On cluster sets of arbitrary functions

by

L. Zajíček (Praha)

Abstract. In the article two general theorems concerning cluster sets of arbitrary functions are proved. The first theorem (Theorem 1) generalizes a theorem of F. Hunter on the sets of asymmetry of arbitrary functions and the second (Theorem 8) is an analogy of Theorem 1. It refers to the set of all points which two cluster sets defined in different ways do not intersect. By these general theorems some theorems on cluster sets and approximate cluster sets of arbitrary functions defined on Euclidean spaces are proved. For example, a characterization of the sets of all the points of asymmetry of arbitrary functions defined on Euclidean spaces is given.

1. Introduction and notation. The symbol $E_2$ denotes the $n$-dimensional Euclidean space with the usual scalar product $\langle \ldots \rangle$ and the norm $|\ldots|$. The origin in $E_2$ is denoted by $0$. We put $E_2^+ = E_2 \cup \{+\infty, -\infty\}$. The vector determined by the points $x \in E_2, y \in E_2$ is denoted by $xy$. The angle determined by the vectors $u, v$ is denoted by $\angle u, v$. For $o \neq t \in E_2$ and $0 < a < \pi$ the open cone $U(t, a)$ is the set of all points $o \neq z \in E_2$ for which $\angle oz, ot < a$. The whole space $E_2$ is also termed a cone. The open sphere of the centre $x \in E$ and the radius $r$ is denoted by $K(x, r)$. The symbol $\mu$ denotes the outer Lebesgue measure in $E_2$. If $M \subset E_2$ and $x \in E_2$, then we denote by $M_x$ the image of $M$ under the translation taking the origin into $x$. If $Y \subset E_2, M \subset E_2, x \in E_2$ and $\mu(K(x, h) \cap Y) > 0$ for arbitrary $h > 0$, then we put

$$
\overline{D}(Y, x, M) = \limsup_{h \to 0} \mu(K(x, h) \cap Y) / \mu(K(x, h) \cap M),
$$

$$
\underline{D}(Y, x, M) = \liminf_{h \to 0} \mu(K(x, h) \cap Y) / \mu(K(x, h) \cap M).
$$

We write $\overline{D}(Y, x) = \overline{D}(Y, x, E_2)$ and in the case of $n = 1$ we put $D^+(Y, x) = D(Y, x, [x, \infty))$ and $D^-(Y, x) = D(Y, x, (-\infty, x])$. If $U$ is the cone in $E_2$, then we put $D^U(Y, x) = \overline{D}(Y, x, U)$. Similarly we define the symbols $D(Y, x), D_+(Y, x), D_-(Y, x), D_{U_2}(Y, x)$.

Let $M \subset E_2, x \in E_2$. Let $f: Y \to T$ be a mapping. Then we define the partial cluster set $C(f, y, M)$ as the set of all points $y \in T$ such that $x \in f^{-1}(y) \cap M$ for any neighbour-
On cluster sets of arbitrary functions

Theorem 1) concerning the sets of points of asymmetry of arbitrary functions is proved. This theorem generalizes the basic theorem of ([7]). We frequently use it to deduce theorems concerning mappings from theorems concerning sets.

In the third part we characterize the sets of points of asymmetry of arbitrary mappings \( E_{n} \to T \) where \( T \) is an infinite locally compact separable metric space.

In the fourth part we prove several theorems concerning approximate cluster sets of arbitrary functions. We prove two theorems (Theorem 3, Theorem 6), which improve Hunter's theorem on the sets of points of approximate asymmetry. In this part we use some well-known theorems concerning the boundary behaviour of arbitrary functions in \( E_{n} \).

It is possible to say that in parts 2-4 the set of all points at which two cluster sets defined in different ways are not equal is investigated.

In parts 5-8 the set of all points at which two cluster sets do not intersect is investigated.

In part five a general theorem concerning sets of this type (Theorem 8) is proved.

In part six several theorems are proved on the basis of that general theorem. We prove that the set of all points \( x \) for which \( W^{+}(f, x) \cap W^{-}(f, x) = \emptyset \) is countable for an arbitrary function \( f: E_{n} \to E_{m} \). This theorem generalizes a theorem of Kempisty (9). Further, we prove an analogy of this theorem for functions defined in \( E_{n}, n > 1 \). The last theorem of this part (Theorem 11) generalizes Bagemihl's theorem concerning crookedly ambiguous points ([1], p. 213).

In part seven several theorems which describe the boundary behaviour of arbitrary functions in \( E_{n} \) in terms of the angle approximate cluster sets are proved. Theorem 12 generalizes both the theorem of Bruckner and Goffman ([4], p. 517), and the theorem of Goffman and Sledj ([6], Theorem 4).

2. A general theorem concerning sets of points of asymmetry. This part is based on Hunter's paper [7]. Theorem 1 strengthens and generalizes the basic theorem of [7]. The main difference between Theorem 1 and Hunter's theorem is that by Hunter's theorem it can only be proved that a certain set is small and Theorem 1 enables us to prove in addition that this set is a Borel set.

An arbitrary mapping \( w: S \to S \), for which \( M \subseteq w(M) \) and \( \bigcup_{k=1}^{n} M_{k} = w \left( \bigcup_{k=1}^{n} M_{k} \right) \), where the sets \( M, M_{1}, \ldots, M_{n} \) are arbitrary subsets of \( S \), is called a closure operation on \( S \). We put \( M^{w} = \{ x \in w(M - \{ x \}) \} \).

Clearly the relation \( \bigcup_{k=1}^{n} M_{k}^{w} = \left( \bigcup_{k=1}^{n} M_{k} \right)^{w} \) holds.
On cluster sets of arbitrary functions

3. A characterization of the sets of points of asymmetry of arbitrary real functions defined on \(E_n\). The main purpose of this part is to characterize the sets of points of asymmetry of arbitrary functions defined on \(E_n\).

We prove a more general theorem from which it follows that the same characterization holds for the sets of points of asymmetry of more general mappings, e.g. of real functions which we consider as mappings \(E_n \rightarrow E_n^*\) (i.e. if we permit the limit values \(+ \infty, - \infty\). Suppose that \(n > 1\) is a fixed integer.

If \(G \subset E_n\) and there exists a system of Cartesian coordinates and a Lipschitz function \(f: E_{n-1} \rightarrow E_n\) such that \(G\) is the set of all points whose coordinates fulfill the equation \(x_n = f(x'_1, \ldots, x'_{n-1})\), then the set \(G\) is called a Lipschitz surface. If \(M \subset E_n\) and there exists a sequence \(\{G_n\}_n\) of Lipschitz surfaces in \(E_n\) such that \(M \subset \bigcup \{G_n\}\), then the set \(M\) is called a sparse set. It is obvious that every sparse set is a set of the first category and of measure zero. Every subset of a sparse set is a sparse set and the union of a sequence of sparse sets is a sparse set.

The essential part of the proof of the following proposition is contained in [11], p. 265.

**Proposition 1.** Suppose we are given \(M \subset E_n\) and a cone \(U = U(t, a)\) \((|t| = 1)\) in \(E_n\). Denote by \(A\) the set of all \(x \in M\) such that \(x \in M \setminus U\). Then if
(i) \( A \) is a sparse set.

(ii) If \( a > \frac{1}{2} \), then the set \( A \) is countable.

