COROLLARY 2. *If T is a universal Horn theory and T', the theory of the infinite models of T, is complete then every model of T is atomic compact.*

Proof. All finite structures are atomic compact so it suffices to consider the models of T'. By the main theorem of [5] T' is $\aleph_1$-categorical. By the corollary to Theorem 1 of [5], T' is almost strongly minimal. T' is model complete by Lindstrom's theorem and the result follows.

The situation regarding possible strengthenings of the last three results is clarified by noticing that the last example in [4] has the following properties. T is a $\forall\exists$ $\aleph_1$-categorical Horn theory which is not almost strongly minimal but each model of T is atomic compact.

References

[1] J. T. Baldwin, *Almost strongly minimal theories I*, J. Symbolic Logic 37 (1972).

[2] — *Almost strongly minimal theories II* (to appear J. Symbolic Logic).

[3] — *The number of automorphisms of a model of an $\aleph$-categorical theory* (to appear Fund. Math.).

[4] — and A. H. Lachlan, *On strongly minimal sets*, J. Symbolic Logic 36 (1971), pp. 79–96.

[5] — — *On universal Horn theories categorical in some infinite power* (to appear Algebra Universalis).

[6] P. Lindstrom, *On model completeness*, Theoria 30 (1964), pp. 183–196.

[7] J. Mycielski, *Some compactifications of general algebras*, Colloq. Math. 13 (1964), pp. 1–9.

[8] — and C. Ryll-Nardzewski, *Equationally compact algebras II*, Fund. Math. 61 (1968), pp. 271–281.

[9] J. R. Shoenfield, *Mathematical Logic*, 1967.

[10] W. Taylor, *Some constructions of compact algebras*, Annals of Math. Log. 3 (1971), pp. 395–437.

[11] B. Węglorz, *Equationally compact algebras* (I), Fund. Math. 59 (1966), pp. 289–298.

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On limit numbers of real functions

by

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Abstract. In this work is given a general way of introducing of limit numbers for real function of real variable. With every real number is connected some family of sets fulfilling two natural conditions. They assure that for arbitrary function at every point x there exists at least one limit number and the set of all limit numbers at a point x is closed. By adequate adjustment of the family $\mathfrak{B}$ one can get usual limit numbers or approximate limit numbers. The main results of the work are concerned with the questions of: the set of points of $\mathfrak{B}$-asymmetry, connections between the ordinary continuity and $\mathfrak{B}$-continuity and $\mathfrak{B}$ semicontinuity of upper and lower $\mathfrak{B}$-functions of Baire.

Introduction. The aim of this work is to generalize the notion of limit numbers and approximate limit numbers and to find some properties of these generalized limit numbers. To obtain this it will be convenient to use the following definition:

If $f\colon R \to R$ and $x_0 \in R$, where R denotes the set of all real numbers, then g is called the *limit number of f at x_0* if and only if, for every $\varepsilon > 0$, x_0 is a point of accumulation of the set $\{x\colon |f(x)-g| < \varepsilon\}$.

The starting point of my considerations is the following remark: the family $\mathfrak{B}$ of all sets having x_0 as a point of accumulation have the following properties:

(1) every set including the set from $\mathfrak{B}$ also belongs to $\mathfrak{B}$,

(2) if $E_1 \cup E_2 \in \mathfrak{B}$, then $E_1 \in \mathfrak{B}$ or $E_2 \in \mathfrak{B}$,

(3) if $E_1 \in \mathfrak{B}$, then for every $t > 0$ also $E_1 \cap (x_0-t, x_0+t) \in \mathfrak{B}$.

There is a very similar situation in the case of approximate limit numbers. Now $\mathfrak{B}$ is the family of all sets for which x_0 is not a point of dispersion.

The foregoing generalization will depend on making use of rather arbitrary families of sets fulfilling only conditions (1)-(3). These conditions seem to be natural, because the set of limit numbers of an arbitrary functions at every point obtained by means of them is non empty and closed.

1. Definition 1. Let $\mathfrak{B}_0^+$ be a non empty family of non empty sets $E \subset R$ such that

(4) if $E \in \mathfrak{B}_0^+$, then, for every $t > 0$, $E \cap (0, t) \in \mathfrak{B}_0^+$,

(5) $E_1 \cup E_2 \in \mathfrak{B}_0^+$ if and only if $E_1 \in \mathfrak{B}_0^+$ or $E_2 \in \mathfrak{B}_0^+$.

For every set $E \subset R$ and $x \in R$ we shall write

$$E + x = \{y: \bigvee_{a \in E} (y = a + x)\}, \qquad -E = \{y: -y \in E\}.$$

Then the family $\mathfrak{B}_0^-$ is defined as

$$\mathfrak{B}_0^- = \{E: -E \in \mathfrak{B}_0^+\}.$$

For every $x \in R$ let

$$\mathfrak{B}_x^+ = \{E: (E - x) \in B_0^+\}, \qquad \mathfrak{B}_x^- = \{E: (E - x) \in \mathfrak{B}_0^-\},$$

and $\mathfrak{B}_x = \mathfrak{B}_x^+ \cup \mathfrak{B}_x^-$. Now let $\mathfrak{B} = \bigcup_{x \in R} \mathfrak{B}_x$.

