(iii) $$\underset{\sim}{H}(\underset{\sim}{S}^k(\underset{\sim}{d}), \underset{\sim}{S}^l(\underset{\sim}{d}), \underset{\sim}{c}) \wedge \underset{\sim}{F}_{k,l}(\underset{\sim}{S}^k(\underset{\sim}{d}), \underset{\sim}{S}^l(\underset{\sim}{d}), \underset{\sim}{c}), \quad k > l,$$
$$\underset{\sim}{H}(\underset{\sim}{S}^k(\underset{\sim}{d}), \underset{\sim}{S}^l(\underset{\sim}{d}), \underset{\sim}{c}) \wedge \neg \underset{\sim}{F}_{k,l}(\underset{\sim}{S}^k(\underset{\sim}{d}), \underset{\sim}{S}^l(\underset{\sim}{d}), \underset{\sim}{c}), \quad k > l.$$

In $R(\mathcal{A}_2) - R(\mathcal{A}_1)$ there exist elements d_1 and d_2 such that for no natural number k is d_1 the kth successor or predecessor of d_2. The formulas $\underset{\sim}{H}(\underset{\sim}{S}^k(\underset{\sim}{d}_1), \underset{\sim}{S}^l(\underset{\sim}{d}_2), \underset{\sim}{v}_0)$ define principal types in $\mathrm{Th}(\mathcal{A}_2, d_1, d_2)$ which are realized by 2 points. That is, $*$ holds in $\mathcal{A}_2$ so $|\mathrm{Aut}(\mathcal{A}_2)| = 2^{\aleph_0}$.

Easy variations on this example will give theories T with $f(0) = 1$, $f(n) = \aleph_0$ $0 < n < N$ and $f(n) = 2^{\aleph_0}$ for $n \geqslant N$, for any choice of N. Is there a theory T such that for some natural number $N > 1$, $f(n)$ is finite for $n < N$, $f(n) = \aleph_0$ for some segment beginning with N, say $N \leqslant n \leqslant M$ and $f(n) = 2^{\aleph_0}$ for $n > M$? That is, is it possible in other than the trivial case when all elements of the prime model are named for the value of f to jump from finite to countable to uncountable?

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On certain homological properties of finite-dimensional compacta. Carriers, minimal carriers and bubbles (*)

by

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Abstract. A compact metric space X is q-cyclic ($q = 1, 2, \ldots$) provided that there exists a coefficient group G such that the Vietoris-Čech homology group $H_q(X, G)$ of X is non-trivial. An irreducibly q-cyclic compact metric space is called a q-dimensional closed Cantorian manifold, or a q-bubble. The following question, asked by P. S. Alexandroff, is considered in the paper: Given a q-dimensional q-cyclic compact metric space X, does X contain a q-bubble? As is known, the answer is not always positive, but by adding some assumptions on X a positive result is obtained. A class of spaces, the so called WSC_q-compacta, is exhibited in the paper and it is proved that each q-dimensional q-cyclic WSC_q-compactum contains at least one and at most countably many q-bubbles. Furthermore, some other properties of WSC_q-compacta are studied.

1. Introduction. By a compactum is meant a compact metric space. As is well known, the Čech and the Vietoris homology theories are equivalent in the algebraic sense (see for instance [13], p. 273). The q-dimensional Vietoris-Čech homology group of a compactum X with coefficients in an Abelian group G will be denoted by $H_q(X, G)$. This group will be sometimes represented as the limit of the inverse system of the (simplicial) qth homology groups of the nerves of all finite open coverings of X, with G as the coefficient group (the well-known Čech construction) and sometimes as the group of homology classes of q-dimensional true cycles in X with coefficients in G (the Vietoris construction). In the first case, the notations, symbols and terminology from [9], chap. IX will be applied here, in the second case, we shall base ourselves on the construction described in [4], chap. II, sec. 3. In particular, the concepts of infinite chains and infinite cycles are very useful. By means of these concepts the following characterization of the dimension of compacta has been established:

(*) This is the present writer's doctoral thesis (in a modified form), defended at the University of Warsaw, Poland, in February 1969. The original title of the thesis was "Dimensional properties of ANR-spaces".

(1.1) THEOREM (Alexandroff [1]). *A compactum X is of dimension $\leqslant n$ if and only if for any closed subset F of X and any Abelian group G, any infinite q-dimensional cycle lying in F is homologous to zero in F whenever it is homologous to zero in X.*

The coefficient group $\mathfrak{R}_1$ of rationals reduced modulo 1 is of special importance for the combinatorial methods in the dimension theory of compacta. In fact, in the above theorem the arbitrary group G can be replaced by $\mathfrak{R}_1$ and the term "infinite cycle" by "true cycle". Thus, the dimension of a compactum can also be characterized as follows:

(1.2) THEOREM (Alexandroff [1]; see also [15], p. 246). *A compactum X is of dimension $\leqslant n$ if and only if, for any closed subset F of X, the homomorphism $i_*\colon H_n(F, \mathfrak{R}_1) \to H_n(X, \mathfrak{R}_1)$, induced by inclusion $i\colon F \to X$, is a monomorphism.*

(1.3) DEFINITION. A compactum X is said to be *cyclic in dimension q* (or shortly: *q-cyclic*) if there exists an Abelian group G such that the group $H_q(X, G)$ is not trivial. Otherwise, X is called *acyclic in dimension q* (or *q-acyclic*). A q-dimensional compactum X is called a *q-dimensional closed Cantorian manifold* (see [1], p. 227) if it is irreducibly q-cyclic, i.e. if X is q-cyclic and if any closed proper subset of X is q-acyclic. A q-dimensional closed Cantorian manifold will be called here, for simplicity, a *q-dimensional bubble*, or shortly: a *q-bubble*.

P. S. Alexandroff [1] has raised the question: Does any q-dimensional, q-cyclic compactum X contain a q-dimensional closed Cantorian manifold? The answer is, in general, negative (see [12]). However, by some additional assumptions on X, a positive result can be obtained. In this paper a class of compacta is studied, namely the so called compacta without small q-dimensional cycles (abbreviated to: WSC_q, see Section 3) and it is proved that if a q-dimensional compactum X is q-cyclic and WSC_q, then X containes a q-bubble. Moreover, the number of all q-bubbles contained in a q-dimensional compactum X, which is WSC_q, is at most countable. The class of compacta WSC_q is rather large; for instance any ANR-space is WSC_q. This paper contains furthermore a construction of a 2-dimensional ANR-space whith an infinite number of 2-bubbles.