Proof. For every positive integer \( k \), we denote by \( P_k \) the set of all points \( x \in M \) such that \( M \cap U_x \cap K(x, 1/k) = \emptyset \). We express each \( P_k \) as the union of a sequence \( \{ P_{k,m} \}_{m=1}^{\infty} \) of sets with diameters less than \( 1/k \).

Clearly \( A = \bigcup_{m=1}^{\infty} P_{k,m} \). Choose a new system of Cartesian coordinates such that \((0, \ldots, 0, 0) \) and \((0, \ldots, 0, 1) \) are the new coordinates of the origin and the point \( t \), respectively. For an arbitrary pair of points \( x \in P_{k,m} \), \( y \in P_{k,m} \) with the new coordinates \((x'_1, \ldots, x'_n) \) and \((y'_1, \ldots, y'_n) \), we have \( x \not\in U_y \) and \( y \not\in U_x \). From this it follows that \( (y-x, t) < \cos \alpha (y-x) \) and \( (y-x, t) > \cos \alpha (y-x) \) and therefore

\[
(y-x, t) \in \{ y \mid y \in M \}.
\]

If \( a > \frac{1}{2} \), (2) is absurd, hence no set \( P_{k,m} \) contains two different points. Therefore the set \( A \) is countable and this proves (ii).

If \( a < \frac{1}{2} \), it follows from (2) that

\[
|x'_m - x'_k| \leq \cos \alpha \left( |y'_m - y'_k| + |x'_k - y'_k| \right),
\]

\[
|y'_m - y'_k| \leq \cos \alpha \left( |y'_m - y'_k| + |x'_k - y'_k| \right).
\]

Hence there exists a Lipschitz function \( f \) defined on a subset of \( \mathbb{E}_n \) such that for every two points \( x \neq y \) we have \( f(x) = f(y) \). Since for every Lipschitz function defined on a subset of a metric space there exists an extension on the whole space \( [3] \), \( P_{k,m} \) is a subset of a Lipschitz surface. Hence the set \( A \) is a sparse set.

**Proposition 2.** Suppose we are given \( M \subset \mathbb{E}_n \) and a cone \( U = U(t, a) \) in \( \mathbb{E}_n \). Then

(i) \( \{ M, \epsilon, \epsilon(U) \} \) is a sparse set.

(ii) If \( a > \frac{1}{ \epsilon } \), then the set \( \{ M, \epsilon, \epsilon(U) \} \) is countable.

Proof. By definition, \( \{ M, \epsilon, \epsilon(U) \} = \{ x \mid x \in M \} \), \( x \not\in \mathbb{E}_n \). Hence \( \{ M, \epsilon, \epsilon(U) \} \) is a sparse set.

**Proposition 3.** Suppose we are given \( M \subset \mathbb{E}_n \) and a cone \( U = U(t, a) \) in \( \mathbb{E}_n \). Then \( \{ M, \epsilon, \epsilon(U) \} \) is a \( \mathbb{E}_n \) set.

Proof. Since \( \{ M, \epsilon, \epsilon(U) \} = M' \), \( \{ x \mid x \not\in M \} \), it is sufficient to prove that the set \( P = \{ x \mid x \not\in M \} \) is a \( \mathbb{E}_n \) set. If for every positive integer \( k \) we denote by \( P_k \) the set of all points \( x \) such that \( U_a \cap K(x, 1/k) \) contains \( \emptyset \), then clearly \( \{ P_k \}_{k=1}^{\infty} \) is a sequence such that

\[
\lim_{k \to \infty} x_k = x,
\]

we have clearly

\[
U_x \cap K(x, 1/k) \subset \bigcup_{m=1}^{\infty} U_{x_m} \cap K(x_m, 1/k).
\]

If \( x_m \in P_k \) for every positive integer \( m \), it follows from (3) that \( U_x \cap K(x, 1/k) \) contains \( \emptyset \) and therefore \( x \not\in P_k \). Hence all the sets \( P_k \) are closed and consequently \( P \) is a \( \mathbb{E}_n \) set.

**Lemma 1.** If \( F \subset \mathbb{E}_n \) is a closed set, then there exists a set \( M \subset \mathbb{E}_n \) such that \( F = M \).

**Proof.** It is sufficient to add to each isolated point \( x \in M \) in an obvious way a sequence of points converging to \( x \).

**Theorem 2.** Let \( F \) be a locally compact topological space having a countable basis of open sets. Then

(i) \( \{ f, \epsilon, \epsilon(U) \} \rightarrow \emptyset \) is an arbitrary mapping, then the set \( \{ f \} \) of all points of asymmetry of the mapping \( f \) is a sparse \( \mathbb{E}_n \) set.

(ii) If \( A \) is a sparse \( \mathbb{E}_n \) set and \( \mathbb{E}_n \subset F \) is an infinite set, then there exists a mapping \( f: \mathbb{E}_n \rightarrow F \) such that \( f(\mathbb{E}_n) \cap N = \emptyset \) and \( A = \{ f \} \).

**Proof.** Let \( U \) be the set of all cones \( U(t, a) \) in \( \mathbb{E}_n \) for which the coordinates of the point \( t \) and the number \( a \) are rational. Then evidently \( A = \{ f, \epsilon, \epsilon(U) \} = \{ x \mid \epsilon(f, x) \leq \epsilon(U) \} \neq \emptyset \). By Theorem 1 there exist sequences \( \{ M_k \}_{k=1}^{\infty} \), \( \{ U_k \}_{k=1}^{\infty} \) of sub-

sets of \( \mathbb{E}_n \) such that \( \{ f, \epsilon, \epsilon(U) \} = \bigcup_{k=1}^{\infty} A(M_k, \epsilon, \epsilon(U)) \). From Propositions 2 and 3 it follows that each set \( \{ f, \epsilon, \epsilon(U) \} \) is a sparse \( \mathbb{E}_n \) set and consequently \( A \) is a sparse \( \mathbb{E}_n \) set.

(ii) Since \( A \) is a sparse set, there exists a sequence \( \{ x_k \}_{k=1}^{\infty} \) of Lipschitz surfaces such that \( A \cap \bigcup_{k=1}^{\infty} H_k = \emptyset \). Further, \( A = \bigcup_{q=1}^{\infty} D_q \) where all \( D_q \) are closed sets. Then \( A = \bigcup_{k=1}^{\infty} (B_k \cap D_k) \) and therefore we may write \( A = \bigcup_{m=1}^{\infty} M_m \), where any set \( M_m \) is a closed subset of a Lipschitz surface \( G_m \). Any set \( G_m \) is closed in \( \mathbb{E}_n \) and there exists a homeomorphism \( h_m: \mathbb{E}_n \to G_m \). Any set \( h_m(G_m) \) is closed in \( \mathbb{E}_n \) and by Lemma 1 there exists \( U_n \subset \mathbb{E}_n \) such that \( h_m(G_m) = \mathbb{E}_n \). Writing \( G_m = h_m(B_n) \), we have \( G_m = h_m(D_n) \), we have \( G_m = h_m(D_n) \). Let \( B = \bigcup_{k=1}^{\infty} (B_k \cap D_k) \) and \( Q \subset \bigcup_{k=1}^{\infty} (B_k \cap D_k) \), we infer that the set \( B' \cap D' \) contains at most one point and \( B' \cap D' = \emptyset \). If \( B' \cap D' \neq \emptyset \), we have

\[ a \]
Evidently there exists a sequence \( \{F_k\}_{k=1}^{\infty} \) of disjoint dense subsets of \( E_s \) such that \( E_s - \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} F_k \).

