According to 6.2 (b) there is an essential map $f\colon X_t \to S$. However, for every closed subset F of X_t distinct from X_t we have $f|F \sim 1$. Indeed, if $f|F \nsim 1$, then, by [4], p. 425, there is a continuum $C \subset F$ such that $f|C \nsim 1$. Since C is distinct from X_t, C is a snake-like continuum. It follows that $f|C \sim 1$, a contradiction. Thus f irr $\nsim 1$. Therefore, by (1) and by [4], p. 421, the continuum X_t is a simple closed curve. This completes the proof of (b).

PROBLEM. Let X be a plane circle-like continuum and let μ be a Whitney map on $C(X)$. Can $\mu^{-1}(t)$ be embedded in the plane for each $t \in I$?

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Measures on bundles and bundles of measures

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Abstract. Various measures on bundles are characterised as a measure on the base space and a measurable section of an associated bundle of measures. In addition, machinery is developed to permit very general constructions of measures in this fashion. Various applications are given, including a new characterization of a method of integration sections of vector bundles.

I. Introduction. Measures on fibre bundles and associated structures have been studied in a variety of settings. For example, Goetz [G] constructed a product measure on a bundle, given Baire measures on the base and fibre with the measure on the fibre being translation invariant under the group of the bundle. He showed that this product measure deserved its name by proving a form of Fubini's theorem. In a similar fashion, Brothers [B; 3.3] makes a construction that lifts a current on the base of a bundle with prescribed fibre and group and which satisfies the conditions that on a product bundle the lifted current is the product of the current and the fibre and that the construction be natural with respect to bundle maps. It follows that such a lifting is unique. If one asks for such a lifting of measures with respect to a preassigned measure on the fibre satisfying Brother's conditions then, provided the base space of the bundle contains a measurable set with positive and finite measure, a necessary and sufficient condition is group invariance of the measure. To accomplish such constructions without group invariance of the measure on the fibre is still possible provided there is some compatibility between the group and measure. However, these liftings are not unique.

In another setting, Allard [A] studies the variational properties of a varifold. A varifold, introduced by Almgren, is defined as a Radon measure on a fibre bundle over a manifold with compact fibre. In Section III, we will characterize Radon measures on bundles with compact fibre as a measure on the base and a measurably varying Radon measure on each fibre. A form of this result for varifolds is in [A]. However, the proof these utilizes the intrinsic geometry of the varifold and various differentiation techniques in [F].

Gelbaum has raised the following problem: given a measure, μ, on the base of a vector bundle which satisfies; there exists a neighborhood system for the topology of the base each of which has boundary of μ-measure zero; then, when can you "integrate" the continuous sections with compact support to obtain, as a result of this integration, an element of the fibre space. In [G1] and [G2], he constructs such an integration process provided the group of the bundle acts as isometries on the fibre with respect to some norm. Again in Section III, we will, in many cases, be able to characterize such an integration process as Lebesgue integration with respect to μ of a section of a certain bundle evaluated on the given section of the original bundle.

The organization and content of this paper is as follows. We will restrict our attention to locally compact separable metric spaces for simplicity. It is clear from Urysohn's metrization theorem that this class contains all 2nd countable, locally compact Hausdorff spaces. Further, we will restrict attention to Borel regular measures. These measures need not be σ-finite. Recall, that in this setting, a Borel regular measure which is finite on compact sets is a σ-finite Radon measure. Further, we will view fibre bundles as maximal coordinate bundles (c.f. [5]).

In Section II, we introduce conditions which are weaker than group invariance. These conditions require that the measure of translates of Borel sets under the group action vary in a Borel or continuous fashion on the group. This requirement implies that under the transition mappings of the bundle, the measures vary in a Borel fashion. One of the requirements, called quasi-invariance, implies that the group actions carry Borel sets of σ-finite or zero measure to Borel sets of σ-finite or zero measure. I do not know what is required in addition to this condition to imply quasi-invariance. Similar questions arise in the work of Reichelderfer ([RR], [R1], [R2]).

In Section III, we discuss two types of measures on bundles. The first type (local product measures) behave locally like a product measure and the second (almost product measures) are linear combinations of product measures. If the measures on the fibre which give the local product measure are sufficiently we behaved and the measure on the base is locally finite then the local product measure is actually a product measure in the sense of [G] with a suitable reduction of the group of the bundle.