Definition 2. The number g is called a $\mathfrak{B}$*-limit number* (a *left-sided $\mathfrak{B}$-limit number, a right-sided $\mathfrak{B}$-limit number*) *of a function f at a point x_0* if for every $\varepsilon > 0$ the set

$$\{x: |f(x) - g| < \varepsilon\}$$

belongs to $\mathfrak{B}_{x_0}$ ($\mathfrak{B}_{x_0}^-$, $\mathfrak{B}_{x_0}^+$ respectively).

The symbol $+\infty$ ($-\infty$) will be called a $\mathfrak{B}$*-limit number of f at a point x_0* if for every real number r

$$\{x: f(x) > r\} \in \mathfrak{B}_{x_0} \qquad (\{x: f(x) < r\} \in \mathfrak{B}_{x_0}).$$

To limit numbers defined in this way applies the main property of limit numbers.

Theorem 1. *For every function $f: R \to R$ and every point $x_0 \in R$ there exists at least one $\mathfrak{B}$-limit number (left-sided $\mathfrak{B}$-limit number, right-sided $\mathfrak{B}$-limit number) of the function f at the point x_0.*

Proof. For an arbitrary number $t > 0$

$$(x_0, x_0 + t) \cap \{x: |f(x)| < \infty\} \in \mathfrak{B}_{x_0}^+.$$

According to (4) we infer that at least one of the sets

$$\{x: f(x) \leqslant 0\}, \qquad \{x: f(x) \geqslant 0\}$$

belongs to $\mathfrak{B}_{x_0}^+$. Let us suppose that the second of these sets is in $\mathfrak{B}_{x_0}^+$. Let $I_n = [n, n+1]$ for every natural n. Then

$$[0, \infty) = \bigcup_{n=0}^{\infty} I_n.$$

Consider two cases:

(i) there exists an n_0 such that $\{x: f(x) \in I_{n_0}\} \in \mathfrak{B}_{x_0}^+$,

(ii) $\{x: f(x) \in I_n\} \notin \mathfrak{B}_{x_0}^+$ for every n.

If the second case is fulfilled, then $+\infty$ is a right-sided $\mathfrak{B}$-limit number of f at the point x_0.

If the first case is fulfilled, then at least one of the sets

$$\{x: f(x) \in [n_0, n_0 + \tfrac{1}{2}]\}, \qquad \{x: f(x) \in [n_0 + \tfrac{1}{2}, n_0 + 1]\}$$

belongs to $\mathfrak{B}_{x_0}^+$. In this manner we obtain a descending sequence of closed intervals $\{J_k\}$ such that

(6) $$|J_k| = 2^{-k}, \qquad \{x: f(x) \in J_k\} \in \mathfrak{B}_{x_0}^+.$$

Let $\{g\} = \bigcap_{k=1}^{\infty} J_k$. Then for every number $\varepsilon > 0$ there exists an index k such that

$$\{x: f(x) \in J_k\} \subset \{x: |f(x) - g| < \varepsilon\}.$$

The first of these sets belongs to $\mathfrak{B}_{x_0}^+$; thus by (5) the second set also belongs to $\mathfrak{B}_{x_0}^+$.

Let the function $h: R \to R$ be defined in the following way:

$$h(x) = f(x_0 - 2x).$$

Then every right-sided $\mathfrak{B}$-limit number of the function h at the point x_0 is a left-sided $\mathfrak{B}$-limit number of the function f at the point x_0. So there exists a left-sided $\mathfrak{B}$-limit number of f at x_0. The unilateral $\mathfrak{B}$-limit number is a $\mathfrak{B}$-limit number; hence the proof is complete.

Let $L_{\mathfrak{B}}(f, x)$, $(L_{\mathfrak{B}}^+(f, x), L_{\mathfrak{B}}^-(f, x))$ denote the set of all $\mathfrak{B}$-limit numbers (right-sided $\mathfrak{B}$-limit numbers, left-sided $\mathfrak{B}$-limit numbers, respectively) of a function f at a point x.

Theorem 2. *For every function $f: R \to R$ and every $x \in R$ all sets $L_{\mathfrak{B}}^+(f, x)$, $L_{\mathfrak{B}}^-(f, x)$, $L_{\mathfrak{B}}(f, x)$ are closed and*

(7) $$L_{\mathfrak{B}}(f, x) = L_{\mathfrak{B}}^+(f, x) \cup L_{\mathfrak{B}}^-(f, x).$$

Proof. Equality (7) is obvious in virtue of Definitions 1 and 2.

Let $\{g_n\}$ be a sequence such that

$$g_n \in L_{\mathfrak{B}}^+(f, x_0) \quad \text{and} \quad g = \lim_{n \to \infty} g_n,$$

where x_0 is a point of the set R. For every positive number there exists a g_n such that

$$\{x: |f(x) - g| < \varepsilon\} \supset \{x: |f(x) - g_n| < \tfrac{1}{2}\varepsilon\} \in \mathfrak{B}_{x_0}^+.$$

Hence $g \in L^+_{\mathfrak{B}}(f, x_0)$. The proof that the set $L^-_{\mathfrak{B}}(f, x_0)$ is closed is similar. From equality (7) we infer that the set $L_{\mathfrak{B}}(f, x_0)$ is also closed.

The foregoing properties are the fundamental properties of limit numbers and approximate limit numbers. This conception of the way of introduction of $\mathfrak{B}$-limit numbers is therefore a natural generalization of the way of introduction of limit numbers. If conditions (4) and (5) are not fulfilled, then Theorems 1 and 2 are not valid.

