Triangle contractive self maps of a Hilbert space

by

D. E. Daykin and **J. K. Dugdale** (Reading)

Abstract. The well known Banach contraction mapping theorem says that if distances go down under a self map of a complete metric space then the self map has a unique fixed point. The purpose of this paper is to generalize to maps which contract three or more points. We show that if triangles go down under a self map of a Hilbert space then the self map has either a fixed point or a fixed line (or both).

1. Introduction. The well known Banach contraction mapping theorem says that if distances go down under a self map of a complete metric space then the self map has a unique fixed point. Roughly speaking the object of this paper is to show that if triangles go down under a self map of a Hilbert space then the self map has either a fixed point or a fixed line (or both). We believe that if tetrahedrons go down then the self map will have a fixed point, line or plane and so on. However, we have not yet pursued this investigation.

2. Triangles and lines. Throughout this paper H denotes a Hilbert space. If $y,\ z \in H$ we define the line $L(y, z)$ passing through y and z by

$$L(y, z) = \{x|\ x = \alpha y + \beta z \text{ for } \alpha, \beta \text{ scalars with } \alpha+\beta = 1\}.$$

When $y \neq z$ the α, β are uniquely determined by the point x of $L(y, z)$, but when $y = z$ then $L(y, z)$ is the single point y. Any two distinct points are sufficient to determine a line, for if $x_1, x_2 \in L(y, z)$ and $x_1 \neq x_2$ then $L(x_1, x_2) = L(y, z)$. Also two distinct lines intersect in at most one point. Thus lines are simply linear varieties, they are closed convex sets, and we get the expected line geometry in H.

For arbitrary points x, y, z of H put $a = x-y$, $b = y-z$ and $c = z-x$, then

$$\tfrac{1}{2}\sqrt{a^2b^2-(a, b)(b, a)} = \tfrac{1}{2}\sqrt{b^2c^2-(b, c)(c, b)} = \tfrac{1}{2}\sqrt{c^2a^2-(c, a)(a, c)}.$$

The distance Π of x from $L(y, z)$ is

$$\Pi\big(x, L(y, z)\big) = \begin{cases} \|a\| & \text{if } \quad y = z, \\ \dfrac{1}{\|b\|}\sqrt{a^2b^2-(a, b)(b, a)} & \text{if } \quad y \neq z, \end{cases}$$

so we define the area $\Delta(x, y, z)$ of the triangle x, y, z to be half the base $\|b\|$ times the height H, and the area is the same whichever side of the triangle is taken as base. Also $x \in L(y, z)$ iff $\Delta(x, y, z) = 0$ when $y \neq z$.

If $s = \frac{1}{2}(\|a\|+\|b\|+\|c\|)$ then

$$A(x, y, z) = \sqrt{s(s-\|a\|)(s-\|b\|)(s-\|c\|)} = \tfrac{1}{2}\sqrt{a^2b^2-\{\tfrac{1}{2}(a, b)+\tfrac{1}{2}(b, a)\}^2}.$$

However $\Delta(x, y, z) \leqslant A(x, y, z)$ with equality iff (a, b) is real, which always holds if H is real. If we were working in a metric space we might have to use A to define the area of a triangle, but we will not do so here.

We finish this section by observing that if H is a complex Hilbert space and H_1 the associated real Hilbert space then for all $x, y, z \in H$

$$\Delta(x, y, z) = \Delta_1(x, y, z)$$

where Δ_1 is Δ calculated in H_1.

3. Properties of fixed points and lines. Let f be a self map of H. A point p of H is a *fixed point of* f if $fp = p$. If $0 < \alpha < 1$ and

$$\|fx-fy\| \leqslant \alpha\|x-y\|$$

for all $x, y \in H$ then f is *contractive*. This is what we meant in the introduction when saying "distances go down".