In attempting to classify Radon measures on bundles, we are motivated by Federer [F; 2.5.20] where he classifies Radon measures on the product of compact spaces using his very general theorem [F; 2.5.12]. This theorem says: if one is given a vector space, Ω, of functions on B to a separable normed linear space, E, so that Ω satisfies some compatibility conditions with a lattice of functions, L, on B then a linear function, T, on Ω which is, in a weak sense, both bounded and continuous, has an associated measure on B, μ, and a μ-measurable and μ-unique section, k, of E^* with the weak topology so that

$$T(\omega) = \int \langle \omega, k \rangle d\mu .$$

We obtain a form of this theorem for vector bundles in Section 3 which allows us to characterize Radon measures on bundles as a measurable section of a bundle of Radon measures on each fibre and to characterize the integral described by Gelbaum in a similar fashion.

It follows that measures on bundles, can, in many cases, be described as integrating over the base a suitable section of a bundle of measures. This can be viewed as a Fubini theorem. In Section IV, we discuss a bundle of measures and conditions on sections of this bundle which enable us to construct measures on the total space by a Fubini process. In many cases, the almost product measures are representable as an integral of a section of the bundle of measures. If one topologizes the set of Borel regular measures, say with the weak topology, then only the trivial group acts continuously on the space of Borel regular measures. Thus we build an untopologized bundle of measures and place restrictions on the sections in terms of families of Borel sets in the total space. Various methods of finding acceptable sections are given. Finally, if the section of measures is suitably restricted, an integral formula is given which enables us to compute the measure locally in a product neighborhood.

The conventions of Chapter 2 of Federer's *Geometric Measure Theory* [F] are adopted and most references to measure theory are to this source.

II. Groups and measures on a space. All topological spaces (except the groups and normed linear spaces of Section 3) are assumed to be locally compact separable metric spaces. We will adopt the following notation for $A \subset X$, a topological space: ChA denotes the characteristic function of A and,

(1) 2^A is the set of subsets of A;

(2) b^{2^A} is the set of Borel subsets of A;

(3) c^{2^A} is the set of closed subsets of A.

Clearly, $2^A \supset b^{2^A} \supset c^{2^A}$.

We will also adopt the conventions of Federer [F; chapter 2] concerning measure theory. Thus, a measure on X is an extended real valued function, φ: $2^X \to \bar{R}$, satisfying

(i) $\varphi(\emptyset) = 0$ and

(ii) $0 \leqslant \varphi(A) \leqslant \sum_{i=1}^{\infty} \varphi(B_i)$, whenever $A \subset \bigcup_{i=1}^{\infty} B_i$.

We will denote by φ^{2^A}, the φ-measurable subsets of A. We will denote by $\mathcal{M}(X)$, the set of Borel regular measures on X and by $\mathcal{R}(X)$, the subset of Radon measures on X. We will call a function, f, *countably φ-measurable* (*φ-integrable*) if f is φ-measurable (φ-integrable) and $\{x|\ f(x) \neq 0\}$ has σ-finite φ-measure. Such countably φ-measurable functions are φ-almost equal to a Borel function ([F; 2.3.6]).

There are three natural operations on measures.

(1) Given $A \subset X$ and φ a measure on X, the φ restriction to A, φLA, is defined as $\varphi LA(C) = \varphi(A \cap C)$;

(2) given φ, a measure on X, and $g\colon X \to Y$, any function, then the g-image of φ, $g_{\#}\varphi$, is a measure on Y given by

$$g_{\#}\varphi(C) = \varphi\big(g^{-1}(C)\big);$$

(3) given φ, a measure on X, and ψ a φ-measurable non-negative real valued function, the measure $\varphi L\psi$ given by

$$\varphi L\psi(C) = \inf\Big\{\int_D \psi\, d\varphi \,\Big|\ D \in \varphi^{2^X} \text{ and } D \supset C\Big\}.$$

It is easy to see that if $\varphi \in \mathcal{M}(X)$ and $A \in b^{2^X}$ or $A \in \varphi^{2^X}$ with σ-finite φ-measure then $\varphi LA \in \mathcal{M}(X)$. Further, if $f\colon X \to Y$ is continuous and locally univalent then $f_{\#}\colon \mathcal{M}(X) \to \mathcal{M}(Y)$ and if, in addition, f is proper then $f_{\#}\colon \mathcal{R}(X) \to \mathcal{R}(Y)$. Finally, if $\varphi \in \mathcal{M}(X)$ then $\varphi L\psi \in \mathcal{M}(X)$.