We put
\[
 f(x) = \begin{cases} 
 t_k & \text{if } x \in \bigcap_{i=1}^{k} G_i, \quad \text{where } k = \min \{t, s \in \mathbb{Z} \} \land G_t \cap T_s = \emptyset, \\
 t_k & \text{if } x \in F_k. 
\end{cases}
\]

If \( y \in A \), writing \( f = \min \{t, s \in \mathbb{Z} \} \), we infer that \( y \in C_{t+k} \) and \( y \in C_{s+k} \)
for \( 1 \leq k < j \). Then \( r_j \in C(f, y) \). Since \( G_j \) is a Lipschitz surface, there exists a cone \( U \) such that \( U \cap G_j = \emptyset \) and therefore \( U \cap C_j = \emptyset \). Then \( r_j \in C(f, y) \) for all integers \( k \) and therefore \( R \cap C(f, y) = \emptyset \). Hence \( C(f, y) = C(f, y) = \emptyset \)
for an arbitrary cone \( U \in E_s \) and therefore \( f \notin A \).

Note. If we replace in Theorem 2 \( E_s \) by \( E_s \) and \( \frac{1}{2} \) by \( \frac{1}{2} \), then we obtain a theorem whose proof is quite analogous
to the proof of Theorem 3.

4. Theorems concerning approximate cluster sets of arbitrary functions.

The main purpose of this part is to prove Theorems 3 and 7 which improve
Hunter’s theorem concerning the sets of approximate symmetry of arbitrary functions in \( E_s \). In this part we shall assume that \( d \geq 1 \) is a fixed number.

The half-space \( E_s \times (0, \infty) \) is denoted by \( H \). Any point \((x, 0) \in E_s \times (0, \infty) \)
is identified with the point \( x \in E_s \). If \( U \) is a cone in \( E_s \times (0, \infty) \),
with the vertex at the origin such that \( U \cap H = \emptyset \), then \( U \) is called a cone in \( H \). If \( t \in H, x \in E_s \) and \( f \) is a function in \( H \),
we put \( C(f, x) = C(f, x, F_t) \), where \( F_t \) is the half-line issuing from the point \( x \) in the direction of the vector \( t \).

A point \( x \in E_s \) is termed a \( P \)-set of a set \( M \subseteq E_s \) if there exists
a \( h > 0 \) such that for any \( s > 0 \) there exists spheres \( K(x, h), K(y, r) \) such that \( K(y, r) \subseteq K(x, h) \subseteq M \). A set \( M \subseteq E_s \) is termed a \( P \)-set if it is an arbitrary point \( x \in M \) is a \( P \)-point of the set \( M \). A set \( M \subseteq E_s \) is termed a \( P \)-set if it is the union of a sequence of \( P \)-sets. Every \( P \)-set is a set of the first category and of measure zero. On the contrary,
there exist sets of the first category and of measure zero which are not
\( P \)-sets. This assertion is stated without proof in [5]. We shall use the
following theorems concerning the boundary behaviour of functions in \( E_s \): coordinate axes.

Theorem A [5]. Let \( f \) be an arbitrary function defined on \( H \). Let \( A \)
be the set of all points \((x, z) \in E_s \) for which there exist cones \( U, V \in H \) such that \( C(f, x) \neq C(f, z) \). Then \( A \) is a \( P \)-set.

Theorem B [4]. Let \( f \) be a continuous function defined on \( H \). Let \( t \in H
and \( A \) be the set of all points \((x, z) \in E_s \) for which \( C(f, x) \neq C(f, z) \). Then \( A \) is a set of the first category.

Denote by \( q \) the mapping defined on \( H \) by the relation \( q(x, z) = K(x, z) \). Let a set \( M \subseteq E_s \) be given. Then we denote by \( f_M \) the function defined on \( H \) by the relation \( f_M(x) = \mu(M \cap K(x, z)) [\mu(z, x)] \).

Lemma 2. Let \( U(t, a, b) \) be cones in \( E_s \), \( 0 < a < b < \infty \). Let \( x \in E_s \), \( y \in U(t, a, b) \), \( |x-y| = r \). Then, writing \( d = \frac{1}{2} \) sin \( p \) \((\beta - a) \),
we have \( K(y, d) \subseteq U(t, b) \cup K(x, 2r) \).

Proof. Let \( z \in K(y, d) \). Then evidently \( d \leq |x-y| \leq a \). Hence \( x, y \) \( \leq \frac{1}{2} \) sin \( p \) \((\beta - a) \) and according to the definition of \( d \) we obtain \( x, y \leq \frac{1}{2} \) sin \( p \) \((\beta - a) \) and \( y \leq \frac{1}{2} \) sin \( p \) \((\beta - a) \) and therefore \( x \in U(t, a, b) \). Consequently \( K(y, d) \subseteq U(t, a, b) \) and, since evidently \( K(y, d) \subseteq U(t, b) \cup K(x, 2r) \), \( K(y, d) \subseteq U(t, b) \cap K(x, 2r) \).

Lemma 3. Let \( U(t, a, b) \) be a cone in \( E_s \), \( 0 < a < b < \infty \). Then there exists a cone \( V \) in \( H \) with the following property: If \( x \in E_s \) and \( (x, x+\epsilon) \subseteq V \),
then \( x \in U(t, a, b) \) and \( |x-z| |a < x < a | \leq |z-z| b \).

Proof. It is evidently valid to put \( V = U(t, a, b), \) \( \beta > 0 \) is a sufficiently small number.

Lemma 4. \( U = U(t, a, b) \) be a cone in \( E_s \). Then there exists a cone \( V \) in \( H \) with the following property: If \( x \in E_s \) and \( (x, x+\epsilon) \subseteq V \),
then \( f_M(x) = 0 \) and \( f_M(x) = 0 \). Then there exist a number \( a > 0 \) and a sequence \( (x_k), (x_k) \subseteq \) of points of \( V \) such that \( \lim a_k = x \) and \( f_M(x_k) = \mu([x, x_k]) \mu([x_k, x]) = 0 \) for all integers \( k > 1 \).

If \( x_k = (x_k, x_k) \), then \( f_M(x_k) = (x_k, x_k) \). Then, by the choice of the cone \( V \), we have \((x_k, x_k) \subseteq U(t, a, b) \) and
\( |x_k-x| - \sin \{a \} < a \in \frac{1}{2} \sin \{a \} \).

Put \( R_k = U(t, a, b) - U(t, a, b) \). Then, according to Lemma 2, \( K(x_k, x_k) \subseteq E_s \). On account of (4) we infer that there exist an \( \epsilon > 0 \) such that \( \mu K(x_k, x_k) > \epsilon \) and therefore
\( \mu(M \cap R_2) \mu K(x_k, x_k) \mu K(x_k, x_k) \mu K(x_k, x_k) \mu K(x_k, x_k) > \epsilon \).

For all \( k > 1 \). Hence \( D_M(x_k) > \epsilon \) and this is a contradiction.

The following lemma is obvious.

Lemma 5. \( M \subseteq E_s \), \( x \in E_s \). Then, putting \( b = (0, \ldots, 0, 1) \in H \),
we have \( D_M(x) = \max C(f_M, x) \).