From the definitions of the family $\mathfrak{B}$ and of $\mathfrak{B}$-limit numbers we immediately obtain.

THEOREM 3. *Let $\mathfrak{B}$ and $\mathfrak{D}$ be families fulfilling conditions* (4) *and* (5). *Then for every function $f\colon R \to R$ and $x \in R$ the inclusion $L_{\mathfrak{B}}(f, x) \subset L_{\mathfrak{D}}(f, x)$ holds if and only if $\mathfrak{B} \subset \mathfrak{D}$.*

DEFINITION 3. The number g is called a $\mathfrak{B}$-*limit of the function f at the point x_0* if $\{g\} = L_{\mathfrak{B}}(f, x_0)$.

Unilateral $\mathfrak{B}$-limits are defined analogously. $\mathfrak{B}$-limits so defined have the same properties as the usual limits.

Let $\mathfrak{B}^*_x$ be a family of sets defined as follows:

$$E \in \mathfrak{B}^*_x \quad \text{if} \quad R \setminus E \notin \mathfrak{B}_x .$$

Then a $\mathfrak{B}$-limit of a function may also be defined in the following way.

The number g is called a $\mathfrak{B}$-*limit of the function f at the point x_0* if, for every number $\varepsilon > 0$, $\{x\colon |f(x) - g| < \varepsilon\} \in \mathfrak{B}^*_{x_0}$. The family $\mathfrak{B}^*_x$ has the following properties:

(8) $\emptyset \notin \mathfrak{B}^*_x$,

(9) if $E_1 \in \mathfrak{B}^*_x$ and $E_1 \subset E_2$, then $E_2 \in \mathfrak{B}^*_x$,

(10) if $E_1 \in \mathfrak{B}^*_x$, then for every $t > 0$ also $E_1 \cap (x-t, x+t) \in \mathfrak{B}^*_x$.

(11) if $E_1, E_2 \in \mathfrak{B}^*_x$, then $E_1 \cap E_2 \in \mathfrak{B}^*_x$.

Hence the family $\mathfrak{B}^*_x$ is a filter of subsets of the set R. This conception of limit is equivalent to the conception of limit with the aid of a filter (see Bourbaki [2]). Using a family of filters one can define the set of limit numbers of a function. Moreover, remark that if we have a family of filters $\{\mathfrak{F}_x\}_{x \in R}$ fulfilling the conditions

(12) if $E \in \mathfrak{F}_0$, then $E + x \in \mathfrak{F}_x$,

(13) if $E \in \mathfrak{F}_0$, then for every positive number t also $E \cap (-t, t) \in \mathfrak{F}_0$

then with the aid of the family of filters $\{\mathfrak{F}_x\}_{x \in R}$ the family $\mathfrak{B}$ fulfilling conditions (4) and (5) can be defined. We define this family as follows:

$$\mathfrak{B}_0 = \{E\colon \bigwedge_{A \in \mathfrak{F}_0} (A \cap E \neq \emptyset)\} .$$

The family $\mathfrak{B}^*_0$ obtained from the family $\mathfrak{B}_0$ coincides with the filter $\mathfrak{F}_0$.

Now we give some examples of families $\mathfrak{B}$.

EXAMPLE 1. Let $\mathfrak{N}^+_0$ be the family of sets for which 0 is a point of right-sided accumulation. It is obvious that the family $\mathfrak{N}$ defines ordinary limit numbers and the $\mathfrak{N}$-limit is a limit in the usual sense.

EXAMPLE 2. Let us consider the family $\mathfrak{A}^+_0$ of all sets for which 0 is not a point of right-sided dispersion. Limit numbers with respect to this family $\mathfrak{A}$ are approximate limit numbers.

EXAMPLE 3. Let $\mathfrak{M}^+_0$ be the family of sets E such that, for all numbers $t > 0$, $|E \cap (0, t)| > 0$, where $|E|$ denotes the outer Lebesgue measure of a set E.

EXAMPLE 4. The qualitative limit numbers are obtained from a family $\mathfrak{C}$ defined as follows: a set E belongs to the family $\mathfrak{C}^+_0$ if for every $t > 0$ the set $E \cap (0, t)$ is of the second category.

EXAMPLE 5. Let $\mathfrak{H}^+_0$ denote the family of all sets E such that a set $E \cap (0, t)$ is non denumerable for an arbitrary number $t > 0$.

EXAMPLE 6. Let $\{p_k\}$ be the sequence of the prime numbers and $P_k = \{p_k^{-1}, p_k^{-2}, \ldots, p_k^{-n}, \ldots\}$. We shall say that a set E belongs to the family $\mathfrak{G}^+_0$ if that set contains infinite subsequences of infinitely many sequences P_k.

It is easy to remark that the above families fulfil conditions (4), (5).