By a mapping we shall always understand a continuous function. The qth homology group of a compactum X with the coefficient group $\mathfrak{R}_1$ will be denoted, for simplicity, by $H_q(X)$.

2. Carriers of an element of the homology group. Irreducible and minimal carriers. Let X be a compactum and G an Abelian group, and let a be an element of the group $H_q(X, G)$. A closed subset F of X is called a *carrier of the element* a if there is an element $a' \in H_q(F, G)$ such that $i_*(a') = a$, that is, if $a \in \operatorname{Im} i_*$, where $i_*\colon H_q(F, G) \to H_q(X, G)$ is the homomorphism, induced by the inclusion $i\colon F \to X$. A carrier F of a is said to be *irreducible* if no proper subset of F is a carrier of a. A carrier F of a is said to be *minimal* if any carrier of a contains F. Thus, any minimal carrier of a is irreducible (but not conversely). Observe that if there exists a minimal carrier of a, then it is unique. It is easy to prove (for instance, by the continuity of the Vietoris-Čech homology theory) that

(2.1) any carrier F of an element $a \in H_q(X, G)$ contains an irreducible carrier of a.

The proof of the following theorem will be based on the Vietoris construction of the homology groups.

(2.2) THEOREM. *Suppose that X is a q-dimensional compactum. If F_1 and F_2 are carriers of an element $a \in H_q(X, G)$, then the intersection $F_1 \cap F_2$ is also a carrier of a.*

Proof. Suppose that a is the homology class of a q-dimensional true cycle α in X. By the assumptions of the theorem, there are true cycles α_1 and α_2 lying in F_1 and F_2, respectively, such that α_1 and α_2 are homologous to α in X. Thus, $\alpha_1 - \alpha_2 \sim 0$ in X. The q-dimensional true cycle $\alpha_1 - \alpha_2$ lies in $F_1 \cup F_2$ and $\dim X = q$; therefore, by (1.1), $\alpha_1 - \alpha_2$ is homologous to zero even in $F_1 \cup F_2$. Let β be a $(q+1)$-dimensional infinite chain lying in $F_1 \cup F_2$ and such that $\partial\beta = \alpha_1 - \alpha_2$. Now, define an infinitely small displacement (see [1], p. 180) of β as follows: for any vertex v of the chain β, if v belongs to a simplex of β which meets both F_1 and F_2, then replace v by a point $\bar{v}$ of $F_1 \cap F_2$ with the minimal distance from v; otherwise, let v stay at the same place, that is, $\bar{v} = v$. Let $\bar{\beta}$ denote the displaced chain. Observe that if a vertex v of β lies in F_i $(i = 1, 2)$, then the vertex $\bar{v}$ of $\bar{\beta}$ also lies in F_i. Thus $\partial\bar{\beta} = \bar{\alpha}_1 - \bar{\alpha}_2$ where $\bar{\alpha}_i$ $(i = 1, 2)$ is the displaced true cycle α_i. Therefore $\bar{\alpha}_i$ is a true cycle lying in F_i and homologous to α_i in F_i. On the other hand, the infinite chain $\bar{\beta}$ can be presented in the form $\bar{\beta} = \beta_1 + \beta_2$, where β_i is an infinite chain lying in F_i $(i = 1, 2)$. For instance, β_1 can be defined as that infinite chain which consists of all $(q+1)$-dimensional simplexes of $\bar{\beta}$ lying in F_1 and β_2 — all other simplexes of $\bar{\beta}$. The equalities

$$\partial\bar{\beta} = \bar{\alpha}_1 - \bar{\alpha}_2 \quad \text{and} \quad \partial\bar{\beta} = \partial\beta_1 + \partial\beta_2$$

imply that $\bar{\alpha}_1 - \partial\beta_1 = \bar{\alpha}_2 - \partial\beta_2$.

Therefore the infinite cycle $\bar{\alpha} = \bar{\alpha}_1 - \partial\beta_1 = \bar{\alpha}_2 - \partial\beta_2$ lies in $F_1 \cap F_2$ and $\bar{\alpha} - \bar{\alpha}_1 \sim 0$ in F_1. In particular, $\bar{\alpha}$ is a true cycle in F_1. But $\dim F_1 \leqslant q$, which implies that the infinite cycle $\bar{\alpha}$ is true even in $F_1 \cap F_2$. Thus, $F_1 \cap F_2$ is a carrier of a, since $\bar{\alpha} \sim \bar{\alpha}_1 \sim \alpha_1 \sim \alpha$ in X.

(2.3) COROLLARY. *If $\dim X = q$, then any irreducible carrier of an element $a \in H_q(X, G)$ is minimal.*

Proof. Let F_0 be an irreducible carrier of $a \in H_q(X, G)$ and let F be an arbitrary carrier of the same element a. Then, by (2.2), the set $F_0 \cap F$ is a carrier of a and, by the irreducibility of F_0, the equality $F_0 \cap F = F_0$ holds, that is, $F_0 \subset F$, which proves the minimality of F_0.

It follows by (2.1) and (2.3) that if $\dim X = q$, then for any element a of $H_q(X, G)$ there exists a (unique) minimal carrier of a; denote this carrier by $C(a)$. It is easy to see that the following formulas hold:

(2.4) $$C(a+b) \subset C(a) \cup C(b)\,,$$

(2.5) $$C(-a) = C(a)\,.$$

Formulas (2.4) and (2.5) yield:

(2.6) $$C(a) \dot{-} C(b) \subset C(a+b) \subset C(a) \cup C(b)\,,$$

where the sign $\dot{-}$ denotes the symmetric difference between sets.

3. Compacta without small q-dimensional cycles. In this section the Čech construction of the homology theory will be used. All notations related to this construction are adopted from [9], chap. IX.

By a covering of a compactum we shall always mean a finite open covering. Let α be a covering of a compactum X. Then X_α denotes the nerve of α (as a simplicial complex) and $H_q(X_\alpha, G)$ denotes the qth simplicial homology group of X_α with the coefficient group G. For any refinement β of α there is a simplicial projection $p\colon X_\beta \to X_\alpha$ and the homomorphism $\pi_\alpha^\beta\colon H_q(X_\beta, G) \to H_q(X_\alpha, G)$, induced by p. The collection $\{H_q(X_\alpha, G), \pi_\alpha^\beta\}$ is an inverse system of groups, and the homology group $H_q(X, G)$ is the limit of this system. The canonical projection of $H_q(X, G)$ into $H_q(X_\alpha, G)$ is denoted by π_α.