We will denote by $K(X)$, the set of continuous real valued functions with compact support on X with the sup-norm topology. It follows from standard techniques that $K(X)$ is a separable normed linear space. We will denote by $\mathfrak{D}(X)$ the space of Daniell integrals on $K(X)$ with the weak topology. The subspace of $\mathfrak{D}(X)$, $\mathfrak{D}_b(X)$, is the set of bounded linear functionals on $K(X)$ with norm less or equal to one. It is a standard theorem ([F; 2.5.5]) that given $L \in \mathfrak{D}(X)$ then there exists a unique pair of Radon measures ψ^+ and ψ^- so that

(1) $$L(f) = \int f\, d\psi^+ - \int f\, d\psi^- \quad \text{for} \quad f \in K(X);$$

and

(2) $$\int f\, d\psi^+ = \sup\{L(k)|\ 0 \leqslant k \leqslant f,\ k \in K(X) \text{ for every } f \in K(X) \text{ with } 0 \leqslant f(x)\}.$$

DEFINITION II.1. Let $g\colon X \to X$ be a homeomorphism and $\varphi \in \mathcal{M}(X)$, then g is called a *φ-absolutely continuous action* (φ-a.c.a.) *on* X if and only if for every $B \in b^{2^X}$ with σ-finite φ-measure, there exists a φ-integrable function $g'(x, B)$ so that $g_{\#}(\varphi LB)(C) = \int_C g'(x, B)\, d\varphi$ for every $C \in \varphi^{2^X}$.

It is easy to see that $g(\varphi^{2^X}) = g_{\#}\varphi^{2^X}$ whenever g is a homeomorphism, and, hence, $f \circ g^{-1}$ is $g_{\#}\varphi$-measurable and that

$$\int f\, d\varphi = \int f \circ g^{-1} d g_{\#}\varphi$$

whenever f is φ-integrable ([F; 2.4.18]).

LEMMA II.1. *Let $\varphi \in \mathcal{M}(X)$, g be a φ-a.c.a. on X, and $\{B_i\}_{i=1}^{\infty}$ be a sequence of Borel sets in X, each of σ-finite φ-measure, then:*

(a) $g'(x, \bigcup_{i=1}^{\infty} B_i) = \sup\{g'(x, B_i)|\ i = 1, 2, \ldots\}$ *for φ-almost all x; and*

(b) *if the B_i's are pairwise disjoint,* $g'(x, \bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} g'(x, B_i)$ *for φ-almost all x.*

Proof. Given φ, g, and $\{B_i\}$ as above, then $\bigcup_{i=1}^{\infty} B_i = B_0$ has σ-finite φ-measure. Observe that $g'(x, B_i) = 0$ φ-almost everywhere in $X \setminus g(B_i)$. Further, if $C \in b^{2^X}$ and $C \subset g(B_i)$ then $\int_C g'(x, B_0)\, d\varphi = \int_C g'(x, B_i)\, d\varphi$ and $g'(x, B_0) = g'(x, B_i)$ φ-almost everywhere in $g(B_i)$. Thus (a) and (b) follow.

The following Theorem characterizes φ-absolutely continuous actions on X. It was suggested by a particularly elegant but as yet unpublished result of Reichelderfer in transformation theory. Theorems much like this appear in [RR], [R1], [R2], etc.

THEOREM II.1. *Let $g\colon X \to X$ be a homeomorphism and $\varphi \in \mathcal{M}(X)$; then, g is a φ-a.c.a. on X if and only if for every $B \in b^{2^X}$ with σ-finite φ-measure:* (i) *$g(B) = C \cup D$ where C and D are disjoint Borel sets with C having σ-finite φ-measure and $g_{\#}\varphi(D) = 0$; and* (ii) *if $F \in b^{2^X}$ with $\varphi(F) = 0$ then $g_{\#}(\varphi LB)(F) = 0$.*

Proof. Assume g is a φ-a.c.a. and $B \in b^{2^X}$ with σ-finite measure then $B = \bigcup_{i=1}^{\infty} B_i$ with $\varphi(B_i) < \infty$. As $g'(x, B_i)$ is countably φ-measurable, by Lemma II.1 so in $g'(x, B)$. Thus $G = \{x|\ g'(x, B) \neq 0\} \cap g(B)$ has σ-finite φ-measure. Let $C \in b^{2^X}$ with $C \supset G$ and $\varphi(C \setminus G) = 0$ and $D = g(B) \setminus C$. As $\varphi\big(g^{-1}(D)\big) = \int_D g'(x, B)\, d\varphi = 0$, $g_{\#}\varphi(D) = 0$. Finally, if $\varphi(F) = 0$ then $g_{\#}(\varphi LB)(F) = \int_F g'(x, B)\, d\varphi = 0$.