THEOREM 1 (Contraction mapping principle). *If f is contractive then f has a unique fixed point p and iteration from any point leads to p.*

We will call a line or part of a line L a *fixed line of* f if $fL \subset L$. If f has a fixed point or a fixed line or both we say it has a *fixture*. We say that f is *triangle expansion bounded*, abbreviated to f is TEB, if there is a positive constant α such that for every three points x, y, z of H

$$(1)\qquad \begin{aligned} &\text{either } \Delta(fx, fy, fz) \leqslant \alpha\Delta(x, y, z)\\ &\text{or}\quad \|fx-fy\| \leqslant \alpha\|x-y\|\\ &\text{and}\quad \|fy-fz\| \leqslant \alpha\|y-z\|\\ &\text{and}\quad \|fz-fx\| \leqslant \alpha\|z-x\|. \end{aligned}$$

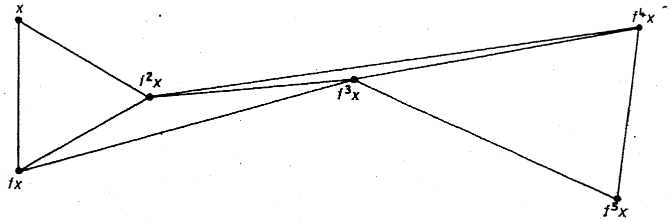
If f is TEB with $0 < \alpha < 1$ we say that f is *triangle contractive* and write f is TC.

Finally f is *triangle perimeter contractive*, abbreviated to f is TPC, if $0 < \alpha < 1$ and for every three points x, y, z of H

$$(2)\qquad \|fx-fy\|+\|fy-fz\|+\|fz-fx\| \leqslant \alpha\{\|x-y\|+\|y-z\|+\|z-x\|\}.$$

Note that asking for f to be TC or TPC are independent requirements and both are weaker than asking for f to be contractive.

It is very easy to prove that if f is a TPC self map (of even a complete metric space) then f has a unique fixed point p, iteration from any point leads to p, and a sequence of iterates is a Cauchy sequence. The difficulty in dealing with TC maps lies in the fact that even under iteration we can't be sure that we won't go from small triangles to big ones as shown in diagram. We could even have made fx, f^2x, f^3x, f^4x collinear in the diagram



Our work really centres on the TC condition and we make the

CONJECTURE. Every TC self map has a fixture. (When H is finite dimensional.)

In table we give some examples of TC self maps f of the two dimensional real (u, v) plane.

Table

$f(u, v)$	No. of fixed points	No. of fixed lines
$(-\frac{1}{2}v, \frac{1}{2}u)$	1	0
$(u+1, \frac{1}{2}v)$	0	1
$(-\frac{1}{2}u, \frac{1}{3}v)$	1	2
$(-\frac{1}{2}u, -\frac{1}{2}v)$	1	∞
$(u, \frac{1}{2}v)$	∞	∞

Clearly (1) is satisfied for three points x, y, z if fx, fy, fz are collinear. Hence one can make trivial examples in H as follows. Choose any line N and any set P of points on N. Next choose any family Q of lines such that firstly, each line of Q meets N at a point of P, and secondly, if two distinct lines of Q intersect at a point p then p is in P. Then map the whole of the space H onto the line N in any way that keeps points of P and lines of Q fixed. These examples show that the following theorem which describes the arrangement of fixed points and fixed lines of a TC map is the best possible.

THEOREM 2. *Let f be* TC.

(i) *If two different fixed lines of f meet at a point p then p is a fixed point.*

(ii) *If f has no fixed points then it has at most one fixed line and if it does have one such line L then $(f^n w)\to L$ for every $w\in H$.*

(iii) *If f has exactly one fixed point p then its fixed lines, if any, all pass through p.*

(iv) *If f has two or more fixed points then they will all lie on a fixed line L. Moreover any other fixed line M will intersect L and $(f^n w)\to L$ for every $w\in H$.*

Part (i) of this theorem is trivial but before dealing with the other parts we prove some lemmas.

LEMMA 1. *If f is* TC *and p, q, r are fixed points then they are collinear.*

Proof. Since $\|p-q\| = \|fp-fq\| > a\|p-q\|$ if $p\neq q$ we have by (1)

$$\Delta(p, q, r) = \Delta(fp, fq, fr) \leqslant a\Delta(p, q, r)$$

giving $\Delta(p, q, r) = 0$ and hence that p, q, r are collinear.