(3.1) Definition. A compactum X is said to be *without small q-dimensional cycles* (shortly: X is WSC_q) if there is a covering α of X such that, for any coefficient group G, the projection $\pi_\alpha\colon H_q(X, G) \to H_q(X_\alpha, G)$ is a monomorphism. If X is WSC_q for any $q = 0, 1, 2, \ldots$ then X is said to be *without small cycles* (shortly: WSC).

(3.2) Example. Any polyhedron is WSC. Moreover, if P is a polyhedron with a triangulation T and if τ is the open star covering of P, then $\pi_\tau\colon H_q(P, G) \to H_q(P_\tau, G)$ is an isomorphism for any q and any coefficient group G (see [9], pp. 250–251).

(3.3) Theorem. *Let X and Y be a pair of compacta and suppose that Y is WSC_q. If there exists a mapping $f\colon X \to Y$ such that, for any coefficient group G, the homomorphism $f_*\colon H_q(X, G) \to H_q(Y, G)$ induced by f is a monomorphism, then X is WSC_q.*

Proof. Let α be a covering of the compactum Y such that $\pi_\alpha\colon H_q(Y, G) \to H_q(Y_\alpha, G)$ is a monomorphism. By the definition of the homomorphism f_* induced by f there exist a covering α' of X and a simplicial mapping f_α of $X_{\alpha'}$ into Y_α such that the diagram

$$\begin{array}{ccc} H_q(X, G) & \xrightarrow{f_*} & H_q(Y, G) \\ \Big\downarrow{\scriptstyle \pi_{\alpha'}} & & \Big\downarrow{\scriptstyle \alpha} \\ H_q(X_{\alpha'}, G) & \xrightarrow[f_{\alpha *}]{} & H_q(Y_\alpha, G) \end{array}$$

is commutative. The composition $f_{\alpha *}\pi_{\alpha'}$ is a monomorphism, since $\pi_\alpha f_*$ is a monomorphism. Therefore $\pi_{\alpha'}$ is a monomorphism, which completes the proof.

(3.4) Corollary. *If a q-dimensional compactum X is WSC_q, then any closed subset of X is WSC_q.*

(3.5) Corollary. *If X is WSC_q and Y is acyclic in all dimensions, then the product $X \times Y$ is WSC_q. In particular, any prism (that is, the product of a polyhedron and the Hilbert-cube) is* WSC.

(3.6) Corollary. *If X is WSC_q, then any retract of X is WSC_q.*

(3.7) Corollary. *Any* ANR*-space is* WSC.

Corollary (3.7) is a simple consequence of (3.5), (3.6), and the characterization of ANR-spaces as retracts of prisms, [4], p. 105.

(3.8) Theorem. *Supposes that X is a compact subset of a locally compact metric space M and X is WSC_q. Then there exists a compact neighbourhood W of X in M such that the inclusion $i\colon X \to W$ induces a monomorphism $i_*\colon H_q(X, G) \to H_q(W, G)$ for any coefficient group G.*

Proof. Let $\alpha = \{U_1, U_2, \ldots, U_n\}$ be a covering of X such that $\pi_\alpha\colon H_q(X, G) \to H_q(X_\alpha, G)$ is a monomorphism. By a known Čech theorem ([8], p. 171) there exist open subsets $U_1', U_2', \ldots, U_n'$ of M such that $U_k = U_k' \cap X$ $(k = 1, 2, \ldots, n)$ and $U_{k_1}' \cap U_{k_2}' \cap \ldots \cap U_{k_m}' = \emptyset$ if and only if $U_{k_1} \cap U_{k_2} \cap \ldots \cap U_{k_m} = \emptyset$ for any system $k_1, k_2, \ldots, k_m$ of indices $(1 \leqslant k_i \leqslant n$ for $i = 1, 2, \ldots, m)$. The set

$$U = \bigcup_{k=1}^{n} U_k'$$

is a neighbourhood of X in M. By the local compactness of M and the compactness of X, the neighbourhood U contains a compact neighbourhood W of X. The family

$$\alpha' = \{U_1' \cap W, U_2' \cap W, \ldots, U_n' \cap W\}$$

is a covering of W and the simplicial mapping $i_\alpha\colon X_\alpha \to W_{\alpha'}$ defined by $i_\alpha(U_k) = U_k' \cap W$ $(k = 1, 2, \ldots, n)$ is a simplicial homeomorphism. Thus, the homomorphism $i_{\alpha *}\colon H_q(X_\alpha, G) \to H_q(W_{\alpha'}, G)$, induced by i_α, is an isomorphism. By the definition of a homomorphism induced by a mapping, the diagram

$$\begin{array}{ccc} H_q(X, G) & \xrightarrow{\ i_*\ } & H_q(W, G) \\ {\scriptstyle \pi_\alpha}\downarrow & & \downarrow{\scriptstyle \pi_{\alpha'}} \\ H_q(X_\alpha, G) & \xrightarrow[\ i_{\alpha*}\ ]{} & H_q(W_{\alpha'}, G) \end{array}$$

is commutative. The composition $i_{\alpha*}\pi_\alpha$ is a monomorphism, which implies that i_* is a monomorphism.

(3.9) THEOREM. *Suppose that X is a closed subset of the Hilbert-cube Q. Then X is* WSC_q *if and only if there exists a compact neighbourhood W of X in Q such that, for any coefficient group G, the homomorphism i_*: $H_q(X, G) \to H_q(W, G)$, induced by the inclusion i: $X \to W$, is a monomorphism.*

Proof. Necessity follows immediately by (3.8).

Sufficiency. Let P be a prismatic neighbourhood of X in Q (see [4], p. 105) contained in W and let i_1: $X \to P$ and i_2: $P \to W$ denote the respective inclusions. Then $i_* = i_{2*}i_{1*}$, where i_{1*}: $H_q(X, G) \to H_q(P, G)$ and i_{2*}: $H_q(P, G) \to H_q(W, G)$ are homomorphisms induced by i_1 and i_2, respectively, and therefore i_{1*} is a monomorphism. Thus, by (3.5) and (3.3), the compactum X is WSC_q.

Theorem (3.9) makes possible the statement that the property "to be WSC_q" is a so called shape invariant in Borsuk's theory of shape (see, for instance, [5], or [6]). Moreover, Theorem (3.3) has the following generalization (with a similar proof) in the theory of shape:

(3.10) COROLLARY. *Let X and Y be a pair of compacta and let Y be* WSC_q. *If there exists a fundamental sequence $\underline{f}$ from X to Y such that the homomorphism $\underline{f}_*$: $H_q(X, G) \to H_q(Y, G)$ induced by $\underline{f}$ is a monomorphism (for any coefficient group G), then X is* WSC_q.