Lemma 6. If \( M \subseteq E_s \), \( x \in E_s \), and \( f_M \in M \cap K(x, h) > 0 \) for any \( h > 0 \),
then \( 1 \leq C(f_M, x) \).
Proof. Since \( \mu(M \cap K(x, h)) > 0 \) for any integer \( h \geq 1 \), there exists a point \( y_k \in K(x, h) \) such that \( D_M(y_k) = 1 \). Hence there exists an \( 0 < a_k < 1/k \) such that \( \mu(M \cap K(y_k, a_k)) \mu(K(y_k, a_k)) > 1 - 1/k \). Then \( \lim (y_k, a_k) = x \) and \( \lim x_k = \mu K(y_k, a_k) = 1 \) and therefore \( \epsilon \in G_1(f, x) \).

**Proposition 4.** Let \( M \subset E_n \). Denote by \( A \) the set of all \( x \in E_n \) for which \( D_M(x) > 0 \) and \( D_V M(x) = 0 \). Then \( A \) is a \( P_s \)-set.

**Proof.** Let \( x \in A \). By Lemma 3 \( G_1(f, x) \neq \emptyset \) and therefore there exists a cone \( W \subset H \) such that \( G_1(f, x) \neq \emptyset \). On the other hand, by Lemma 4 there exists a cone \( V \subset H \) such that \( G_1(f, x) = \emptyset \). Therefore \( G_1(f, x) \neq \emptyset \), \( G_1(f, x) \neq \emptyset \), and consequently, on account of Theorem A, the set \( A \) is a \( P_s \)-set.

**Proposition 5.** Let \( M \subset E_n \). Denote by \( A \) the set of all \( x \in E_n \) such that \( D_M(x) = 1 \) and \( \mu(M \cap K(x, h)) > 0 \) for any \( h > 0 \). Then \( A \) is a set of the first category.

**Proof.** Let \( x \in A \). From Lemma 6 it follows that \( 1 \notin C(f, x, H) \) and from Lemma 5 it follows that \( 1 \notin C(f, x, H) \). Therefore \( C(f, x, H) \) is not a cone \( H \) and, since \( f \) is evidently continuous in \( H \), from Theorem B it follows that \( A \) is a set of the first category.

**Proposition 6.** Let \( M \subset E_n \). Denote by \( A \) the set of all \( x \in E_n \) for which \( 1 \notin D_M(x) \). Then \( A \) is a set of the first category and of measure zero.

**Proof.** From Proposition 5 it follows that \( A \) is a set of the first category. Choose a measurable set \( G \subset M \) such that \( \mu(M \cap G) = \mu(K \cap M) \) for any sphere \( K \). Then \( D(G, x) \neq 1 \) and \( D(E_n - G, x) \neq 1 \) for any point \( x \in A \). Hence, by the density theorem, \( A \) is a set of measure zero.

**Lemma 7.** Let \( M \subset E_n \) and let \( U(t, a) \) be a cone in \( E_n \). Then the function \( f(t, a) = D_M(x) \) is of Baire class 2.

**Proof.** Put \( g(x, r) = \mu(U(t, a) \cap K(x, r)) \cap M \mu(U(t, a) \cap K(x, r)) \text{ for } x \in E_n \) and \( r > 0 \). The function \( g \) is evidently continuous in \( E_n \times (0, \infty) \) and \( f(s) = \lim_{r \to 0} g(x, r) \). Put

\[
\alpha_{x}(a) = \max_{0 < a < 1} g(x, a) \quad \text{and} \quad \beta_{x}(a) = \sup_{0 < a < 1} g(x, a).
\]

Then \( f(a) = \lim_{r \to 0} \beta_{x}(a) \) and \( \beta_{x}(a) = \lim_{r \to 0} \alpha_{x} \).

Since all the functions \( \alpha_{x} \) are continuous, all the functions \( \beta_{x} \) are of Baire class 1. Therefore the function \( f(a) \) is of Baire class 2.

**Theorem 3.** Let \( P \) be a separable locally compact metric space. Let \( f: E_n \to P \) be an arbitrary mapping. Then the set \( A_1(f) \) of all points of approximate asymmetry of \( f \) is a \( P_s \)-set of the class \( P_{\infty} \).

**Proof.** Let \( \epsilon \) be the set of all cones \( U(t, a) \) in \( E_n \) for which the coordinates of the point \( t \) and the number \( a \) are rational. Since \( A(f, d, U) = \{ x: W(f, x) \wedge W_0(f, x) \} \), clearly

\[
A_{\text{eff}}(f) = \bigcup_{d \in \epsilon} A(f, d, U(d)).
\]

By Theorem 4 there exist sequences \( \{ M_{k}, n_{k}, \}_{k=1}^{\infty}, \{ L_{k}, n_{k}, \}_{k=1}^{\infty} \) of subsets of \( E_n \) such that

\[
A(f, d, U) = \bigcup_{k=1}^{\infty} \{ A(M_{k}, d, U(d)) \} \cup \{ A(L_{k}, d, U(d)) \}.
\]

By definition, for any \( M \subset E_n \) we have

\[
A(M, d, U) = \{ x: D_M(x) > 0 \} \cup \{ x: D_V M(x) = 0 \}.
\]

From Proposition 4 and (7) it follows that each set \( A(M, d, U) \) is a \( P_s \)-set. From Lemma 7 and (7) it follows that \( A(M, d, U) \) is the intersection of a \( G_{\delta} \) set and a \( F_{\sigma} \) set. Hence \( A(M, d, U) \) is a \( F_{\sigma} \) set. Consequently the theorem follows from (5) and (8).

From Propositions 5 and 6 which refer to sets we deduce, by a usual method, two theorems which refer to mappings.

In the rest of this part, let \( P \) be a fixed topological space having a countable basis of open sets and let \( f: E_n \to P \) be an arbitrary mapping. If \( x \in E_n \), we denote by \( M(f, x) \) the set of all points \( y \in P \) such that \( \mu(f^{-1}(V) \cap K(x, a)) > 0 \) for any \( a > 0 \) and any neighborhood \( V \) of the point \( y \). Evidently \( M(f, x) \) is the set of all points \( y \in P \) for which there exists a \( B \subset E_n \) such that \( \lim_{a \to 0} f(a) = \mu(B \cap K(x, a)) = 0 \) for any \( h > 0 \). Further, we denote by \( H(f, x) \) the set of all points \( y \in P \) such that \( \mu(f^{-1}(V) \cap K(x, a)) = 1 \) for any neighborhood \( V \) of the point \( y \). The set \( H(f, x) \) coincides with the set of all points \( y \in P \) for which there exists a \( B \subset E_n \) such that \( \mu f(t) = \mu(B) \) for any \( t \in P \).

**Lemma 8.** Let \( Z \subset E_n \) be a measurable set, \( D_Z(x) > 0 \). Then \( \mu f(t) \in W(f, x) \). Z \subset W(f, x) \subset M(f, x) \).

**Proof.** It is clearly sufficient to prove \( \mu f(t) \in W(f, x) \subset M(f, x) \). For an arbitrary set \( M \subset E_n \) we clearly have

\[
D_M(x) = D(M - Z(x)) + D(M \cap Z(x)) < 1 - D(Z(x)) + D(M \cap Z(x)).
\]

Therefore the relation \( D_M(x) = 1 \) implies \( D(M \cap Z(x)) = 0 \). From this and from the definitions our assertion immediately follows.