An immediate consequence of f being TC is that if $\|fx-fy\| > a\|x-y\|$ then $\Delta(x, y, z) = 0$ implies $\Delta(fx, fy, fz) = 0$ or in other words $fL(x, y) \subset L(fx, fy)$. When $L(x, y) = L(fx, fy)$ this means that $L(x, y)$ is fixed and gives us the first part of

LEMMA 2. *If f is* TC *and L is a line with $x, y\in L$ such that $fx, fy\in L$ and $\|fx-fy\| \geqslant \beta\|x-y\| > 0$ where $\beta > a$ then L is a fixed line. Further, $(f^n w)\to L$ for every $w\in H$.*

Proof. We have just seen that L is fixed. Since (1) must hold, for any $w\in H$ and any positive integer n we have

$$\tfrac{1}{2}\|fx-fy\|\Pi(f^{n+1}w, L) = \Delta(fx, fy, f^{n+1}w) \leqslant a\Delta(x, y, f^n w) = \tfrac{1}{2}a\|x-y\|\Pi(f^n w, L).$$

Hence by induction $\Pi(f^{n+1}w, L) \leqslant (a/\beta)^n\Pi(fw, L)$ and the right hand side tends to 0 as n tends to infinity, giving that $(f^n w)\to L$.

Lemma 2 has two obvious corollaries which we state as Lemmas 3 and 4.

LEMMA 3. *If f is* TC *and p, q are distinct fixed points then $L = L(p, q)$ is a fixed line. Further $(f^n w)\to L$ for every $w\in H$.*

LEMMA 4. *If the conditions of Lemma 2 hold for two distinct lines L, M then f has one and only one fixed point p and L and M both pass through p. Further, $(f^n w)\to p$ for every $w\in H$.*

LEMMA 5. *If f is* TC *and L, M are distinct fixed lines of f then there are fixed points p, q on L, M respectively, possibly $p = q$.*

Proof. Case (i). $\|fx-fy\| \leqslant a\|x-y\|$ if x, y are either both on L or both on M. Thus f restricted to either L or M is a contractive self map of L or M and so by Theorem 1 there exist fixed points p, q on L, M respectively.

Case (ii). The conditions of Lemma 2 are satisfied for either L or M. Without loss of generality we assume they are satisfied for L. Thus $(f^n w)\to L$ for every $w\in M$. Since M is a fixed line this means that L and M intersect. Theorem 2(i) tells us that this point of intersection is a fixed point.

LEMMA 6. *If f is* TC *with a fixed line L and a fixed point p not on L then f is contractive on L (and so has a fixed point on L).*

Proof. If f is not contractive on L then the conditions of Lemma 2 are satisfied and we have the contradiction $(f^n p)\to L$ but $f^n p = p\notin L$.

LEMMA 7. *If f is* TC *then limits of convergent sequences of iterates are collinear when such sequences exist.*

Proof. Suppose $(f^n w_i)\to v_i$ in H for $i = 1, 2, 3$. Put $\Delta = \Delta(v_1, v_2, v_3)$, $\sigma = \|v_1-v_2\|$ and for $n = 1, 2, 3, \ldots$ let $\Delta_n = \Delta(f^n w_1, f^n w_2, f^n w_3)$ and $\sigma_n = \|f^n w_1 - f^n w_2\|$. Then $(\Delta_n)\to\Delta$ and $(\sigma_n)\to\sigma$. Hence $\Delta = 0$ for otherwise $\Delta, \sigma > 0$ and (since $a < 1$) for n sufficiently large $\Delta_{n+1} > a\Delta_n$ and $\sigma_{n+1} > a\sigma$ contradicting (1).

LEMMA 8. *If f is* TC *and p, q are distinct points of H with $fp = q$ and $fq = p$ then the line $L(p, q)$ is fixed.*

Proof. Suppose $w\in L = L(p, q)$ but $fw\notin L$. Then triangle p, q, w violates (1) and so the lemma is proved.