(3.11) THEOREM. *If a compactum X is* WSC_q, *then the group $H_q(X)$ is at most countable.*

Proof. Suppose that X is a closed subset of the Hilbert-cube Q. By (3.9), there exists a compact neighbourhood V of X in Q such that the inclusion of X into V induces a monomorphism of the respective homology groups with any coefficient group, in particular with $\mathfrak{R}_1$ as the coefficient group. Let P be a prismatic neighbourhood of X contained in V. Then the inclusion i: $X \to P$ induces a monomorphism i_*: $H_q(X) \to H(P)$. The group $H_q(P)$ is at most countable; hence $H_q(X)$ is at most countable.

4. q-cyclicity of q-dimensional compacta. P. S. Alexandroff [2] has proved the following so called convergence theorem, formulated in the

language of ε-chains, ε-cycles and ε-homologies (see the Vietoris construction of the homology theory) with coefficients in $\mathfrak{R}_1$ (or certain other coefficient groups):

(4.1) THEOREM ([2], p. 31, the Convergence theorem). *If X is a compactum, then for any number $\varepsilon > 0$ there is a $\delta > 0$ ($\delta \leqslant \varepsilon$) such that any δ-cycle in X is ε-homologous to a true cycle in X.*

(4.2) COROLLARY (see [2], p. 31). *If there exists an infinite q-dimensional cycle in X with coefficients in $\mathfrak{R}_1$ which is not homologous to zero in X, then there exists a true q-dimensional cyle in X with coefficients in $\mathfrak{R}_1$ which is not homologous to zero in X, that is, the group $H_q(X)$ is not trivial.*

If X is a compact subset of a finite-dimensional Euclidean space E, or of the Hilbert-cube Q, then the Convergence theorem (4.1) can be equivalently expressed in the following form:

(4.3) CONVERGENCE THEOREM. *For any compact neighbourhood V of X (in E, or in Q, resp.) there exists a compact neighbourhood W of X contained in V and such that $j_*(H_q(W)) = i_*(H_q(X))$, where j_*: $H_q(W) \to H_q(V)$ and i_*: $H_q(X) \to H_q(V)$ are homomorphisms induced by the respective inclusions j: $W \to V$ and i: $X \to V$.*

As in (4.1), the coefficient group $\mathfrak{R}_1$ can be replaced in (4.2) and in (4.3) by certain other groups. The proof of the equivalence between the expressions (4.1) and (4.3) of the Convergence theorem is rather simple, and thus we omit it.

Let S^q denote the q-dimensional sphere. A mapping f: $X \to S^q$ is called *inessential* if it is homotopic to a constant mapping; otherwise it is called *essential.*

Suppose that X is a subset of a metric space M with a distance ϱ. A mapping f of X into M is called an *ε-displacement* ($\varepsilon > 0$) if $\varrho(x, f(x)) < \varepsilon$ for any $x \in X$. As is well known, any finite-dimensional compactum can be imbedded into a finite-dimensional Euclidean space, and moreover (see [1], p. 73):

(4.4) THEOREM. *If X is at most q-dimensional compact subset of a finite-dimensional Euclidean space E, then for any $\varepsilon > 0$ there exists a q-dimensional polyhedron $P \subset E$ and an ε-displacement of X into P.*

Let X be a compact subset of a locally compact metric space M and let G be an Abelian group. Then the following theorem holds (see, for instance, [4], p. 39):

(4.5) THEOREM. *For any element $a \in H_q(X, G)$, $a \neq 0$, there exists a compact neighbourhood V of X in M such that $i_*(a) \neq 0$, where i_*: $H_q(X, G) \to H_q(V, G)$ is the homomorphism induced by the inclusion i: $X \to V$.*

(4.6) COROLLARY. *Let X be a closed subset of a compact metric space M.*

Then, for any non-zero element $a \in H_q(X, G)$ there exists an $\varepsilon > 0$ such that, for any ε-displacement f of X onto $f(X) \subset M$, the element $f_*(a)$ of the group $H_q(f(X), G)$ is non-zero.

Proof. Assume that M is a subset of the Hilbert-cube Q. Certainly, this assumption does not restrict the generality of the proof. By (4.5), there exists a compact neighbourhood V of X in Q such that $i_*(a) \neq 0$, where $i_*\colon H_q(X, G) \to H_q(V, G)$ is the homomorphism induced by the inclusion $i\colon X \to V$. There exist an $\varepsilon > 0$ and a compact neighbourhood $W \subset V$ of X in Q such that any linear segment of length less than ε and with an endpoint in X is contained in W. Let $j\colon X \to W$ be the inclusion and $j_*\colon H_q(X, G) \to H_q(W, G)$ the homomorphism induced by j. Clearly $j_*(a) \neq 0$ since $W \subset V$. Now, let $f\colon X \to f(X) \subset M$ be an ε-displacement. Observe that $f(X) \subset W$; let j_1 denote the inclusion of $f(X)$ into W. For any $x \in X$, the linear segment with endpoints x and $f(x)$ is contained in W; therefore the composition $j_1 f\colon X \to W$ is homotopic to the inclusion j. Thus, $j_* = j_{1*} f_*$, which yields $f_*(a) \neq 0$.

(4.7) Theorem. *If X is a q-dimensional compactum, then the following conditions are equivalent:*

(i) *X is q-cyclic;*

(ii) *There exists an essential mapping of X into S^q;*

(iii) *$H_q(X) \not\approx 0$.*

Proof. Suppose that X is a subset of a finite-dimensional Euclidean space E.

1° (i) $\Rightarrow$ (ii). Suppose that G is an Abelian group such that $H_q(X, G) \not\approx 0$ and let a be a non-zero element of the group $H_q(X, G)$. Let $f_\varepsilon\colon X \to f_\varepsilon(X)$ be an ε-displacement of X into a q-dimensional polyhedron $P \subset E$ (see (4.4)), where ε is a positive number such that $f_{\varepsilon *}(a) \neq 0$ (see (4.6)). Let $j\colon f(X) \to P$ be the inclusion and let $f = jf_\varepsilon$, $f\colon X \to P$. By (1.1), the homomorphism $j_*\colon H_q(f(X), G) \to H_q(P, G)$ is a monomorphism, since $\dim P = q$. Thus, the element $f_*(a)$ of the group $H_q(P, G)$ is non-zero. There exists a mapping $g\colon P \to S^q$ such that $g_*(f_*(a)) \neq 0$, where $g_*\colon H_q(P, G) \to H_q(S^q, G)$ is the homomorphism induced by g (see [3], p. 514) since the polyhedron P is q-dimensional. Therefore, the composition $gf\colon X \to S^q$ is essential.