Assume for each $B \in b^{2^X}$, (i) and (ii) are fulfilled. Observe that with $g(B) = C \cup D$, $g_{\#}(\varphi LB)$ is absolutely continuous with respect to φLC. Hence the Radon-Nikodym gives the result.

The following transformation rules are extremely useful. Further, in Theorem III.3, a chain rule is established.

THEOREM II.2. *If*

(i) $\varphi \in \mathcal{M}(x)$,

(ii) g *is a* φ-a.c.a. *on* X,

(iii) $B \in b^{2^X}$ *has* σ*-finite* φ*-measure, and*

(iv) f *is a* φ*-measurable function, then:*

(a) $f \circ g$ *is* φLB*-measurable and*

$$\int_B f \circ g\, d\varphi = \int f(x) g'(x, B)\, d\varphi\,,$$

provided either integral exists;

(b) $f \circ g^{-1}(x) \cdot g'(x, B)$ *is* φ*-measurable and*

$$\int_B f\, d\varphi = \int f \circ g^{-1}(x) g'(x, B)\, d\varphi\,,$$

provided either integral exists.

Proof. (a) As g^{-1} carries φ-measurable sets to φLB-measurable sets, $f \circ g$ is φLB-measurable. Using the remark after Definition II.1, f is then $g_{\#}\varphi LB$-measurable and

$$\int_B f \circ g\, d\varphi = \int f\, d g_{\#}\varphi LB\,.$$

The result follows from the Radon-Nikodym Theorem.

(b) Observe that if $A \in b^{2^X}$ with $\varphi(A) = 0$ then $g(A)$ has φLC-measure zero where $g(B) = C \cup D$. As $g'(x, B) = 0$ φ-almost everywhere off C, it suffices to show that $f \circ g^{-1}$ is φLC-measurable. However every $S \in \varphi^{2^X}$ has $g(S)$ φLC-measurable from the observation above and $f \circ g^{-1}$ is φLC-measurable. The integral formula follows from the Radon-Nikodym theorem as $\int_B f\, d\varphi = \int f\, d\varphi LB = \int f \circ g^{-1} d g_{\#}\varphi LB$, from the remark after Definition II.1.

THEOREM II.3. *If* $\varphi \in \mathcal{M}(X)$ *and* g *and* h *are* φ-a.c.a.'s *on* X *then* $g \circ h$ *is a* φ-a.c.a. *on* X *and if* $B \in b^{2^X}$ *with* σ*-finite* φ*-measure,*

$$(g \circ h)'(x, B) = g'(x, C) \cdot h'\big(g^{-1}(x), B\big)$$

for φ*-almost all* x *where* $h(B) = C \cup D$ *as in Theorem* II.1.

Proof. Following Theorem II.1, $g \circ h(B) = g(C) \cup g(D) = C' \cup D' \cup g(D)$ where C' has σ-finite φ-measure. As $g^{-1}(C)$ has φ-measure zero it follows that $h^{-1} \circ g^{-1}(C) \cup h^{-1}(D)$ has measure zero. Now if $F \in b^{2^X}$ with $\varphi(F) = 0$ then

$$\begin{aligned}(g \circ h)_{\#}(\varphi LB)(F) &= \varphi\Big(B \cap h^{-1}\big(g^{-1}(F)\big)\Big)\\ &= \varphi\Big(B \cap h^{-1}\Big(\big(g^{-1}(F) \cap C\big) \cup \big(g^{-1}(F) \cap D\big)\Big)\Big)\\ &= \varphi\Big(B \cap h^{-1}\big(g^{-1}(F) \cap C\big)\Big) + \varphi\Big(B \cap h^{-1}\big(g^{-1}(F) \cap D\big)\Big)\,.\end{aligned}$$

Observe that $\varphi\big(g^{-1}(F) \cap C\big) = 0$ and, hence, $\varphi\Big(B \cap h^{-1}\big(g^{-1}(F) \cap C\big)\Big) = 0$. Further, $\varphi\big(h^{-1}(D)\big) = 0$ so $(g \circ h)_{\#}(\varphi LB)(F) = 0$.

The integral formula follows from Theorem II.2.