Proof of Theorem 2. We have already said that (i) is trivial. If f has two fixed lines then by Lemma 5 it has a fixed point. Now if f has a fixed line L but no fixed point it can't be contractive on L and so by Lemma 2 we see that (ii) holds.

If f has a fixed line not passing through p then f has a second fixed point by Lemma 6. Thus (iii) is proved.

By Lemma 1 all fixed points lie on a line L which by Lemma 3 is fixed. If M is another fixed line then L and M intersect, since otherwise Lemma 5 would give a contradiction. By Lemma 3 we see that $(f^n w)\to L$ for every $w\in H$. Thus (iv) is proved.

THEOREM 3. *If f is* TEB *but not continuous then fH is part of a fixed line.*

Proof. Suppose f is discontinuous at the point q. Then there is an $\varepsilon > 0$ and a sequence $(x_n)\to q$ with $\|fx_n - fq\| \geqslant \varepsilon$ for $n = 1, 2, \ldots$ For any point y of H we have $\|x_n - q\|\to 0$ so $\Delta(y, x_n, q)\to 0$ and $\Delta(fy, fx_n, fq)\to 0$. If we take x_1 for y and let $L = L(fx_1, fq)$ then

$$\Delta(fx_1, fx_n, fq) = \tfrac{1}{2}\|fx_1 - fq\|\Pi(fx_n, L) \geqslant \tfrac{1}{2}\varepsilon\Pi(fx_n, L)$$

so $\Pi(fx_n, L)\to 0$. Suppose now that for some y we have $fy\notin L$, that is $\Pi(fy, L) > 0$. Then $\|fq - fx_n\| \geqslant \varepsilon$ and $\Pi\big(fy, L(fq, fx_n)\big)\to L(fy, L)$ so $\Delta(fy, fx_n, fq) \not\to 0$. Since this is impossible, the theorem follows.

We conclude this section by remarking that if p is a fixed point of f and L is a fixed line of f and if f is non-expansive then the foot q of the perpendicular from p to L is also a fixed point of f. We recall that f is non-expansive if $\|fx-fy\| \leqslant \|x-y\|$ for all $x, y \in H$.

4. Existence of fixtures. In this section we will give conditions which ensure the existence of a fixture. Of course Lemmas 2 and 8 are results of this nature.

THEOREM 4. *If f is* TEB *and has a convergent sequence of iterates then it has a fixture.*

In fact we prove slightly more than this below.

LEMMA 9. *Suppose f is* TEB *and p is a point such that every neighbourhood of p contains a point x and its image fx then either p is a fixed point or $L(p, fp)$ is a fixed line containing fH.*

Proof. Suppose $fp \neq p$ and let $L = L(p, fp)$. Suppose further that $z \in L$ but $fz \notin L$. Choose a sequence (x_n) of points such that $(x_n) \to p$ and $(fx_n) \to p$. Thus

$$\big(\Delta(x_n, p, z)\big) \to \Delta(p, p, z) = 0 \quad \text{but} \quad \big(\Delta(fx_n, fp, fz)\big) \to \Delta(p, fp, fz) > 0$$

and

$$\|x_n - p\| \to 0 \quad \text{but} \quad \|fx_n - fp\| \to \|p - fp\| > 0 .$$

Hence for n sufficiently large (1) is violated for the triangle x_n, p, z and so the lemma is proved.

LEMMA 10. *Let $0 < a < 1$ and x, y, z be three non-collinear points of H such that if x_1 is the foot of the perpendicular from x to $L = L(y, z)$ then $\max\{\|x_1 - y\|, \|x_1 - z\|\} \leqslant 2\|y - z\|$. Put $b = \|y - z\|$ and $h = \Pi(x, L)$ and suppose that $0 < \varepsilon < \frac{1}{2}(1-a)bh/(2b+h)$. Then $\Delta(x', y', z') > a\Delta(x, y, z)$ for any three points x', y', z' of H such that $\max\{\|x - x'\|, \|y - y'\|, \|z - z'\|\} < \varepsilon$.*

Proof. First we note that $b, h, b-2\varepsilon, h-4\varepsilon$ are positive. Then

$$\Delta(x', y', z') = \tfrac{1}{2}\|y' - z'\| \Pi\big(x', L(y', z')\big) > \tfrac{1}{2}(b - 2\varepsilon)(h - 4\varepsilon)$$
$$> \tfrac{1}{2}abh = a\Delta(x, y, z)$$

as required.