Remark. It is easy to see that if there exists a set $E \in \mathfrak{B}^+_{x_0}$ such that

$$\lim_{x \to x_0,\, x \in E} f(x) = g ,$$

then $g \in L^+_{\mathfrak{B}}(f, x_0)$. The converse theorem is not true. Let $\mathfrak{A}$ be the family defined in Example 2 and let $\{E_n\}$ be a descending sequence of sets having the upper density equal to $1/n$ at the point 0. Then a function f defined as

$$f(x) = \begin{cases} n^{-1} & \text{for} \quad x \in E_n \setminus E_{n+1} , \\ 2 & \text{for} \quad x \notin E_1 , \end{cases}$$

has at the point 0 a limit number equal to 0, and, on the contrary, there exists no set E such that $\overline{D}(E, 0) > 0$ and

$$\lim_{x \to x_0,\, x \in E} f(x) = 0 .$$

However, we have the following

THEOREM 4. *Let $f\colon R \to R$ be an arbitrary function and $x_0 \in R$. Then the conditions*

(i) $g \in L^+_{\mathfrak{B}}(f, x_0)$,

(ii) *there exists a set $E \in \mathfrak{B}^+_{x_0}$ such that* $\lim\limits_{x \to x_0,\, x \in E} f(x) = g$

are equivalent if and only if the family $\mathfrak{B}$ fulfils the following condition:

(W) *For every descending sequence of sets* $\{E_n\}$ *such that* $E_n \in \mathfrak{B}_0^+$, *there exists a decreasing sequence* $\{x_n\}$ *converging to* 0 *and such that*

$$\bigcup_{n=1}^{\infty} (E_n \cap [x_{n+1}, x_n)) \in \mathfrak{B}_0^+ .$$

Proof. Necessity. Let us suppose that condition (W) is not fulfilled. Then there exists a descending sequence of sets $\{E_n\}$ such that $E_n \in \mathfrak{B}_0^+$ and, for every decreasing sequence $\{x_n\}$ converging to 0, $\bigcup_{n=1}^{\infty} (E_n \cap [x_{n+1}, x_n)) \notin \mathfrak{B}_0^+$. Let us define a function f as follows:

$$f(x) = \begin{cases} n^{-1} & \text{for} \quad x \in E_n \setminus E_{n+1} , \\ 2 & \text{for} \quad x \notin E_1 . \end{cases}$$

Then, for all $x \in E_n$, $f(x) \leqslant n^{-1}$, and 0 is a limit number of f at the point 0. On the contrary, there exists no set E fulfilling condition (ii). In fact let us suppose that there is a set $E \in \mathfrak{B}_0^+$ such that $\lim_{x\to 0,\, x\in E} f(x) = 0$. Hence for all natural n there exist numbers $t_n > 0$ such that for $x \in E \cap (0, t_n)$ the inequality $f(x) \leqslant n^{-1}$ holds. The sequence $\{t_n\}$ may be chosen in such a way that, for all n, $t_{n+1} < t_n$. For $x \in E_n \cap [t_{n+1}, t_n)$ also $f(x) \leqslant n^{-1}$. From the definition of function f it follows that

$$E \cap [t_{n+1}, t_n) \subset E_n \cap [t_{n+1}, t_n)$$

and

$$E \cap (0, t_1) \subset \bigcup_{n=1}^{\infty} (E_n \cap [t_{n+1}, t_n)) \notin \mathfrak{B}_0^+ .$$

This contradicts the choice of the set E.

Sufficiency. By the foregoing considerations it is sufficient to prove that condition (i) implies (ii).

Let $f\colon R \to R$ be an arbitrary function and $g \in L_{\mathfrak{B}}^+(f, x_0)$. Then

$$E_n = \{x\colon |f(x) - g| < n^{-1}\} \in \mathfrak{B}_{x_0}^+ .$$

From condition (W) it follows that there exists a sequence $\{x_n\}$ such that $x_0 < x_{n+1} < x_n$, $\lim_{n\to\infty} x_n = x_0$ and

$$E = \bigcup_{n=1}^{\infty} (E_n \cap [x_{n+1}, x_n)) \in \mathfrak{B}_{x_0}^+ .$$

It is easy to see that $\lim_{x\to x_0,\, x\in E} f(x) = g$. This completes the proof.

As for $\mathfrak{B}$-limit numbers, we have the following

Remark. If there exists a set $E \in \mathfrak{B}_{x_0}^*$ such that

$$\lim_{x\to x_0,\, x\in E} f(x) = g ,$$

then $g = (\mathfrak{B}) \lim_{x\to x_0} f(x)$.

The converse theorem is not true in the general case. In the case where for the family $\mathfrak{B}$ we have the family $\mathfrak{A}$ from Example 2 these two conditions are equivalent. It is interesting to see in what cases this equivalence holds. Let us remark that in the proof of Theorem 4 we need only those properties of the family $\mathfrak{B}$ which the family $\mathfrak{B}^*$ has also. These are (8)-(10). Hence we obtain the following

THEOREM 5. *For a function* $f\colon R \to R$ *and* $x_0 \in R$ *the conditions*

(i) $g = (\mathfrak{B}) \lim_{x\to x_0} f(x)$,

(ii) *there exists a set* $E \in \mathfrak{B}_{x_0}^*$ *such that* $g = \lim_{x\to x_0,\, x\in E} f(x)$,

are equivalent if and only if the family $\mathfrak{B}$ *has the following property*

(W*) *For every descending sequence of sets* $\{E_n\}$ *such that* $E_n \in \mathfrak{B}_0^{+*}$, $n = 1, 2, \ldots$ *there exists a decreasing sequence* $\{x_n\}$ *converging to* 0 *and such that*

$$\bigcup_{n=1}^{\infty} (E_n \cap [x_{n+1}, x_n)) \in \mathfrak{B}_0^{+*} .$$

2. DEFINITION 4. We shall say that the *family* $\mathfrak{B}$ *fulfils condition* M if for arbitrary sequences $\{x_n\}$ (of numbers) and $\{E_n\}$ (of sets) such that

$$\lim_{n\to\infty} x_n = 0 , \quad x_n > 0,\ E_n \in \mathfrak{B}_{x_n}$$

the set $E = \bigcup_{n=1}^{\infty} E_n$ belongs to the family $\mathfrak{B}_0^+$.

