline{V}_s \cup Q'_1 \cup \ldots \cup Q'_s$. We see that C' is a continuum, $Q_1 \cup C \subset C'$ and $C' \cap D = \emptyset$. With C' replacing C, we proceed by induction.

Lemma 9. *Let X be a locally connected, paracompact Hausdorff space and F be a closed subspace of X having the property that X is locally compact at each point of F. Then there exists a collection $\{R_\alpha\}$ of conditionally compact regions of X covering F, such that $\{R_\alpha \cap F\}$ is a locally finite subset of X and each $R_\alpha \cap F$ intersects at most finitely many elements of $\{R_\alpha \cap F\}$.*

Proof. We see that F is a locally compact, paracompact subspace of X. By [2, XI.7.3] F is a free union $\{F_\beta\}$ of σ-compact spaces. Let $F_\beta \in \{F_\beta\}$ and using [2, XI.7.2] write $F_\beta = \bigcup_1^\infty U_i$ where U_i is open in F_β, $\overline{U}_i$ is compact, and $\overline{U}_i \subset U_{i+1}$. Let $U_{-1} = U_0 = \emptyset$. We then write $F_\beta = \bigcup_1^\infty (U_i \setminus U_{i-1})$. For each $i \geqslant 1$ there is a finite open cover $\{R_1^i, R_2^i, \ldots, R_{m_i}^i\}$ of $\overline{U}_i \setminus U_{i-1}$ where each R_j^i is a conditionally compact region of X and $R_j^i \cap F_\beta = R_j^i \cap F \subset U_{i+1} \setminus \overline{U}_{i-2}$. It is clear that $\{R_j^i \cap F\} = \{R_j^i \cap F_\beta\}$ is a locally finite subset of F_β and hence of X because F_β is closed in X. Furthermore $R_j^i \cap F_\beta$ can meet $R_m^k \cap F_\beta$ only if $i-1 \leqslant k \leqslant i+1$. So each $R_j^i \cap F_\beta = R_j^i \cap F$ intersects at most finitely many elements of $\{R_j^i \cap F\}$. Let us construct such open sets $\{R_j^i\}$ for each F_β, and call the totality of these sets $\{R_\alpha\}$. Then $\{R_\alpha\}$ satisfies the required conditions.

Lemma 10. *Let X be a paracompact, locally connected, connected Hausdorff space and suppose P is a countable, discrete, closed subspace of X. If $X \setminus P$ is locally compact, then P is C-separated.*

Proof. Let $\{P_i \mid i = 1, 2, \ldots\}$ be a countable, locally finite collection of open subsets of X with pairwise disjoint closures and such that for each i, P_i contains exactly one element of P. If $\{\overline{P}_i\}$ is C-separated then so is P; we shall prove $\{\overline{P}_i\}$ is C-separated.

By Lemma 9 there exists a collection $\{R_\alpha\}$ of relatively compact regions of X covering $F = X \setminus \bigcup P_i$ such that $\{R_\alpha \cap F\}$ is a locally finite subset of X and each $R_\alpha \cap F$ intersects at most finitely many elements of $\{R \cap F\}$. We note that if $Y \subset X$ can be written as a union of a collection $\{Y_\beta\}$ of closed subsets of F such that each R_α intersects at most finitely many Y_β, then $\{Y_\beta\}$ is locally finite and hence Y is closed in X [2, III.9.2].

Since $\{P_i\}$ is countable, $\{P_i\} \cup \{R_\alpha\}$ is an open cover of X, and X is connected, there is a countable subset, say $\{R_i\}$ of $\{R_\alpha\}$ such that $(\bigcup_1^\infty P_i) \cup (\bigcup_1^\infty R_i)$ is connected. Each P_j must intersect at least one R_i and since each R_i is conditionally compact, it can intersect only finitely many P_j. Let $C_0 = \overline{P}_1$ and $D_0 = \overline{P}_2$. Let S_1 be the first element of $\{R_i\}$ that intersects $C_0 \cup D_0$. Let $A_1 = \{\overline{P}_j |\ \overline{P}_j \not\subset C_0 \cup D_0 \text{ and } \overline{P}_j \cap S_1 \neq \emptyset\}$. Let (C_1, D_1) be a division of A_1 in S_1 relative to (C_0, D_0), which is possible by Lemma 8. Thus C_1 is the union of C_0, elements of $\{\overline{P}_j\}$, and a closed subset C_1' of X with $C_1' \subset S_1 \cap F$. Similarly D_1 is the union of D_0, elements of $\{\overline{P}_j\}$, and a closed subset D_1' of X with $D_1' \subset S_1 \cap F$.