2° (ii) $\Rightarrow$ (iii). Let $f\colon X \to S^q$ be an essential mapping. There exist a compact neighbourhood V of X in E and an extension $\varphi\colon V \to S^q$ of f, since S^q is an ANR-space. Let W be that compact neighbourhood of X contained in V the existence of which is stated by the Convergence theorem (4.3). Let ε be a positive number such that any linear segment in E of length less than ε and with an endpoint in X is contained in W; furthermore, let $g\colon X \to P$ be an ε-displacement of X into a q-dimensional polyhedron P. Let $i\colon X \to V$, $j\colon W \to V$, $k\colon X \to W$, and $k_1\colon P \to W$ denote the respective inclusions. The composition $k_1 g\colon X \to W$ is homotopic to the inclusion k; therefore the restriction $h = \varphi_{|P}\colon P \to S^q$ is essential. Thus, by the well-known Hopf theorem ([3], p. 513), the homomorphism $h_*\colon H_q(P) \to H_q(S^q)$ is not trivial, i.e. $\operatorname{Im} h_* \not\approx 0$. On the other hand, h is equal to the composition fjk_1, which implies that j_* is a non-trivial homomorphism. This implies that $\operatorname{Im} i_* \not\approx 0$, since $\operatorname{Im} i_* = \operatorname{Im} j_*$, by the Convergence theorem. Therefore, the group $H_q(X)$ is not trivial.

3° The implication (iii) $\Rightarrow$ (i) is obvious.

Let X be a compactum, let X_1 and X_2 be closed subsets of X with $X_1 \cup X_2 = X$ and write $X_0 = X_1 \cap X_2$.

(4.8) Proposition. *If $H_q(X_0) \not\approx 0$ and $H_q(X_1) \approx H_q(X_2) \approx 0$, then $H_{q+1}(X) \not\approx 0$.*

Proof. Let a be a q-dimensional true cycle in X_0 with coefficients in $\mathfrak{R}_1$, which is not homologous to zero in X_0. By the assumptions on X_1 and X_2, a is homologous to zero both in X_1 and in X_2, that is, for $i = 1, 2$, there is an infinite $(q+1)$-dimensional chain λ_i in X_i with coefficients in $\mathfrak{R}_1$ such that $\delta\lambda_i = a$. By the well-known so called Phragmen-Brouwer theorem (see [7], p. 546), the infinite cycle $\lambda_1 - \lambda_2$ is not homologous to zero in X and, by (4.2), the group $H_{q+1}(X)$ is not trivial.

(4.9) Proposition. *If $H_q(X_0) \approx H_{q+1}(X_1) \approx H_{q+1}(X_2) \approx 0$, then $H_{q+1}(X) \approx 0$.*

Proof. Let a be a $(q+1)$-dimensional true cycle in X with coefficients in $\mathfrak{R}_1$. By means of an infinitely small displacement of a, a true cycle a' homologous to a can be obtained such that $a' = \beta_1 - \beta_2$, where β_i is a $(q+1)$-dimensional infinite chain in X_i $(i = 1, 2)$ with coefficients in $\mathfrak{R}_1$. The q-dimensional infinite cycle $\partial\beta_1 = \partial\beta_2$ lies in X_0 and, by (4.2) it is homologous to zero in X_0. Thus, there is a $(q+1)$-dimensional infinite chain γ in X_0, with coefficients in $\mathfrak{R}_1$, such that $\partial\gamma = \partial\beta_1$. The infinite cycle $\gamma_i = \beta_i - \gamma$ lies in X_i $(i = 1, 2)$ and therefore it is homologous to zero in X_i. Thus, $a' = \gamma_1 - \gamma_2$ is homologous to zero in X, which completes the proof.

(4.10) Theorem. *If X is a q-acyclic compactum of dimension at most q $(q \geqslant 1)$ and if Y is a compactum of dimension at most p, then the product $X \times Y$ is $(p+q)$-acyclic.*

Proof. Suppose first that Y is a polyhedron with a fixed triangulation T.

I. For $p = 0$ the theorem is obvious (the assumption $q \geqslant 1$ is needed here).

II. Let an integer $k \geqslant 1$ be given and suppose that the theorem is true for any $p < k$. Now, put $p = k$ and denote by n the number of all p-dimensional simplexes of T.

1° If $n=0$, then $\dim Y<p$, $\dim(X\times Y)<p+q$ and therefore $X\times Y$ is $(p+q)$-acyclic.

2° Suppose that $n>0$ and the theorem is true for any polyhedron of dimension at most p which contains less than n p-dimensional simplexes. Choose a p-dimensional simplex Δ of the triangulation T and denote $Y_1=|\Delta|$, $Y_2=\overline{Y\setminus|\Delta|}$. The polyhedron Y_2 contains $n-1$ simplexes of dimension p; hence, by assumption 2°, the compactum $X\times Y_2$ is $(p+q)$-acyclic. The compactum $X\times Y_1$ is $(p+q)$-acyclic, since Y_1 is contractible. Moreover, $\dim(Y_1\cap Y_2)\leqslant p-1$; hence, by assumption II, the compactum $X\times(Y_1\cap Y_2)=X\times Y_1\cap X\times Y_2$ is $(p+q-1)$-acyclic. Thus, by (4.9), $H_{p+q}(X\times Y)\approx 0$, which means, by (4.7), that $X\times Y$ is $(p+q)$-acyclic.

Thus, by induction, the theorem is true for any polyhedron Y. Suppose now that X and Y are arbitrary compacta contained in finite-dimensional Euclidean spaces E' and E'', respectively. Then $X\times Y$ is contained in $E=E'\times E''$. Let ε be a positive number and let f be an ε-displacement of Y into a p-dimensional polyhedron $P\subset E'$. The mapping $g\colon X\times Y\to E$ defined by the formula $g(x,y)=\big(x,f(y)\big)$ is an ε-displacement of $X\times Y$ into the set $X\times P$. By the first part of the proof, the compactum $X\times P$ is $(p+q)$-acyclic. The image $g(X\times Y)$ is therefore also $(p+q)$-acyclic, since $\dim(X\times P)\leqslant p+q$. Thus, by (4.6), the compactum $X\times Y$ is $(p+q)$-acyclic, since the number ε can be arbitrarily small.