If in this condition the sequence of sets $\{E_n\}$ is replaced by a sequence of intervals of the form $(x_n - t_n, x_n)$, $(x_n, x_n + t_n)$ or $(x_n - t_n, x_n + t_n)$, then we shall say that the *family* $\mathfrak{B}$ *fulfils condition* M′.

It is easy to see that families $\mathfrak{N}$, $\mathfrak{M}$, $\mathfrak{C}$, $\mathfrak{H}$ fulfil condition M and of course M′ also, and the family $\mathfrak{A}$ does not fulfil the condition M′ and M, either.

For a function f the set $\{x\colon L_{\mathfrak{B}}^+(f, x) \neq L_{\mathfrak{B}}^-(f, x)\}$ is called the *set of* $\mathfrak{B}$-*asymmetry of the function* f. It is well known that the set of asymmetry (in the usual sense) of an arbitrary function is at most denumerable (see W. H. Young [7]). The set of approximate asymmetry need not be denumerable (see L. Belowska [1]); however, it must be a set of the first category and of Lebesgue measure 0 (M. Kulbacka [4]). In the general case we have the following

THEOREM 6. *If the family $\mathfrak{B}$ fulfils condition* M, *then the set of $\mathfrak{B}$-asymmetry of an arbitrary function f is at most denumerable.*

Proof. It suffices to show that the set

$$A = \{x\colon L_{\mathfrak{B}}^-(f, x) \setminus L_{\mathfrak{B}}^+(f, x) \neq \emptyset\}$$

is at most denumerable. Let $x_0 \in A$. Then there exists a point $p \in L_{\mathfrak{B}}^-(f, x_0) \setminus L_{\mathfrak{B}}^+(f, x_0)$. By Theorem 2 there exists a number $\varepsilon_0 > 0$ such that

$$(p-\varepsilon_0, p+\varepsilon_0) \cap L_{\mathfrak{B}}^+(f, x_0) = \emptyset, \quad \{x\colon |f(x)-p| < \varepsilon_0\} \in \mathfrak{B}_{x_0}^+.$$

Hence there exist three rational numbers $a_{x_0}, b_{x_0}, c_{x_0}$ such that

$$a_{x_0} < p < b_{x_0}, \quad c_{x_0} > x_0, \quad \{x\colon a_{x_0} < f(x) < b_{x_0}\} \cap (x_0, c_{x_0}) \notin \mathfrak{B}_{x_0}^+.$$

From condition M it follows that, for different points $x_1, x_2 \in A$, $(a_{x_1}, b_{x_1}, c_{x_1}) \neq (a_{x_2}, b_{x_2}, c_{x_2})$. From this if follows that the set of $\mathfrak{B}$-asymmetry of a bounded function is at most denumerable.

If f is an unbounded function, then let us consider a function defined as follows:

$$u(x) = \frac{f(x)}{1+|f(x)|}.$$

For functions f and u the sets of $\mathfrak{B}$-asymmetry coincide. This completes the proof.

T. Świątkowski in [5] has also studied the set of asymmetry of a function in the general case.

3. In this part of the paper we shall study the connections between the $\mathfrak{B}$-continuity and the continuity of functions in the ordinary sense.

DEFINITION 5. A function f is called *$\mathfrak{B}$-continuous at a point* x_0 if $f(x_0) = (\mathfrak{B})\lim\limits_{x\to x_0} f(x)$.

Of course, if a function f is continuous at x_0, then it is $\mathfrak{B}$-continuous for every family $\mathfrak{B}$ fulfilling conditions (4) and (5). It is easy to see that the converse theorem is not true. However, we have the following

LEMMA 1. *Let a family $\mathfrak{B}$ fulfils condition* M. *If x_0 is a point of $\mathfrak{B}$-continuity of a function f and, for every x from some interval (x_0-t, x_0+t), $f(x) \in L_{\mathfrak{B}}(f, x)$, then f is continuous at the point x_0 (in usual sense).*

Proof. Let us suppose that f is not continuous at x_0. There exist a sequence $\{x_n\}$ converging to x_0 and a number $r > 0$ such that for every n

$$|f(x_n) - f(x_0)| \geq r > 0. \tag{14}$$

Then there exists a sequence $\{k_n\}$ such that there exists a limit $\lim\limits_{n\to\infty} f(x_{k_n}) = y_0$. Suppose that y_0 is a finite number. In the case of $y_0 = \pm\infty$ the proof is similar. Let ε be an arbitrary positive number. There exists an n_0 such that for $n \geq n_0$ we have the following inequalities:

$$|x_{k_n} - x_0| < \varepsilon, \quad |f(x_{k_n}) - y_0| < \tfrac{1}{3}\varepsilon.$$

From the assumption it follows that

$$E_{k_n} = \{x\colon |f(x) - f(x_{k_n})| < \tfrac{1}{3}\varepsilon\} \in \mathfrak{B}_{x_{k_n}}.$$

The set $E = \{x\colon |f(x) - y_0| < \varepsilon\}$ includes a set $\bigcup\limits_{n\geq n_0} E_{k_n}$ which belongs to $\mathfrak{B}_{x_0}^+$ by virtue of condition M. Hence the set E also belongs to $\mathfrak{B}_{x_0}^+$. In this way we have obtained that $y_0 \in L_{\mathfrak{B}}(f, x_0)$. This contradicts (14) and the assumptions.