**Theorem 4.** The set \( B = \{ x: H(f, x) \neq M(f, x) \} \) is a set of the first category.
Proof. Let a sequence \((G_k)_{k=1}^\infty\) form a basis of open sets of \(P\). Denote by \(B_k\) the set of all \(x \in E_n\) such that \(\bar{B}(f^{-1}(G_k))(x) < 1\) and \(\mu f^{-1}(G_k) \cap K(x, h) > 0\) for any \(h > 0\). Clearly \(B \subseteq \bigcup_{k=1}^\infty B_k\). From Proposition 5 it follows that each \(B_k\) is a set of the first category and therefore \(B\) is also a set of the first category.

**Theorem 5.** The set \(C = \{x \in E_n : (f, x) \notin W(f, x)\}\) is a set of the first category and of measure zero.

Proof. Let a sequence \((G_k)_{k=1}^\infty\) form a basis of open sets of \(P\). Denote by \(C_0\) the set of all \(x \in E_n\) such that \(0 < \bar{B}(f^{-1}(G_k))(x) < 1\). Clearly \(C \subseteq \bigcup_{k=1}^\infty G_k\). By account of Proposition 6 we immediately obtain our assertion.

On the basis of Lemma 8 and Theorem 5 we immediately obtain the following theorem.

**Theorem 6.** The set of all points \(x \in E_n\) for which

\[W(f, x) \neq \bigcap \{W(f, x, Z) : Z \text{ is a measurable set, } DZ(x) > 0\}\]

is a set of the first category and of measure zero.

**Proposition 7.** Let \(U = U(t, \beta)\) be a cone in \(E_n, n > 1, M \subseteq E_n\).

Put \(A = \{x \in E_n : DZ(x) > 0\}\). Then

(i) \(A\) is a sparse set.

(ii) If \(\beta > \frac{1}{2}\), then \(A\) is countable.

Proof. Denote by \(A_m\) the set of all points \(x \in E_n\) such that

\[DZ(x) > 0\]

and

\[\mu(K(y, r) \cap M) > 1/m\]

for \(r < \frac{1}{m}\).

Clearly \(A = \bigcup_{m=1}^\infty A_m\). Put \(V = U(t, a)\), where \(a\) is a number such that

\[0 < a \leq \beta \text{ if } \beta \leq \frac{1}{2}\]

and \(\frac{1}{4} < a < \beta \text{ if } \beta > \frac{1}{2}\).

We shall prove that the relation \(x \in A_m\) implies \(x \notin A_m \cap V\). Suppose that this assertion does not hold. Then there exists a sequence \((\xi_k)_{k=1}^\infty\) such that \(x_k = x\) and \(x_k \in V_k \cap A_m\) for each \(k\). Put

\[\xi_k = \frac{1}{2}\xi_k - x_k \sin(\beta - a)\]

On account of Lemma 2 we obtain

\[E_k = K(\xi_k, \delta_k) \subseteq \{U \cap K(x, 2|x - \xi_k|)\} = B_k\]

The number \(p = \mu K(\mu B_k)\) clearly does not depend on \(k\). On account of (9) we infer that, for sufficiently large numbers \(k\), \(\mu(K \cap M) = \mu(K) > 1/m\) and consequently

\[\mu(B_k \cap M) = \mu(B_k) > \mu(K \cap M) = \mu(K) > 1/m\]

Then \(\mu(B_k \cap M) = 0\) and this contradicts (8).

Consequently from Proposition 1 it follows that each set \(A_m\) is a sparse set and in the case of \(\beta > \frac{1}{2}\) each set \(A_m\) is countable. Therefore the relation \(A = \bigcup_{m=1}^\infty A_m\) implies (i) and (ii).

**Theorem 7.** For all points \(x \in E_n\), except those of a sparse set \(A\), \(a\) countable set \(A\), respectively, the following assertion holds:

If there exists an \(a = \limsup f(t)\) for a cone \(U \in E_n\) (a cone \(U = U(t, a)\) in \(E_n\) such that \(a > \frac{1}{2}\), respectively), then \(x \in H(f, x)\).

Proof. Let a sequence \((G_k)_{k=1}^\infty\) form a basis of open sets of \(P\). Let \(U\) be the set of all cones \(U(t, a)\) in \(E_n\) such that the coordinates of the point \(t\) and number \(a\) are rational. Denote by \(A_U\) the set of all \(x \in E_n\) such that there exists an \(a = \limsup f(t)\) and \(x \in H(f, x)\). Denote by \(A_{U, k}\) the set of all \(x \in E_n\) for which \(D^k(E_n - f^{-1}(G_k))(x) = 0\) and \(D(f^{-1}(G_k))(x) \neq 1\). Then, for each cone \(U \in E_n\), \(A_U \subseteq \bigcup_{k=1}^\infty A_{U, k}\). If \(x \in A_{U, k}\), then

\[D(f^{-1}(G_k))(x) > 0\]

Therefore, by Proposition 7, each set \(A_{U, k}\) is a sparse set (in the case of \(U = U(t, a)\) where \(a > \frac{1}{2}\) it is countable). Hence it clearly suffices to put

\[A = \bigcup \{A_U : U \in \mathcal{U}\}, \quad A = \bigcup \{A_U : U \in \mathcal{U}, U = U(t, a), \ a > \frac{1}{2}\},\]

respectively.

5. A general theorem concerning the intersection of cluster sets. Let \(u, v\) be closure operations on a set \(S\). If \(M \subseteq S\), then we put

\[D(M, u, v) = \{S - M^* u \cup (S - (S - M^*) v)\} \cup \{S - M^* u \cup (S - (S - M^*) v)\} \setminus \{S - M^* u \cup (S - (S - M^*) v)\}\]

If \(T\) is a topological space and \(f : S \to T\) is an arbitrary mapping, then we put

\[D(f, u, v) = \{x : C(f, x, u) \cup C(f, x, v) = \emptyset\} \setminus \{x : C(f, x, u) \cup C(f, x, v) = \emptyset\}\]

**Theorem 8.** Suppose we are given closure operations \(u, v\) on a set \(S\) and a compact topological space \(T\) having a countable basis of open sets. Let \(f : S \to T\) be an arbitrary mapping. Then there exist sequences \((M_k)_{k=1}^\infty\), \((L_k)_{k=1}^\infty\), of subsets of \(S\) such that

\[D(f, u, v) = \bigcup_{k=1}^\infty D(M_k, u, v) \cap D(L_k, u, v)\].

Proof. Let $S$ be a countable basis of open sets of $T$. Let $x \in D(f, u, v)$.
Then for each point $y \in T$ there exists a $U_y \in S$ such that $y \in U_y$ and $x \notin f^r(U_y)$ or $x \notin f^l(U_y)$. Since $T$ is compact, there exists a finite set $K \subseteq T$ such that $T = \bigcup_{y \in K} U_y$. Put $G = \bigcup_{y \in K} U_y$ and $H = \bigcup_{y \in K} U_y$, where $K$ is the set of all $y \in K$ such that $x \notin f^r(U_y)$. Then clearly the following relations hold:

$$f^r(G) = \bigcup_{y \in K} f^r(U_y), \quad f^l(H) = \bigcup_{y \in K} f^l(U_y), \quad f^r(G) \cap f^l(H) = \emptyset.$$ 

Therefore

$$x \notin D(f^r(U_y), u, v) \cap D(f^l(U_y), u, v).$$

Let $((G_0, H_0))_{k=1}^n$ be a sequence of all pairs $(G, H)$ such that $G$ and $H$ are finite unions of elements of $S$ and $G \cap H = T$. Put $M_k = f^r(G_k)$, $L_k = f^l(H_k)$. Then

$$D(f, u, v) = \bigcup_{k=1}^n D(M_k, u, v) \cap D(L_k, u, v)$$

On the other hand, suppose that $x \in D(M_k, u, v) \cap D(L_k, u, v)$ for a positive integer $k$. Then for $y \in T$, either $y \notin G_k$ or $y \notin H_k$. In both cases we infer that either $y \notin f^r(f(s, x, v))$ or $y \notin f^l(f(s, x, v))$. Hence $y \notin f(s, x, v) \cap f(s, x, v)$. Therefore we have $x \notin D(f(s, x, v))$. Consequently

$$D(f, u, v) \subseteq \bigcup_{k=1}^n D(M_k, u, v) \cap D(L_k, u, v).$$

and this completes the proof.