(4.11) THEOREM. *Suppose that a compactum X is q-dimensional and* WSC_q *and let $X_1\supset X_2\supset X_3\supset\ldots$ be a decreasing sequence of closed subsets of X. Then, the intersection $X_0=\bigcap_{n=1}^{\infty}X_n$ is q-cyclic whenever all X_n are q-cyclic.*

Proof. Suppose that X is a subset of a finite-dimensional Euclidean space E and let V be a compact neighbourhood of X in E such that the inclusion $i\colon X\to V$ induces a monomorphism $i_*\colon H_q(X)\to H_q(V)$. By the Convergence theorem (4.3), there exists a compact neighbourhood W of X_0 contained in V and such that $\mathrm{Im}\,j_*=\mathrm{Im}\,i_{0*}$, where $j_*\colon H_q(W)\to H_q(V)$ and $i_{0*}\colon H_q(X_0)\to H_q(V)$ are homomorphisms, induced by the respective inclusions $j\colon W\to V$ and $i_0\colon X_0\to V$. Choose a positive integer k such that $X_k\subset W$. Then, the following diagram, consisting of inclusions

$$\begin{array}{ccc} X_k & \xrightarrow{\;j_k\;} & X \\ {\scriptstyle i_k}\downarrow & & \downarrow{\scriptstyle i} \\ W & \xrightarrow[\;j\;]{} & V \end{array}$$

is commutative, which implies that the diagram consisting of the homomorphisms induced by these inclusions

$$\begin{array}{ccc} H_q(X_k) & \xrightarrow{\;j_{k*}\;} & H_q(X) \\ {\scriptstyle i_{k*}}\downarrow & & \downarrow{\scriptstyle i_*} \\ H_q(W) & \xrightarrow[\;j_*\;]{} & H_q(V) \end{array}$$

is also commutative. The homomorphism j_{k*} is a monomorphism, since $\dim X=q$. Consequently,

$$H_q(X_k)\approx \mathrm{Im}\,i_*j_{k*}=\mathrm{Im}\,j_*i_{k*}\subset\mathrm{Im}\,j_*=\mathrm{Im}\,i_{0*},$$

which implies, by the non-triviality of the group $H_q(X_k)$, that the homomorphism i_{0*} is not trivial. In particular, the group $H_q(X_0)$ is not trivial.

Remark. In addition to the above proof, consider the commutative diagram

$$\begin{array}{ccc} H_q(X_k) & \xrightarrow{\;(j_k i)_*\;} & H_q(V) \\ & {\scriptstyle j_{0*}}\nwarrow \quad \nearrow{\scriptstyle i_{0*}} & \\ & H_q(X_0) & \end{array}$$

consisting of homomorphisms induced by the respective inclusions $j_k i\colon X_k\to V$, $i_0\colon X_0\to V$ and $j_0\colon X_0\to X_k$. All homomorphisms of this diagram are monomorphisms and, moreover, $\mathrm{Im}\,i_{0*}=\mathrm{Im}(j_k i)_*$, which implies that j_{0*} is an isomorphism.

5. Irreducibly q-cyclic q-dimensional compacta (q-bubbles). By (4.7) a q-dimensional compactum X is a q-bubble if and only if there exists an essential mapping of X into the q-dimensional sphere S^q and for any closed proper subset X' of X there is no essential mappings of X' into S^q or (equivalently: if and only if the group $H_q(X)$ is non-trivial and, for any closed proper subset X' of X, the group $H_q(X')$ is trivial).

In this section some properties of bubbles will be studied and, in particular, the problem of the existence of q-bubbles in a q-dimensional compactum without small q-dimensional cycles will be solved.

Let SX denote the suspension (see [10], p. 336) of a compactum X with the suspension-vertices v and w and let $\sigma\colon X\times[0,1]\to SX$ be a mapping such that $\sigma(X\times\{0\})=\{v\}$, $\sigma(X\times\{1\})=\{w\}$ and the restriction $\sigma|_{X\times(0,1)}$ is a homeomorphism of $X\times(0,1)$ onto $SX\setminus\{v,w\}$. Denote by X_t the set $\sigma(X\times\{t\})$ for any $t\in(0,1)$.

(5.1) PROPOSITION. *The suspension SX of a compactum X is a $(q+1)$-bubble if and only if X is a q-bubble.*

Proof. As is known, $\dim SX = \dim X + 1$.

1° Suppose that X is a q-bubble. The decomposition of SX into two cones $\sigma(X\times[0,\frac{1}{2}])$ and $\sigma(X\times[\frac{1}{2},1])$ shows, by (4.8), that SX is $(q+1)$-cyclic. Suppose now that Y is a closed proper subset of SX. Then there exists a $t\in(0,1)$ such that $X_t\cap Y$ is a proper subset of X_t. The set $X_t\cap Y$ is q-acyclic, since X_t is a q-bubble; in particular $H_q(X_t\cap Y)\approx 0$. The set $\sigma(X\times[0,t])\cap Y$ is $(q+1)$-acyclic as a closed subset of the $(q+1)$-dimensional, $(q+1)$-acyclic compact set $\sigma(X\times[0,t])$. Analogously, $\sigma(X\times[t,1])\cap Y$ is $(q+1)$-acyclic. The decomposition

$$Y = [\sigma(X\times[0,t])\cap Y]\cup[\sigma(X\times[t,1])\cap Y]$$

shows by (4.9) that $H_{q+1}(Y)\approx 0$ and, by (4.7), Y is $(q+1)$-acyclic. Thus, SX is a $(q+1)$-bubble.

2° Assume now that SX is a $(q+1)$-bubble. The decomposition of SX into the cones $\sigma(X\times[0,\frac{1}{2}])$ and $\sigma(X\times[\frac{1}{2},1]$ shows by (4.9) that $H_q(X)\not\approx 0$. On the other hand, if a closed proper subset X' of X would be q-cyclic, then the suspension SX' would be a $(q+1)$-cyclic closed proper subset of SX, which is impossible.