The following theorem is an immediate consequence of the above lemma.

THEOREM 7. *If the family $\mathfrak{B}$ fulfils condition* M, *then, for an arbitrary function $f\colon R \to R$, the $\mathfrak{B}$-continuity of f in an interval (a, b) is equivalent to the continuity of f in that interval.*

Remark. Let $\mathfrak{B}$ and $\mathfrak{D}$ be two families fulfilling conditions (4) and (5). We shall say that *families $\mathfrak{B}$ and $\mathfrak{D}$ fulfil condition $M_{\mathfrak{B}\mathfrak{D}}$* if for every set $E \in \mathfrak{D}_0^+$ and every family of sets $\{E_x\}_{x\in E}$ such that $E_x \in \mathfrak{B}_x$, $x \in E$, the set $\bigcup\limits_{x\in E} E_x$ belongs to the family $\mathfrak{B}_0^+$, and for every set $E \in \mathfrak{B}_0^+$ and every family of sets $\{E_x\}_{x\in E}$ such that $E_x \in \mathfrak{D}_x$, $x \in E$ the set $\bigcup\limits_{x\in E} E_x$ belongs to the family $\mathfrak{D}_0^+$.

Theorem 7 can be generalized in the following way:

If families $\mathfrak{B}$ and $\mathfrak{D}$ fulfil condition $M_{\mathfrak{B}\mathfrak{D}}$, then, for an arbitrary function f, $\mathfrak{B}$-continuity in an interval (a, b) is equivalent to the $\mathfrak{D}$-continuity of f in this interval.

The proof of this fact is similar to the proof of Lemma 1.

THEOREM 8. *If, for arbitrary function f, $\mathfrak{B}$-continuity coincides with continuity in an interval (a, b), then the family $\mathfrak{B}$ fulfils condition* M′.

Proof. Let us suppose that $\mathfrak{B}$ does not fulfil condition M′. Then there exist sequences $\{x_n\}$ of numbers and $\{I_n\}$ of intervals of the form (for example) (x_n, x_n+t_n) such that

$$\bigcup_{n=1}^{\infty} I_n \notin \mathfrak{B}_0^+.$$

Moreover, one can assume that these intervals are pairwise disjoint. Let the function f be defined as

$$f(x) = \begin{cases} \frac{2}{t_n}x - \frac{2}{t_n}x_n\,, & \text{for} \quad x \in (x_n, x_n + \tfrac{1}{2}t_n]\,, \\ -\frac{2}{t_n}x - \frac{2}{t_n}x_n + 2\,, & \text{for} \quad x \in (x_n + \tfrac{1}{2}t_n, x_n + t_n)\,, \\ 0 & \text{for} \quad \text{remaining } x\,. \end{cases}$$

It is easy to see that the function f is $\mathfrak{B}$-continuous in R and, on the contrary, it is not continuous at the point 0.

4. Let us write for a function $f\colon R \to R$

$$\varphi_{\mathfrak{B}}(f, x) = \min L_{\mathfrak{B}}(f, x)\,, \qquad \Phi_{\mathfrak{B}}(f, x) = \max L_{\mathfrak{B}}(f, x)\,,$$
$$m_{\mathfrak{B}}(f, x) = \min L^*_{\mathfrak{B}}(f, x)\,, \qquad M_{\mathfrak{B}}(f, x) = \max L^*_{\mathfrak{B}}(f, x)\,,$$

where $L^*_{\mathfrak{B}}(f, x) = L_{\mathfrak{B}}(f, x) \cup \{f(x)\}$.

DEFINITION 6. We shall say that the function f is *upper $\mathfrak{B}$-semicontinuous* (*lower $\mathfrak{B}$-semicontinuous*) *at a point* x_0 if

$$f(x_0) \geqslant \Phi_{\mathfrak{B}}(f, x_0) \qquad \big(f(x_0) \leqslant \varphi_{\mathfrak{B}}(f, x_0)\big)\,.$$

For the function f functions $m_{\mathfrak{N}}$ and $M_{\mathfrak{N}}$ are the lower and the upper functions of Baire respectively. These functions are semicontinuous. Now we shall study the properties of functions $\varphi_{\mathfrak{B}}$, $\Phi_{\mathfrak{B}}$, $m_{\mathfrak{B}}$ and $M_{\mathfrak{B}}$.