6. Applications of Theorem 8.

LEMMA 9. The set $D(M, d^*, d^-)$ is countable for each set $M \subseteq E_k$.

Proof. If $z \in D(M, d^*, d^-)$, then either $D^+(z) = 0$ and $d^-(z) = 0$, or $D^+(z) = 0$ and $d^-(z) = 1$. Set

$$f(y) = \begin{cases} \mu(M \cap (0, y)) & \text{if } y > 0, \\ -\mu(M \cap (y, 0]) & \text{if } y < 0. \end{cases}$$

Then either $1 = f^r(x) > f^l(x) = 0$ or $0 = f^r(x) < f^l(x) = 1$, where $f^r(x)$, $f^l(x)$, $f^r(x)$, $f^l(x)$ are Dini derivatives of $f$ at the point $x$. On account of the well-known theorem on Dini derivatives ([11], p. 261) we infer that $D(M, d^*, d^-)$ is countable.

THEOREM 9. Let $T$ be a compact topological space having a countable basis of open sets. Let $f: E \to T$ be an arbitrary mapping. Then the set $D = \{z: W^+(f, x) \cap W^-(f, x) = \emptyset\}$ is countable.

Proof. By Theorem 8 there exist sequences $(M_k)_{k=1}^\infty$, $(L_k)_{k=1}^\infty$ of subsets of $E_k$ such that

$$D = \cap_{k=1}^\infty D(M_k, d^*, d^-) \cap D(L_k, d^*, d^-).$$

On account of this relation and Lemma 9 we obtain the assertion of the theorem.

LEMMA 10. Let $V$ and $U = U(t, \beta)$ be cones in $E_n$, $n > 1$, and let $M \subseteq E_n$. Then

(i) $D(M, d(U), d(V))$ is a sparse set of type $F_{\sigma0}$.

(ii) If $\beta > \frac{1}{\pi}$, then $D(M, d(U), d(V))$ is countable.

Proof. By the definitions

$$D(M, d(U), d(V)) = \{z: D^+(M, z) = 0, D^-(E_n - M, z) = 0 \} \cup \{z: D^+(M, z) = 0, D^-(E_n - M, z) = 0 \}.$$

On account of Lemma 7 we infer that $D(M, d(U), d(V))$ is of the type $F_{\sigma0}$.

The rest of the assertion of the lemma follows from Proposition 7.

THEOREM 10. Let $T$ be a compact topological space having a countable basis of open sets. Let $n > 1$, and let $f: E_n \to T$ be an arbitrary mapping. Let $D$ be the set of all $x \in E_n$ for which there exist cones $U, V$ in $E_n$ (cones $U = U(t, \alpha)$ and $V = U(t, \beta)$, where $\max(\alpha, \beta) > 1$, respectively) such that $W(f, x) \cap W(f, x) = \emptyset$ and $D$ is an arbitrary mapping. Then

$$D = \bigcup_{(U, V) \in R} \{x: W(f, x) \cap W(f, x) = \emptyset\}$$

where $R = \bigcup_{(U, V) \in F}$, respectively. On account of Lemma 10 and Theorem 8 we infer that each set $D_{(U, V)}$ is a sparse set of the type $F_{\sigma0}$ and is countable (for $U$, $V$ in $F$). Since the sets $\mathcal{F}$ and $\mathcal{F}$ are countable, the assertion of the theorem follows from (11). By an arc at a point $x \in E_k$ we shall mean a simple continuous curve

$$q: z = q(t) \quad (0 < 1)$$

such that $q(t) \neq x$ for $0 < t < 1$ and $\lim_{t \to 1} q(t) = x$. We put $k(p) = q(0)$.

We say that the arcs $q, p$ at a point $x$ are associated provided there exists an angle $U = U(t, \alpha)$ in $E_k$ such that $\alpha < \frac{1}{\pi}$, $p \subseteq U$ and $p \subseteq U$. 

\[ \text{[end of page]} \]
Lemma 11. Let \( M \subseteq E_k \). Denote by \( D \) the set of all points \( x \in E_k \) for which there exist associated arcs \( \varphi, \psi \) at \( x \) such that \( \varphi \subseteq M \) and \( \psi \subseteq E_k - M \). Then the set \( D \) is a sparsely set.

Proof. Denote by \( \mathcal{F} \) the set of all \( F = (K, U, A, B) \) with the following properties:

(i) \( K = K(s, r) \) is a disc in \( E_k \), \( U = U(t, a) \) is an angle in \( E_k \) and \( a < \frac{r}{t} \).

(ii) \( A \cup B = \emptyset \) and there exist angles \( V = U(v, \beta) \), \( W = U(w, \gamma) \) such that \( A = Fr_K \cap V_x \), \( B = Fr_K \cap W_x \).

(iii) The coordinates of the points of \( s, t, v, w \) and the numbers \( r, \beta, \gamma \) are rational.

If \( F = (K, U, A, B) \in \mathcal{F} \), we denote by \( D_F \) the set of all points \( x \in E_k \) for which there exist arcs \( \varphi_x, \psi_x \) at \( x \) such that the following relations hold:
\[ \varphi_x \subseteq U_x \setminus K_x, \quad \psi_x \subseteq U_x \setminus K_x, \quad \varphi_x \cap M_x \subseteq E_k - M_x, \quad \psi_x \cap K_x = \{k(x)\}, \]
\[ k(x) \subseteq U_x, \quad Fr_K \cap \psi_x = \{k(x)\}, \quad k(x) \subseteq B \subset U_x. \]
It is easy to see that \( D = \bigcup D_F \). Let \( F \in \mathcal{F} \) be given. Let us use the same notation as in the definition of the set \( \mathcal{F} \). Then without loss of generality we may suppose, considering the points \( t, s, v, w \) as complex numbers, that \( v = kx^a \) for some \( k > 0 \) and \( 0 < a < \pi \). Let \( Z = U(x, \delta) \) be the complementary angle to \( U \) such that \( x = \Re v \) and \( \delta = \frac{\pi}{2} - a \). Let \( x \in D_F \). Then we shall prove that \( x \in D_F \subseteq Z_j \). Assume, on the contrary, that \( x \in D_F \setminus Z_j \).

Then clearly there exists a \( y \in D_F \setminus Z_j \) such that \( (A \cup B) \cap U_y \). Now it is easy to see that the arcs \( \varphi_x \) and \( \psi_x \) intersect, and this is a contradiction. Therefore by Proposition 1 the set \( D \) is a sparsely set. Since the set \( \mathcal{F} \) is countable, the set \( D \) is a sparsely set as well.