(5.2) Theorem. *The product $X\times Y$ of a q-bubble X $(q\geqslant 1)$ and a p-bubble Y $(p\geqslant 1)$ is either $(p+q)$-acyclic, or a $(p+q)$-bubble.*

Proof. Let Z denote the product $X\times Y$ and suppose that Z is $(p+q)$-cyclic. Let Z' be a closed proper subset of Z. In order to prove that Z' is $(p+q)$-acyclic, it is sufficient to show that $H_{p+q}(Z')\approx 0$, since $\dim Z'\leqslant p+q$. Let $(x,y)\in Z$ be a point of the complement of Z'. There is an open neighbourhood U of x in X with $\dim \mathrm{Bd}\, U\leqslant q-1$ and an open neighbourhood V of y in Y such that the sets $U\times V$ and Z' are disjoint. In order to show that $H_{p+q}(Z')\approx 0$, it is sufficient to show that $H_{p+q}(Z\setminus U\times V)\approx 0$. Write $Z_1=(X\setminus U)\times Y$ and $Z_2=\overline{U}\times(Y\setminus V)$. Then $Z\setminus U\times V = Z_1\cup Z_2$ and $Z_1\cap Z_2 = \mathrm{Bd}\, U\times(Y\setminus V)$. Observe that $X\setminus U$ is q-acyclic, since X is a q-bubble. Therefore, by (4.10), Z_1 is $(p+q)$-acyclic. By the same argumentation, Z_2 is $(p+q)$-acyclic and $Z_1\cap Z_2$ is $(p+q-1)$-acyclic. Thus, by (4.9), $Z_1\cup Z_2$ is $(p+q)$-acyclic, which completes the proof.

(5.3) Proposition. *Let X be a closed q-cyclic subset of a compactum M and suppose that for any positive number ε there exist a q-bubble $X_\varepsilon\subset M$ and an ε-displacement f_ε of X into X_ε. Then X is a q-bubble.*

Proof. It is easy to see that $\dim X = q$. Indeed: $\dim X\geqslant q$, since X is q-cyclic. On the other hand, for any $\varepsilon>0$ there exists an ε-displacement of X into a q-dimensional set; thus $\dim X\leqslant q$.

In order to prove that X is a q-bubble, suppose, on the contrary, that a closed proper subset X' of X is q-cyclic. Let $x_0\in X$ be a point of the complement of X' and put $\varepsilon_1 = \mathrm{Inf}\{\varrho(x_0,x')\colon x'\in X'\}$. Let $a\in H_q(X')$

be a non-zero element. By (4.6), there exists an $\varepsilon_2>0$ such that, for any ε_2-displacement $f\colon X'\to f(X')\subset M$, the element $f_*(a)\in H_q(f(X'))$ is non-zero. Put $\varepsilon=\min\{\frac{1}{2}\varepsilon_1,\varepsilon_2\}$ and consider the ε-displacement f_ε of X into X_ε. The formula $f(x')=f_\varepsilon(x')$ for any $x'\in X'$ defines an ε-displacement $f\colon X'\to f(X')\subset X_\varepsilon$. Observe that $f_\varepsilon(x_0)\notin f(X')$, since $\varrho(x_0,x')>2\varepsilon$ for any $x'\in X'$. Thus, the set $f(X')$ is q-acyclic as a closed proper subset of the q-bubble X_ε. But on the other hand, $f_*(a)$ is a non-zero element of the group $H_q(f(X'))$. This contradiction completes the proof.

(5.4) Corollary. *The inverse limit of an inverse sequence of q-bubbles is either q-acyclic or a q-bubble.*

(5.5) Theorem. *Any q-dimensional, q-cyclic compactum X which is WSC_q contains a q-bubble. Moreover, the number of all q-bubbles contained in X is at most countable.*

Proof. Consider the family $\mathcal{F}$ of all q-cyclic closed subsets of X with the inclusion-relation $\subset$ as a partial ordering on $\mathcal{F}$. Any q-bubble contained in X is a minimal element of $\mathcal{F}$. In order to prove that there exists a minimal element in $\mathcal{F}$, it is sufficient to show by the well-known so called Brouwer reduction theorem (see [11], p. 161) that, for any decreasing sequence $F_1\supset F_2\supset\ldots$ of elements of $\mathcal{F}$, the intersection $\bigcap_{n=1}^{\infty} F_n$ belongs to $\mathcal{F}$. But just this is stated by (4.11).

Concerning the second part of the theorem, observe that any q-bubble contained in X is a minimal carrier (see sec. 2) of an element of the group $H_q(X)$. This group is, by (3.11), at most countable; therefore the family of all q-bubbles contained in X is at most countable.

(5.6) Remark. Any q-bubble contained in a q-dimensional polyhedron with a fixed triangulation is a subpolyhedron under the same triangulation. Thus, the number of all q-bubbles contained in a q-dimensional polyhedron is finite.

The conclusion of Remark (5.6) cannot be extended onto the class of all q-dimensional compacta WSC_q, not even onto all q-dimensional ANR-spaces. The aim of the next section is to describe a suitable counter-example.

6. An example of a 2-dimensional ANR-space which contains infinitely many 2-bubbles.. Let $\{p_n\}$ be the increasing sequence of all primes. Consider the following subsets of the complex plane C:

$$D=\{z\in C\colon |z|\leqslant 2\},\qquad D'=\{z\in C\colon |z|\leqslant 1\}$$

and

$$A_n=\{z\in C\colon z=0,\ \text{or}\ 0\leqslant \arg z^{p_n}\leqslant\pi\quad\text{and}\ |z|\leqslant 1\}$$

(see Fig. 1 for $n=3$ i.e. $p_n=5$). Let $\varphi_n\colon A_n\to C$ be a mapping defined by the formula $\varphi_n(z)=z^{p_n}$ for any $z\in A_n$. Observe that both A_n and the image $\varphi_n(A_n)$ are AR-sets $(n=1,2,\ldots)$.

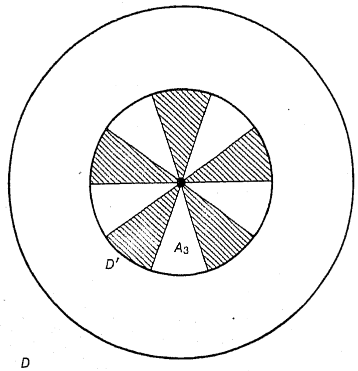

Fig. 1

Let S^2 denote the unit 2-dimensional sphere in the Euclidean 3-space, $S^2=\{(x_1,x_2,x_3)\colon\ x_1^2+x_2^2+x_3^2=1\}$. Consider the sequence of points $c_n=(1/2^n,\sqrt{1-1/4^n},0)$ lying on S^2 and the sequence of positive numbers