Now suppose $A_n, C_n, D_n, C_n', D_n', S_n$ have been chosen. Let S_{n+1} be the first element of $\{R_i\} \setminus \{S_1, S_2, \ldots, S_n\}$ that intersects $C_n \cup D_n$. Let $A_{n+1} = \{\overline{P}_j |\ \overline{P}_j \not\subset C_n \cup D_n \text{ and } \overline{P}_j \cap S_{n+1} \neq \emptyset\}$. Let (C_{n+1}, D_{n+1}) be a division of A_{n+1} in S_{n+1} relative to (C_n, D_n). Thus C_{n+1} is the union of C_n, elements of $\{\overline{P}_j\}$ and a closed subset C_{n+1}' of X with $C_{n+1}' \subset S_{n+1} \cap F$. Similarly D_{n+1} is the union of elements of $\{\overline{P}_j\}$, D_n, and a closed subset D_{n+1}' of X with $D_{n+1}' \subset S_{n+1} \cap F$.

Let $C = \bigcup C_i$ and $D = \bigcup D_i$. Then $C \cap D = \emptyset$, C and D are connected, and $\overline{P}_1 \subset C$ and $\overline{P}_2 \subset D$. By a straightforward inductive argument it can be proved that $\bigcup \overline{P}_i \subset C \cup D$. If we can prove C and D are closed in X, the proof will be complete.

But C can be written as $(\bigcup C_i') \cup B$ where B is the union of a collection of elements of $\{\overline{P}_i\}$. Thus B is closed. Now each R_α can intersect at most finitely many elements of $\{C_i'\}$ because each $C_i' \subset S_i \cap F$, $S_i \in \{R_\alpha\}$, and $R_\alpha \cap F$ intersects at most finitely many elements of $\{R_\alpha \cap F\}$. Thus by a previous remark, $\bigcup C_i'$ is a closed subset of X. Therefore C is closed. By a similar argument D is closed. The proof is complete.

LEMMA 11. *Let X be a locally compact, locally connected, Lindelöf, Hausdorff space and suppose A is a disconnected closed subset of X. Then there exists a closed neighborhood B of A whose components form a locally finite set of at least two and at most countably many elements each of which intersects A.*

Proof. Let $A = H \cup K$ where H and K are disjoint, non-empty closed sets. Since X is locally compact and Hausdorff, X is regular; and this condition along with the Lindelöf property implies X is paracompact and hence normal. Furthermore there is a sequence $\{V_i\}$ of conditionally compact open subsets of X such that $X = \bigcup V_i$ and for each i, $\overline{V}_i \subset V_{i+1}$ [2, XI.7.2].

Using the normality, let E and F be open sets with $\overline{E} \cap \overline{F} = \emptyset$, $H \subset E$ and $K \subset F$. Let $V_0 = V_{-1} = \emptyset$. For each $i > 0$, let $A_i = A \cap (\overline{V}_i \setminus V_{i-1})$. Let B_i be the closure of a finite union of conditionally compact regions such that every component of B_i meets A_i and $A_i \subset \operatorname{Int} B_i \subset B_i \subset (V_{i+1} \setminus \overline{V}_{i-2}) \cap (E \cup F)$. Define $B = \bigcup B_i$. It is routine to check that B satisfies the required conditions.

THEOREM 3. *Let X be a locally compact, locally connected, connected, Lindelöf, Hausdorff space. Then X has Property C.*

Proof. Let A be a separated closed subset of X. By Lemma 11 there is a closed set B containing A such that the components of B form a locally finite set of at least two and at most countably many elements each of which intersects A. If B has only finitely many components, then, applying Lemmas 3 and 4, A is C-separated. Thus, suppose B has a countable number of components.