Theorem 11. Let \( T \) be a compact topological space having a countable basis of open sets, and let \( f : E_k \rightarrow T \) be an arbitrary mapping. Denote by \( D \) the set of all points \( x \in E_k \) for which there exist associated arcs \( \varphi_x \) and \( \psi_x \) at \( x \) such that \( G(f(x), \varphi_x) \cap G(f(x), \psi_x) = \emptyset \). Then the set \( D \) is a sparsely set.

Proof. Set
\[ M^x = \varphi_x \quad \text{and} \quad N^x = \psi_x \quad \text{if} \quad x \in D \]
and
\[ M^x = N^x = E_k \quad \text{if} \quad x \notin D. \]
Set \( u = u([M^x]) \) and \( v = u([N^x]) \). The definition of these symbols is in the first part. From Lemma 11 it follows that \( D(M, u, v) \) is a sparsely set for an arbitrary \( M \subseteq E_k \). Since \( D = \{x : G(f(x), u) \cap G(f(x), v) = \emptyset\} \), we obtain by Theorem 8 that \( D \) is a sparsely set.

Note. If we replace the word "associated arcs \( \varphi_x \) and \( \psi_x \) at \( x \)" by "arcs \( \varphi_x \) and \( \psi_x \) at \( x \) having non-collinear semitangents at \( x \)" in Theorem 11, then we obtain Bagemihl's theorem on crookedly ambiguous points [1].

7. The boundary behavior of arbitrary functions in the plane and the approximate cluster sets. The main purpose of this part is to prove Theorem 12. Theorem 12 improves both a theorem from [4] which asserts that the set \( A \) from Theorem 12 is a set of the first category and a theorem from [6] which asserts that this set is a zero measure. Theorem 12 is a further application of Theorem 8.

In this part we shall use the following notation. If \( \alpha \in E_k \), then we denote by \( P(\alpha) \) the half-line in \( E_k \) issuing from the origin and containing the point \( \alpha \). If \( \alpha \in E_k \), \( \beta \in E_k \), \( 0 < a < \pi \), \( 0 < \beta < \tau \), \( \alpha, \beta \), are given, then we denote by \( U(\alpha, \beta) \) the angle determined by half-lines \( P(\alpha) \) and \( P(\beta) \). If the numbers \( \alpha \) and \( \beta \) are rational, we shall say that the angle \( U(\alpha, \beta) \) is rational.

Lemma 12. Let \( U = U(\alpha, \beta) \) and \( V = U(y, \delta) \) be angles such that \( 0 < a < \beta < \gamma < \delta < \pi \). Then there exist positive integers \( k, l, m, n, u \) and a number \( 0 < p < 1 \) such that the following assertion holds: If \( x \in E_k, \ y \in E_k, \) and \( y - x = r > 0 \), then

(i) \( U \subseteq V \subset K(y, l\tau) \) and \( \mu(K(y, l\tau) \cap V_x) \mu(U_x \cap V_y) < m \).

(ii) \( U \subseteq V \subset K(x, r\tau) \) and \( \mu(T_x \cap K(x, r\tau) \cap U_x) \leq n \), where \( T \) is the triangle determined by half-lines \( P_x \), \( P(\beta) \), \( P(\gamma) \).

(iii) If we put \( z = x + pr \), then \( P(z) \cap P(\delta) \cap P(\gamma) \neq \emptyset \).

(iv) Let \( (x_i)_{i=1}^\infty \) be a sequence of real numbers such that \( p < a_i < \frac{1}{2}(y+1) \) for each positive integer \( i \). Define the sequence \( (y_i)_{i=1}^\infty \) by relations \( y_i = x + r \) and \( y_i = x + a_i(y_i-1) \). Then \( \lim_{i \to \infty} y_i = T \subset \bigcup_{i=1}^\infty V_{y_i} \). Further, there exists a positive integer \( N \) such that any point \( z \in T \) is contained in at most \( N \) different angles \( U_{y_i} \).

Proof. The existence of the numbers \( k, l, m, n, u \) is obvious. Condition (iii) determines the number \( p \). We shall prove that the number \( p \) satisfies condition (iv). Clearly \( y_i = x + r \sum_{k=1}^i a_k \) and therefore \( x + rp \) \( \leq y_i \). Consequently \( \lim_{i \to \infty} y_i = x \). From (iii) it follows that \( T \subset \bigcup_{i=1}^\infty V_{y_i} \). Clearly there exists a number \( \tau \) such that \( V_x \cap V_y = \emptyset \), where \( \tau = x + \frac{1}{2}(y+1) \). It is obviously sufficient to choose \( \tau = \tau \).

Lemma 13. Let \( W \subseteq H \) be an angle in \( E_k \) and let \( M \subseteq C \) be an arbitrary set. Put \( A = \{x \in E_k : \text{D}(W) = 0 \}, \mu(M, x, H) > 0 \). Then \( A \subseteq \text{D}_{PH} \) of type \( F \).

Proof. Let \( x \in A \). Then clearly there exists a rational angle \( V = U(y, \delta) \) such that \( 0 < \gamma < \delta < \pi \) and \( \text{D}(W) = 0 \). If we put
$Z = U(s, r-e) \cup U(\delta+e, \pi-e)$, then for any sufficiently small $\varepsilon > 0$ we have

$D(\xi, M, a) \geq D(M, a) - D(M - Z, a) \geq D(M) - 2s/\pi > 0$,

and therefore $D^2M(a) > 0$ for an angle $U$ in $H$. We may clearly suppose that the cone $U$ is rational. Put $A_{U,V} = \{x: D^2M(x) > 0, D^2M(x) = 0\}$ for any pair of angles $U, V$ in $H$. Let $\mathcal{U}$ be the set of all pairs of rational angles in $H$ such that $U \cup V = \{0\}$. Then $A = \bigcup_{(U, V) \in \mathcal{U}} A_{U,V}$. We shall prove that $A_{U,V}$ is a $P_\varepsilon$-set for any pair of angles $(U, V) \in \mathcal{U}$. Assume, on the contrary, that there exists a pair $(U, V) \in \mathcal{U}$ such that $A_{U,V}$ is not a $P_\varepsilon$-set. Then we can suppose without loss of generality that $U = U(\alpha, \beta)$ and $V = U(\gamma, \delta)$, where $0 < \alpha < \beta < \gamma < \delta < \pi$. Let the sense of the letters $k, l, m, n, p, T$ be the same as in Lemma 12 and put $d = 1/\max N$. For any positive integers $a, b$ denote by $B_{a,b}$ the set of all points $x \in E$ such that $D^2M(x) > 1/a$ and

$\mu(V_a \cap M \cap K(x, h)) \mu(V_b \cap M \cap K(x, h)) < d/a$ for any $k < 1/b$.

Then clearly $A_{U,V} \subset \bigcup_{a=1}^{\infty} B_{a,b}$. Since $A_{U,V}$ is not a $P_\varepsilon$-set, there exist positive integers $a, b$ such that $B_{a,b}$ is not a $P_\varepsilon$-set. Hence there exists a point $x \in B_{a,b}$ which is not a $P_\varepsilon$-point of the set $B_{a,b}$. Then there exists a number $s_0 > 0$ such that

$B_{a,b} \cap x + ps \in \{x + (p+1)s/2\} \neq \emptyset$ for any $s < s_0$.