LEMMA 11. *Suppose f is* TC *and L is a line in H such that for each positive integer n there is a pair u_n, x_n of points in L, H respectively with $\|u_n - x_n\| < 1/n$ and $\|u_n - fx_n\| < 1/n$. Then f has a fixture.*

Proof. If (u_n) is bounded then it has a point of accumulation p. This point satisfies the conditions of Lemma 9 and so f has a fixture.

Thus we suppose that (u_n) is unbounded. If L is not fixed then there is a point $w \in L$ with $fw \notin L$. Now there exists a positive integer m such that $1/m$ is very small and $\|x_m - w\|$ very large relative to both $\|w - fw\|$ and $\Pi(fw, L)$. Further there exists a positive integer k such that $1/k$ is very small and $\|x_k - x_m\|$ is very large relative to all these distances. We consider w, x_m, x_k. Now $\|fx_m - fx_k\| > a\|x_m - x_k\|$ and $\Delta(w, x_m, x_k)$ is approximately $\frac{1}{2}\|x_m - x_k\|\Pi(x_m, L) \leqslant \frac{1}{2}m\|x_m - x_k\|$ and $\Delta(fw, fx_m, fx_k)$ is approximately $\frac{1}{2}\|x_m - x_k\|\Pi(fw, L)$. Thus $\Delta(fw, fx_m, fx_k) > a\Delta(w, x_m, x_k)$. This contradicts (1) and therefore we conclude that L is fixed.

THEOREM 5. *If f is* TC *and there is a sequence (x_n) of points in a finite dimensional H with $\|x_n - fx_n\| \to 0$ then f has a fixture.*

Proof. If (x_n) has a cluster point, i.e. if there exists a point p such that every neighbourhood of p contains infinitely many of the x_n's then by Lemma 9 f has a fixture.

If (x_n) has a cluster line, i.e. if there exists a line L such that every neighbourhood of L contains infinitely many of the x_n's then by Lemma 11 f has a fixture.

Thus we assume that (x_n) has no cluster point and no cluster line and obtain a contradiction. If we project (x_n) onto the unit sphere centred at the origin to get (x_n'), then (x_n') has a cluster point. Let L be the line through the origin and this point. Let C be a cylinder with L as axis and whose radius is very large relative to $\Pi(x_1, L)$ and $\|x_1 - fx_1\|$. Since (x_n) has no cluster point or line there exists a positive integer l such that $x_l \notin C$ and $\|x_l - fx_l\|$ is very small but $\|x_l - x_1\|$ is very large relative to the radius of C. Now a very narrow cone centred at 0 with L as axis will contain infinitely many of the points (x_n). So we can lastly find a positive integer k such that $\|x_k - fx_k\|$ is very small and $\|x_k - x_l\|$ and $\|x_k - x_1\|$ are very large, and $\Pi\big(x_l, L(x_1, x_k)\big)$ is at least of the same order as the radius of C. By Lemma 10 $\Delta(fx_1, fx_l, fx_k) > a\Delta(x_1, x_l, x_k)$ and $\|fx_k - fx_l\| > a\|x_k - x_l\|$ so (1) is violated and we have our contradiction.

There is no reason why the (x_n) of Theorem 5 should not be a sequence of iterates of f and the next theorem discusses such a sequence.

THEOREM 6. *If H is finite dimensional, f is* TC *and has a sequence of iterates which converges to a line then f has a fixture.*

Proof. By hypothesis there is a line L and a point x in H such that $\Pi(x_n, L) \to 0$ where $x_n = f^n x$ for $n = 1, 2, \ldots$ We assume $\|x_n - fx_n\| \neq 0$ for all n as otherwise x_n is a fixed point of f. Also we assume $\liminf\|x_n - fx_n\| \neq 0$ for otherwise Theorem 5 applies to a subsequence of (x_n). Then there is a $\lambda > 0$ such that $\|x_n - fx_n\| > \lambda$ for all n. Also $\|x_{n+1} - fx_{n+1}\| > a\|x_n - fx_n\|$ infinitely often. We shall show that L is fixed by assuming there is a point $w \in L$ with $fw \notin L$ and obtaining a contradiction.