Let G be the decomposition of X obtained from the relation, x is equivalent to y if and only if x and y are in the same component of B. Since the set of components of B is locally finite, G is upper semicontinuous. Thus the quotient map $p\colon X \to X/G$ is a closed map. By Theorem VIII.6.5 of [2], we know that X is paracompact and that since p is closed, X/G is paracompact and Hausdorff. The quotient map p is monotone, X/G is connected and locally connected, and X/G is locally compact outside the countable, closed, discrete subspace $p(B)$. By Lemma 10, $p(B)$ is C-separated. Under a monotone identification the inverse image of a connected set is connected. Therefore B is C-separated. By Lemma 3, A is C-separated and the proof is complete.

In the following we shall use implicitly results and terminology from [3]. By a polyhedron we mean a space homeomorphic to the space $|K|$ of a simplicial complex K. We note that although every polyhedron is paracompact and Hausdorff, polyhedra need not be locally compact or Lindelöf.

THEOREM 4. *Every connected polyhedron has Property C.*

Proof. Let X be a connected polyhedron and assume $X = |K|$ where K is a simplicial complex. Suppose A is a disconnected, closed subset of X. Let H and K be disjoint closed and non-empty sets such that $A = H \cup K$. Since X is paracompact it is normal, so let E and F be neighborhoods of H and K respectively such that $\overline{E} \cap \overline{F} = \emptyset$. There exists a subdivision L of K such that every simplex of L that meets A is contained in $E \cup F$. Let C_H be the subcomplex of L consisting of all simplexes of L that intersect H and all faces of such simplexes; let C_K be defined similarly for K. Then $H \subset |C_H| \subset E$ and $K \subset |C_K| \subset F$.

Let C_0 be a component of C_H and D_0 be a component of C_K. Let $\mathfrak{M}$ be the set of all ordered pairs (M, M_0) such that M is a component of $C_H \cup C_K$ and M_0 is a closed path in L intersecting M and C_0 but not intersecting D_0. Let C be the union of all the subcomplexes of L which are elements of an ordered pair of $\mathfrak{M}$. Then C is connected, $C_0 \subset C$, and $C \cap D_0 = \emptyset$. Now if M is a component of $C_H \cup C_K$ and there is not a closed path in L intersecting D_0 and M but not C_0, then there is a closed path M_0 in L intersecting D_0 and M but not intersecting C. Thus, let $\mathfrak{N}$ be the set of all ordered pairs (M, M_0) such that M is a component of $C_H \cup C_K$ and M_0 is a closed path in L intersecting M and D_0 but not intersecting C. Let D be the union of all the subcomplexes of L which are elements of an ordered pair of $\mathfrak{N}$. Then D is connected, $D_0 \subset D$ and $D \cap C = \emptyset$.

We see that every component of $C_H \cup C_K$ is either in C or D. Since C and D are connected subcomplexes of L, both $|C|$ and $|D|$ are continua of X. Since $C \cap D = \emptyset$, $|C| \cap |D| = \emptyset$. But $A = H \cup K \subset |C| \cap |D|$ and $|C| \cap H \neq \emptyset \neq K \cap |D|$. Therefore A is C-separated.

Corollary. *Let X be a connected polyhedron or a locally connected, connected, Lindelöf, locally compact Hausdorff space. Then the following are equivalent.*

a. *X is unicoherent.*

b. *Every pair of disjoint continua in X can be separated by a connected set.*

If in addition X is separable, then using Theorem 2 *of* [1], (a) *and* (b) *are equivalent to*:

c. *For every pair of disjoint non-empty closed sets A and B of X there exists a continuous function f of X onto* $[0,1]$ *such that* $f(A \cup B) = \{0,1\}$ *and for some dense subset D of* $[0,1]$, $f^{-1}(d)$ *is connected for every* $d \in D$.

Remark. Recently the authors have obtained an example of a locally connected, connected separable metric space that fails to have Property C. They also have an example of a connected CW-complex that fails to have Property C.

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VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY
THE UNIVERSITY OF OKLAHOMA

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