For a set $A(x) \subset R$, where $x \in R$, we shall write

$$\mathcal{A}(x) = \{x\} \times A(x)\,,$$
$$\mathcal{K}(p_0, r) = \{(x, y)\colon\ p_0 = (x_0, y_0),\ |x - x_0| < r,\ |y - y_0| < r\}\,.$$

DEFINITION 7. We shall say that a *point* $p_0 = (x_0, y_0)$ *belongs to the upper topological $\mathfrak{B}$-limit of a family of sets* $\{E_x\}_{x \in X}$ if for every number $r > 0$

$$\{x\colon\ x \in X,\ E_x \cap \mathcal{K}(p_0, r) \neq \emptyset\} \in \mathfrak{B}_{x_0}\,.$$

We shall denote the upper topological $\mathfrak{B}$-limit of the family of sets $\{E_x\}_{x \in X}$ by $(\mathfrak{B}) \operatorname*{ls}_{x \in X} E_x$.

The notion of the upper topological $\mathfrak{B}$-limit of a family of sets permits us to give some characterization of sets of $\mathfrak{B}$-limit numbers in an analogous form to that used in [3] for sets of ordinary limit numbers. As in [3], $\operatorname*{ls}_{x \in X} E_x$ denotes the usual upper topological limit of a family of sets.

LEMMA 2. *For every bounded function $f\colon R \to R$ and $x_0 \in R$ the inclusion*

$$\mathcal{T}(x_0) \cap \operatorname*{ls}_{x < x_0} \mathfrak{L}_{\mathfrak{B}}(f, x) \subset \mathfrak{L}^-_{\mathfrak{B}}(f, x_0)\,, \tag{15}$$

where $\mathcal{T}(x_0) = \{(x, y)\colon\ x = x_0\}$, holds if and only if the family $\mathfrak{B}$ fulfils condition M.

Proof. In the first part of the proof let us assume that the family $\mathfrak{B}$ fulfils condition M. Let

$$(x_0, y_0) \in \mathcal{T}(x_0) \cap \operatorname*{ls}_{x < x_0} \mathfrak{L}_{\mathfrak{B}}(f, x)\,.$$

Then there exist two sequences $\{x_n\}$, $\{y_n\}$ such that

$$x_n < x_0\,, \qquad x_n \to x_0\,, \qquad y_n \to y_0\,, \qquad y_n \in L_{\mathfrak{B}}(f, x_n)\,.$$

Let ε be an arbitrary positive number. There exists an index n_0 such that for all $n \geqslant n_0$ we have

$$|x_n - x_0| < \varepsilon\,, \qquad |y_n - y_0| < \tfrac{1}{3}\varepsilon\,, \qquad \{x\colon\ |f(x) - y_n| < \tfrac{1}{3}\varepsilon\} \in \mathfrak{B}_{x_n}\,.$$

According to condition M

$$E = \bigcup_{n \geqslant n_0} \{x\colon\ |f(x) - y_n| < \tfrac{1}{3}\varepsilon\} \in \mathfrak{B}^-_{x_0}\,.$$

Hence the set $\{x\colon\ |f(x) - y_0| < \varepsilon\}$ containing the set E belongs to $\mathfrak{B}^-_{x_0}$. Thus $(x_0, y_0) \in L^-_{\mathfrak{B}}(f, x_0)$.

Let us suppose now that the family $\mathfrak{B}$ does not fulfil condition M. Then there exist two sequences $\{x_n\}$ of numbers and $\{E_n\}$ of sets such that

$$x_n < 0\,, \qquad x_n \to 0\,, \qquad E_n \in \mathfrak{B}_{x_n} \quad \text{and} \quad E = \bigcup_{n=1}^{\infty} E_n \notin \mathfrak{B}^-_0\,.$$

For the characteristic function of the set E (15) does not hold. This completes the proof.

For sets of ordinary limit numbers we have (in [3])

$$\mathcal{T}(x_0) \cap \operatorname*{ls}_{x < x_0} \mathfrak{L}^*(f, x) = \mathfrak{L}^-(f, x_0)\,.$$

For a family $\mathfrak{B} \neq \mathfrak{N}$ this equality does not hold. However, one can obtain a very similar equality for sets of $\mathfrak{B}$-limit numbers.

LEMMA 3. *If the family $\mathfrak{B}$ fulfils condition $\mathrm{M}_{\mathfrak{B}\mathfrak{B}}$, then for every bounded function f the equality*

$$\mathcal{T}(x_0) \cap (\mathfrak{B}) \operatorname*{ls}_{x < x_0} \mathfrak{L}^*_{\mathfrak{B}}(f, x) = \mathfrak{L}^-_{\mathfrak{B}}(f, x_0)$$

holds for every point $x_0 \in R$.

Proof. From the definition of $\mathfrak{B}$-limit numbers and of the upper topological $\mathfrak{B}$-limit of a family of sets it follows that

$$\mathcal{T}(x_0) \frown (\mathfrak{B}) \operatorname*{ls}_{x<x_0} \{(x, f(x))\} = \mathfrak{L}_{\mathfrak{B}}^{-}(f, x_0) .$$

Now we shall show that the following inclusion is fulfilled:

$$\mathcal{T}(x_0) \frown (\mathfrak{B}) \operatorname*{ls}_{x<x_0} \mathfrak{L}_{\mathfrak{B}}(f, x) \subset \mathfrak{L}_{\mathfrak{B}}^{-}(f, x_0) . \tag{16}$$

Let $p_0 = (x_0, y_0) \in \mathcal{T}(x_0) \frown (\mathfrak{B}) \operatorname*{ls}_{x>x_0} \mathfrak{L}_{\mathfrak{B}}(f, x)$. For every number $r > 0$

$$E = \{x\colon \mathcal{K}(p_0, r) \frown \mathfrak{L}_{\mathfrak{B}}(f, x) \neq \emptyset\} \in \mathfrak{B}_{x_0}^{-} .$$