Case (i). (x_n) bounded. Choose $\rho > 0$ so that (x_n) is contained in the ball B centred on w with radius ρ. Then choose $\varepsilon > 0$ sufficiently small that there is a $\delta > 0$ with the following property. If u, v are two points of B with $\|u - v\| \geqslant \lambda$ and $\Pi(u, L) < \varepsilon$ and $\Pi(v, L) < \varepsilon$ then $\Delta(u, v, fw) \geqslant \delta$. It is easy to see that such a choice is always possible.

Next we note that there is an integer $k = k(\varepsilon)$ such that $\Pi(x_n, L) < \varepsilon$ for $n \geqslant k$ and hence $\Delta_{fn} = \Delta(fx_n, fx_{n+1}, fw) \geqslant \delta$ for $n \geqslant k$. However $\Delta_n = \Delta(x_n, x_{n+1}, w) \to 0$ and so there is an integer h such that $\Delta_{fn} > \alpha\Delta_n$ for $n \geqslant h$. Since as we have already stated $\|fx_n - fx_{n+1}\| > \alpha\|x_n - x_{n+1}\|$ infinitely often we can obtain a contradiction of (1). Thus in this case L is fixed.

Case (ii). (x_n) unbounded. Choose $\varepsilon > 0$ but very small when compared with $\Pi(fw, L)$. Then there is an integer k such that $\Pi(x_k, L) < \varepsilon$ and $\Pi(fx_k, L) < \varepsilon$. Since (x_n) is unbounded there must be infinitely many integers n such that $\|x_k - fx_n\| > \|x_k - x_n\|$. In fact there will be a strictly increasing sequence $(n(i))$ of integers such that $\|x_k - x_{n(i)}\| \to \infty$ as $i \to \infty$ and $\|x_k - fx_{n(i)}\| > \|x_k - x_{n(i)}\|$ for every i. Hence $\|fx_k - fx_{n(i)}\| > \alpha\|x_k - x_{n(i)}\|$ for all i sufficiently large. Now for i large $\Delta(x_k, x_{n(i)}, w)$ is approximately

$$\tfrac{1}{2}\|x_k - x_{n(i)}\|\Pi(x_k, L) < \tfrac{1}{2}\|x_k - x_{n(i)}\|\varepsilon$$

and $\Delta(fx_k, fx_{n(i)}, fw)$ is approximately

$$\tfrac{1}{2}\|fx_k - fx_{n(i)}\|\Pi(fw, L) > \tfrac{1}{2}\alpha\|x_k - x_{n(i)}\|\Pi(fw, L)\,.$$

Therefore (1) is violated and the proof of the theorem is complete.

LEMMA 12. *Suppose f is* TEB, *that $(x_n) \to p$ and $\|x_n - fx_n\| \to \lambda > 0$ then*

(i) *there are at most two accumulation points of (fx_n);*

(ii) *if (fx_n) has two accumulation points r, s then $fH \subset L(r, s)$.*

Proof. If q_1, q_2, q_3 are distinct accumulation points of (fx_n) then $\|p - q_i\| = \lambda$ for $i = 1, 2, 3$, and $\Delta(q_1, q_2, q_3) > 0$. Also there are integers k, l, m with the triangle x_k, x_l, x_m arbitrarily small but triangle fx_k, fx_l, fx_m arbitrarily close to triangle q_1, q_2, q_3. This contradicts (1).

When $r \neq s$ are accumulation points, considering the images of triangles x_k, x_m, y with fx_k, fx_m near r, s respectively reveals that r, s, fy are collinear for all y.