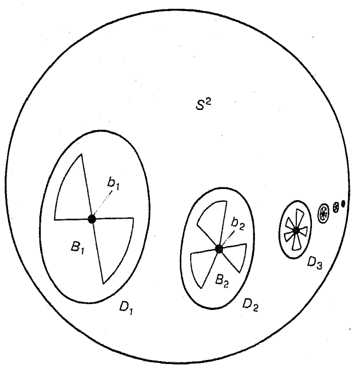

Fig. 2

$r_n=1/2^{n+2}$, $n=1,2,\ldots$ The set $D_n=\{x\in S^2\colon\ \varrho(x,c_n)\leqslant r_n\}$ is a disk lying on S^2. Let h_n be a homeomorphism of D onto D_n and write $B_n=h_n(A_n)$, $D_n'=h_n(D')$ and $b_n=h_n(0)$ (see Fig. 2). Now, define an equivalence relation $\equiv$ on S^2 as follows: for $x\neq y$ the relation $x\equiv y$ holds if and only if there exists an integer n such that x and y are in B_n and $\varphi_n(h_n^{-1}(x))=\varphi_n(h_n^{-1}(y))$. The decomposition of S^2 into equivalence classes is upper semi-continuous. Let K denotes the factor-space $S^2_{/\equiv}$ and let $\eta\colon S^2\to K$ be the natural projection. Observe that $B_1,B_2,\ldots$ are mutually disjoint AR-sets, the sets $\eta(S^2\setminus\bigcup_{n=1}^{\infty}B_n)$, $\eta(B_1)$, $\eta(B_2)$, ... are mutually disjoint and their diameters converge to zero. Moreover, the restriction $\eta_{|(S^2\setminus\bigcup_{n=1}^{\infty}B_n)}$ is a homeomorphism, the image $\eta(B_n)$, for any $n=1,2,\ldots$, is an AR-set, and $K=\eta(S^2)$ is a finite-dimensional compactum (indeed, $\dim K=2$). Thus, the following theorem can be applied here:

(6.1) THEOREM (Lelek [14]). *Let X be an ANR-space and $\{X_i\}$ a sequence of disjoint AR-sets lying in X. Let f be a mapping of X onto a finite-dimensional compactum Y such that the sets $f(X\setminus\bigcup_{i=1}^{\infty}X_i)$, $f(X_1)$, $f(X_2)$, ... are disjoint and their diameters converge to zero. If the restriction $f_{|(X\setminus\bigcup_{i=1}^{\infty}X_i)}$ is a homeomorphism and if $f(X_i)\in$ AR for $i=1,2,\ldots$, then $Y\in$ ANR.*

This theorem implies that the compactum K constructed above is an ANR-space. Write $C_n=\eta(B_n)$ and $K_n=K\setminus\operatorname{Int}C_n$ $(n=1,2,\ldots)$. It will be proved that each K_n is a 2-bubble.

The restriction $\eta_{|(S^2\setminus\bigcup_{n=1}^{\infty}\operatorname{Int}D_n)}$ is a homeomorphism; hence, let any point $x\in(S^2\setminus\bigcup_{n=1}^{\infty}\operatorname{Int}D_n)$ be identified with the point $\eta(x)\in K$. By means of this identification a compactum $M=S^2\cup K$ is obtained such that the equality

$$S^2\setminus\bigcup_{n=1}^{\infty}\operatorname{Int}D_n=K\setminus\bigcup_{n=1}^{\infty}\eta(\operatorname{Int}D_n)=S^2\cap K$$

holds. Observe that there exists a retraction $r\colon M\to S^2$ with the following properties:

(i) $r\big(\eta(D_n')\big)=\{b_n\}$,
(ii) $r\big(\eta(D_n)\big)=D_n$.

The number $\varepsilon_n=\operatorname{Sup}\{\varrho\big(x,r(x)\big)\colon x\in\bigcup_{k=n}^{\infty}\eta(D_k)\}$ is positive $(n=1,2,\ldots)$ and the sequence $\{\varepsilon_n\}$ converges to zero. The subset $P_n=(S^2\setminus D_n)\cup(\eta(D_n)\setminus\operatorname{Int}C_n)$ of M is a polyhedron $(n=1,2,\ldots)$. Denote by $\mathfrak{N}_{p_n}$ the group of integers reduced modulo p_n. It is easy to verify that the group $H_2(P_n,\mathfrak{N}_{p_n})$ is not trivial; indeed, it is isomorphic to the coefficient group $\mathfrak{N}_{p_n}$. Consider the mappings $f\colon P_n \to K_n$ and $g\colon K_n \to P_n$ defined by the formulas

$$f(x) = \begin{cases} \eta(x) & \text{for any } x \in S^2 \setminus D_n, \\ x & \text{for any } x \in \eta(D_n) \setminus \operatorname{Int} C_n \end{cases}$$

and

$$g(x) = \begin{cases} r(x) & \text{for any } x \in K_n \setminus \eta(D_n), \\ x & \text{for any } x \in \eta(D_n) \setminus \operatorname{Int} C_n, \end{cases}$$

and observe that the composition $gf\colon P_n \to P_n$ is homotopic to the identity-mapping on P_n. Thus, the homomorphism $(gf)_*$ induced by gf is the identity-homomorphism on the non-trivial group $H_2(P_n, \mathfrak{N}_{p_n})$. Therefore, the group $H_2(K_n, \mathfrak{N}_{p_n})$ is non-trivial, which yields that

(6.2) the compactum K_n is 2-cyclic $(n = 1, 2, \ldots)$.

The subset

$$P_n^k = (P_n \setminus \bigcup_{i=1}^{k} D_i) \cup \bigcup_{\substack{i=1 \\ i \neq n}}^{k} \eta(D_i)$$

of M is a polyhedron $(k, n = 1, 2, \ldots)$ $(k \geqslant n)$. It is simple to verify that any P_n^k is a 2-bubble (it is important here that the integers p_n and p_m are mutually prime whenever $n \neq m$). Fix an integer $n \geqslant 1$ and define mappings $f_k\colon K_n \to P_n^k$ $(k \geqslant n)$ by the following formula:

$$f_k(x) = \begin{cases} r(x) & \text{for any } x \in \bigcup_{i=k+1}^{\infty} \eta(D_i), \\ x & \text{for any } x \in K_n \setminus \bigcup_{i=k+1}^{\infty} \eta(D_i). \end{cases}$$

The mapping f_k is an ε_k-displacement; hence, it is proved that

(6.3) for any positive number ε there exists an ε-displacement of K_n (the integer n is fixed here) into a 2-bubble lying in M.

It follows by (6.2), (6.3) and (5.3) that K_n is a 2-bubble $(n = 1, 2, \ldots)$, which implies that

(6.4) the compactum K is a 2-dimensional ANR-space which contains an infinite number of 2-bubbles.

(6.5) Problem. Is it true that if a q-dimensional compactum X is WSC_q, then any family of mutually disjoint q-bubbles contained in X is finite?

(6.6) Problem. Does any $(q+1)$-dimensional compactum contain a q-bubble?

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