Let $\varepsilon > 0$ be an arbitrary number. For $x \in E$ let $y_x \in L_{\mathfrak{B}}(f, x)$. Then $\{h\colon |f(h)-y_x| < \frac{1}{3}\varepsilon\} \in \mathfrak{B}_x$ and, moreover, for all $x \in E \frown (x_0-t, x_0)$, where $t > 0$, we have $|y_x-y_0| < \frac{1}{3}\varepsilon$. Hence

$$\{h\colon |f(h)-y_0| < \varepsilon\} \supset \bigcup_{x\in E} \{h\colon |f(h)-y_x| < \tfrac{1}{3}\varepsilon\} \in \mathfrak{B}_{x_0}^{-},$$

and $y_0 \in L_{\mathfrak{B}}^{-}(f, x_0)$. Then by virtue of (16) and the properties of the upper topological $\mathfrak{B}$-limit of a family of sets we have

$$\begin{aligned}\mathcal{T}(x_0) \frown (\mathfrak{B}) \operatorname*{ls}_{x<x_0} \mathfrak{L}_{\mathfrak{B}}^{*}(f, x) &= \mathcal{T}(x_0) \frown \Big[(\mathfrak{B}) \operatorname*{ls}_{x<x_0} \mathfrak{L}_{\mathfrak{B}}(f, x) \smile (\mathfrak{B}) \operatorname*{ls}_{x<x_0} \{(x, f(x))\}\Big] \\ &= \mathcal{T}(x_0) \frown (\mathfrak{B}) \operatorname*{ls}_{x<x_0} \{(x, f(x))\} = \mathfrak{L}_{\mathfrak{B}}^{-}(f, x_0) .\end{aligned}$$

This ends the proof.

The following theorem is an immediate consequence of Lemma 2.

THEOREM 9. *For an arbitrary bounded function f, the function $\varphi_{\mathfrak{B}}$ is lower semicontinuous if and only if the family $\mathfrak{B}$ fulfils condition* M.

THEOREM 10. *For an arbitrary bounded function f, the function $m_{\mathfrak{B}}$ is lower $\mathfrak{B}$-semicontinuous if and only if the family $\mathfrak{B}$ fulfils conditions* $\mathrm{M}_{\mathfrak{B}\mathfrak{B}}$.

Proof. Let us assume that the family $\mathfrak{B}$ fulfils condition $\mathrm{M}_{\mathfrak{B}\mathfrak{B}}$. Then from Lemma 3 it follows that

$$\min L_{\mathfrak{B}}(m_{\mathfrak{B}}(f, \cdot), x_0) \geqslant \min L_{\mathfrak{B}}(f, x_0) \geqslant m_{\mathfrak{B}}(f, x_0) .$$

Hence it follows that $m_{\mathfrak{B}}$ is lower $\mathfrak{B}$-semicontinuous at every point $x_0 \in R$.

Now let us assume that, for every bounded function f, the function $m_{\mathfrak{B}}$ is lower $\mathfrak{B}$-semicontinuous, and let us suppose that the family $\mathfrak{B}$ does not fulfil condition $\mathrm{M}_{\mathfrak{B}\mathfrak{B}}$. There exist a set $E \in \mathfrak{B}_0^{-}$ and a family of sets $\{E_x\}_{x\in E}$ such that

$$E_x \in \mathfrak{B}_x \quad \text{and} \quad E_0 = \bigcup_{x\in E} E_x \notin \mathfrak{B}_0^{-} .$$

Let

$$f(t) = \begin{cases} 0 & \text{for } t \in E_0\setminus\{0\} , \\ 1 & \text{for } t \notin E_0\setminus\{0\} . \end{cases}$$

Then for $x_0 \in E \in \mathfrak{B}_0^{-}$ we have $m_{\mathfrak{B}}(f, x_0) = 0$ and the only $\mathfrak{B}$-limit number of function f at the point 0 is $1 = f(0)$. This contradiction ends the proof.

From [6] it follows that the family $\mathfrak{A}$ (Example 2) fulfils condition $\mathrm{M}_{\mathfrak{A}\mathfrak{A}}$. Then the functions $m_{\mathfrak{A}}$, $M_{\mathfrak{A}}$ are approximately semicontinuous for every bounded function.

References

[1] L. Belowska, *Résolution d'un problème de M. Z. Zahorski sur les limites approximatives*, Fund. Math. 48 (1960), pp. 277–286.

[2] N. Bourbaki, *Topologie générale*, Paris 1961.

[3] J. Jędrzejewski and W. Wilczyński, *On the family of sets of limit numbers*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (8) (1970), pp. 453–460.

[4] M. Kulbacka, *Sur l'ensemble des points de l'asymétrie approximative*, Acta Sci. Math. (Szeged) 21 (1960), pp. 90–95.

[5] T. Świątkowski, *On some generalization of the notion of asymmetry of functions*, Coll. Math. 17 (1967), pp. 77–91.

[6] W. Wilczyński, *On the family of sets of approximate limit numbers*, Fund. Math. 75 (1972), pp. 169–174.

[7] W. H. Young, *La symétrie de structure des fonctions des variables réeles*, Bull. Sci. Math. 52 (2) (1928), pp. 265–280.

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