LEMMA 13. *Suppose f is* TEB *and (x_n) is a sequence of points with $(x_n) \to p$ and $\|x_n - fx_n\| \to \lambda > 0$ and $\Delta(x_n, fx_n, f^2x_n) \to 0$ as $n \to \infty$. Also assume $\|fx_n - f^2x_n\| \geqslant \mu > 0$ for all n. Then f has a fixed line L through p and $(fx_n) \to L$ and $(f^2x_n) \to L$.*

Proof. We know by the last lemma that (fx_n) can't have three accumulation points. Suppose r, s are two accumulation points. Then again by Lemma 12 we have $fx_n, f^2x_n \in fH \subset L(r, s)$. Also because $\Delta(x_n, fx_n, f^2x_n) \to \Delta(p, fx_n, f^2x_n) \to 0$ and $\|fx_n - f^2x_n\| \geqslant \mu$ we have $p \in L(r, s)$. If (fx_n) has only one limit point r then $x_n, fx_n, f^2x_n \to L(p, r)$ and it is easy to see that this line is fixed.

THEOREM 7. *If H is finite dimensional, f is* TC *and (x_n) is a sequence of iterates such that $\liminf\|x_n - x_{n+1}\|$ is finite and $\Delta(x_n, x_{n+1}, x_{n+2}) \to 0$ then f has a fixture.*

Proof. If $\liminf\|x_n - x_{n+1}\| = 0$ then Theorem 5 applies. Therefore we assume that $\liminf\|x_n - x_{n+1}\| = \lambda > 0$. If there exists a line L such that $(x_n) \to L$ then L is fixed by Theorem 6. Thus we assume also that for all lines L we have $(x_n) \not\to L$. If we project (x_n) onto the unit sphere to get (x_n') then (x_n') has a cluster point. Let L be the line through the origin and this point. Let m be a positive integer such that $\|x_m - x_{m+1}\|$ is near λ. Then let C be a cylinder with L as axis and whose radius is very large relative to $\Pi(x_m, L)$ and λ. A sphere centred on x_m contains only finitely many points of (x_n). Also C contains only finitely many of the points. Hence there exists a positive integer l such that $x_l \notin C$ and $\|x_l - x_{l+1}\|$ is near λ and $\|x_l - x_m\|$ is very large relative to the radius of C. Lastly by considering a very narrow cone round L we can find a positive integer k such that firstly $\|x_k - x_{k+1}\|$ is near λ, secondly $\Pi(x_l, L(x_m, x_k))$ is large relative to λ, and lastly $\|x_k - x_l\|$ and $\|x_k - x_m\|$ are large relative to $\|x_n - x_l\|$. By Lemma 10 we have $\Delta(x_{m+1}, x_{l+1}, x_{k+1}) > \alpha\Delta(x_m, x_l, x_k)$ and since $\|x_{l+1} - x_{m+1}\| > \alpha\|x_l - x_m\|$, condition (1) is violated and we have a contradiction. Thus we conclude that f has a fixture.

THEOREM 8. *If H is finite dimensional f has a bounded sequence of iterates and $\Delta(fx, fy, fz) \leqslant \alpha\Delta(x, y, z)$ for all $x, y, z \in H$ where $0 < \alpha < 1$ then f has a fixture.*

Proof. Clearly f is TC and there exists a sequence of iterates (x_n) such that $\liminf\|x_n - x_{n+1}\|$ is finite and $\Delta(x_n, x_{n+1}, x_{n+2}) \to 0$. The result now follows from Theorem 7.

A map f which satisfies the condition of Theorem 8 has the property that fx, fy, fz are collinear whenever x, y, z are collinear and in [1] the problem is raised of characterising such self maps of the plane.

Acknowledgment. The authors would like to thank D. D. Ang of Saigon for his kind suggestions (given directly to one author) and in particular for referring them to a paper [2] of M. Eddlestein which greatly influenced this work.

References

[1] D. E. Daykin, *Problem* 5790, Amer. Math. Monthly 78 (1971), p. 410.

[2] M. Eddlestein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. 37 (1962), pp. 74–79.

UNIVERSITY OF READING

Reçu par la Rédaction le 22. 5. 1972