Choose a number $r > 0$ such that $s_0 > r$, $r < 1/b$ and

$\mu(U_a \cap K(x, r) \cap M) \mu(U_b \cap K(x, r) \cap M) < 1/a$.

On account of (13) we infer that there exists a sequence of numbers $(\varepsilon_i)_{i=1}^{\infty}$ such that $p \leq \varepsilon_i \leq (p+1)/2$ for any integer $i > 1$ and $y_i \in B_{a,b}$ for any integer $i > 0$, where $(y_i)_{i=1}^{\infty}$ is the sequence defined in Lemma 12, (iv). Put $C_t = \varepsilon_i - U_a$ for any integer $i > 0$. On account of Lemma 12, (iv) we have

$T = \bigcup_{i=1}^{\infty} C_t$.

(14) $\mu(C_t) \geq \sum_{i=1}^{\infty} \mu(C_t) / N_i$.

By Lemma 12, (ii) we obtain $K(x, r) \subset \bigcup_{i=1}^{\infty} C_t$ and

(15) $\mu(K(x, r) \cap M) \mu(K(x, r) \cap U_x) \leq \sum_{i=1}^{\infty} \mu(C_t \cap M) / N_i$

$\leq N_\varepsilon \sum_{i=1}^{\infty} \mu(C_t \cap M) / \sum_{i=1}^{\infty} \mu(C_t)$.

Since $lr < 1/b$ and $y_i \in B_{a,b}$ for any positive integer $i > 0$, we obtain by Lemma 15, (i) $\mu(C_t \cap M) / \mu(C_t) \leq 1/a$. Hence by (15)

$\mu(U_x \cap K(x, r) \cap M) \mu(U_x \cap K(x, r) \cap U_x) \leq N_\varepsilon \mu(C_t \cap M) / \sum_{i=1}^{\infty} \mu(C_t)$

and this contradicts (13). Therefore $A_{U,V}$ is a $P_\varepsilon$-set for any $(U, V) \in \mathcal{U}$, and therefore $A$ is a $P_\varepsilon$-set. On account of Lemma 7 we immediately infer that $A$ is a $P_{\varepsilon_0}$-set.

Theorem 12. Let $T$ be a locally compact topological space having a countable basis of open sets. Let $f: H \to T$ be an arbitrary mapping. Denote by $A$ the set of all points $x \in E$, for which there exists an angle $U$ in $H$ such that $W(f(x, U), H) = W(f(x), U)$. Then $A$ is a $P_{\varepsilon_0}$-set of type $P_{\varepsilon_0}$.

Proof. Let $g: H \to T$ be an arbitrary extension of the mapping $f$.

Let $\mathcal{U}$ be the set of all rational angles in $H$. Since

$(x \in E_1: W(f(x, U), H) = W(f(x), U)) = E_1 \cap A(x, d(U), d(U))$

for any angle $U$ in $H$, it is evident that

(16) $E_1 \cap \bigcup_{U \in \mathcal{U}} A(x, d(U), d(U))$.

If $U$ is an angle in $H$, then by Theorem 1 there exist sequences $(M_k)_{k=1}^{\infty}$, $(E_k)_{k=1}^{\infty}$ of subsets of $H$ such that

$A(x, d(U), d(U)) = \bigcup_{k=1}^{\infty} (A(M_k, d(U), d(U)) \cap A(E_k, d(U), d(U)))$.

hence, on account of Lemma 13, we infer that $E_1 \cap A(x, d(U), d(U))$ is a $P_{\varepsilon_0}$-set of type $P_{\varepsilon_0}$. Therefore the assertion of the theorem follows from (16).

The following lemma clearly holds.

Lemma 14. Let $U = U(\alpha, \beta)$ and $V = U(\gamma, \delta)$ be angles in $H$ such that $0 < \alpha < \beta < \gamma < \delta < \pi$. Then there exist positive integers $a, b, s$ such that the following assertion holds: If $x \in E_1, y \in E_0$ and $y - x = r > 0$, then

$\mu(U_x \cap K(x, r) \cap K(y, r) \cap M) \leq \mu(U_y \cap K(y, r) \cap M) \leq \mu(U_x \cap K(x, r) \cap M) \leq \mu(U_y \cap K(y, r) \cap M)$.

Lemma 15. Let $U = U(\alpha, \beta)$ and $V = U(\gamma, \delta)$ be angles in $H$ such that $0 < \alpha < \beta < \gamma < \delta < \pi$, and let $M \subset H$. Then the set $E_1 \subset \cap D(M, d(U), d(V))$ is countable.

Proof. Put $K = \{x \in E_1: D^2M(x) = 0, D^2(M - M)(x) = 0\}$

and $L = \{x \in E_1: D^2M(x) = 0, D^2(M - M)(x) = 0\}$.
Since $E_1 \cap D(M, d(U), d(V)) = K \cup L$, it is clearly sufficient to prove that the set $K$ is countable. Let the sense of the letters $a, b, c$ be the same as in Lemma 14, and let $a$ be an integer. Then we denote by $K_a$ the set of all points $x \in E_1$ such that

$$\mu(U_\alpha \cap K(x, h) \cap M) \mu(U_\alpha \cap K(x, h) < 1/2a$$

and

$$\mu(V_\beta \cap K(y, h) \cap M) \mu(V_\beta \cap K(y, h) < 1/2a$$

for any $h < 1/a$. Clearly $K \subseteq \bigcup_{a=1}^{\infty} K_a$. If $x \in K_x, y \in K_y$ and $0 < y - x < 1/2(\max(s, t))$, we infer by Lemma 14 and by the definition of the sets $K_a$ that

$$\mu(U_\alpha \cap V_\beta \cap M) \mu(U_\alpha \cap V_\beta) < 1/2$$

and

$$\mu(U_\alpha \cap V_\beta \cap \{H-M\}) \mu(U_\alpha \cap V_\beta) < 1/2$$

and this is a contradiction. Hence all the sets $K_a$ are isolated and therefore the set $K$ is countable.

**Theorem 13.** Let $T$ be a locally compact topological space having a countable basis of open sets. Let $f : H \rightarrow T$ be an arbitrary mapping. Denote by $D$ the set of all $x \in E_1$ for which there exist angles $U, V$ in $H$ such that $W(f, x) \cap W(f, x) = \emptyset$. Then $D$ is countable.

**Proof.** Let $g : H \rightarrow E_1$ be an arbitrary extension of the mapping $f$. Let $\lambda$ be the set of all pairs $U = U(a, b), V = U(\gamma, \delta)$ of rational angles in $H$ such that $0 < a < \beta < \gamma < \delta < \pi$. Since

$$\{x \in E_1 : W(f, x) \cap W(f, x) = \emptyset\} = D(g, d(U), d(V)) \cup E_1$$

for any pair $(U, V) \in \lambda$, evidently

$$D = E_1 \cap \bigcup_{(U, V) \in \lambda} D(g, d(U), d(V)).$$

If $(U, V) \in \lambda$, then by Theorem 8 there exist sequences $(M_2)_{n=1}^{\infty}, (L_2)_{n=1}^{\infty}$ of subsets of $H$ such that

$$D(g, d(U), d(V)) = \bigcup_{n=1}^{\infty} D(M_2, d(U), d(V)) \cap D(L_2, d(U), d(V)).$$

Hence, on account of Lemma 15, we infer that $E_1 \cap D(g, d(U), d(V))$ is countable. Consequently, from (17) it follows that the set $D$ is countable.

**References**