COROLLARY II.1. *If* $\varphi \in \mathcal{M}(X)$ *and* g *and* g^{-1} *are both* φ-a.c.a.'s *on* X *then:*

(a) $g(B)$ *has* σ*-finite* φ*-measure whenever* $B \in b^{2^X}$ *has* σ*-finite* φ*-measure,*

(b) $g(C)$ *has* φ*-measure zero; and,*

(c) $g'(x, B) = 1/(g^{-1})'\big(g^{-1}(x), g(B)\big)$ *for* φ*-almost all* $x \in g(B)$.

Recall that a topology group, G, is said to act on X if and only if there is a continuous map $\tau\colon G \times X \to X$ so that:

(1) $$\tau(e, x) = x\,, \quad \text{for every } x \in X;$$

and

(2) $$\tau\big(h, \tau(g, x)\big) = \tau(h \cdot g, x)\,.$$

We will write $g(x)$ for $\tau(g, x)$. The group G is assumed to be a locally compact separable metric space. Several of the following definitions are extensions of definitions in [L.V.], and [L.V.W.].

DEFINITION II.2. Let G act on X.

(a) If $\varphi \in \mathcal{M}(X)$, φ is called *invariant under* G if and only if $g_{\#}\varphi = \varphi$, for all $g \in G$.

(b) If $\varphi \in \mathcal{M}(X)$, φ is called *quasi-invariant under* G if and only if for every $g \in G$, g is a φ-a.c.a. on X and $g'(x, B)$ is Borel measurable on $G \times X$, for every $B \in b^{2^X}$.

(c) If $\varphi \in \mathcal{M}(X)$, φ is called G*-mobile* if and only given $A \subset X$ with A compact then

(i) if $\varphi(A) < \infty$ then $\varphi\big(g(A)\big)$ is continuous in g;

(ii) if $\varphi(A) = \infty$ then $\varphi\big(g(A)\big) = \infty$ for every $g \in G$; and,

(iii) given $B \in b^{2^X}$, $\{g \mid \varphi\big(g(B)\big) = \infty\} \in b^{2^G}$.

(d) If $\varphi \in \mathcal{M}(X)$, φ is given G*-quasi-mobile* if and only if $\varphi\big(g(B)\big)$ is a Borel measurable function in g for every $B \in b^{2^X}$.

Clearly condition (iii) of Definition II.2 (c) is redundant whenever φ satisfies

$$\varphi(B) = \sup\{\varphi(A) \mid A \subset B \text{ and } A \text{ is compact}\}$$

for $B \in b^{2^X}$. Some conditions on φ which imply this requirement are given in [F; 2.10].

LEMMA II.2. *If* $\varphi \in \mathcal{M}(X)$ *and* φ *is* G*-mobile then* φ *is* G*-quasi-mobile.*

Proof. Let $B \in b^{2^X}$ with $\varphi(B) < \infty$ then $\varphi(B) = \sup\{\varphi(A) \mid A$ is compact and $A \subset B\}$. Let $\mathfrak{C}(B) = \{A \mid A \subset B$ with A compact$\}$ then $\varphi\big(g(A)\big)$ is continuous. Further

$$\psi(g) = \sup_{A \in \mathfrak{C}(B)} \{\varphi\big(g(A)\big)\}$$

is lower semi-continuous in g; and, hence, Borel measurable. As $\varphi(g(B)) \geqslant \psi(g)$ and equal to it in case $\varphi(g(B)) < \infty$, the result follows from Definition II.2 (c) (iii).

THEOREM II.4. *If $\varphi \in \mathcal{M}(X)$ is a σ-finite measure on X then φ is G-quasi-mobile.*

Proof. Let H be a Haar measure on G then $H \times \varphi$ is a σ-finite Borel regular measure on $G \times X$ and by [SA; III.9.8] the result follows.

COROLLARY II.2. *If $\varphi \in \mathcal{M}(X)$ and if φ is G quasi-invariant then φLA is G-quasi-mobile for every $A \in b^{2^X}$ with σ-finite φ-measure.*

It is interesting to note that if $\varphi \in \mathcal{R}(X)$ and φ is quasi-invariant then φ is mobile [L.V.]. Further, if $\varphi \in \mathcal{R}(X)$ then φ is mobile if and only if there exists $\mu \in \mathcal{R}(X)$ which is quasi invariant and so that φ is absolutely continuous with respect to μ [L.V.].

III. Almost product measures on bundles. Recall that a (locally trivial) fibre bundle ([S]) is (X, π, B, F, G) where X, B, and F are topological spaces (called the total space, base space and fibre), π: $X \to B$ is a continuous onto map called the projection, and G is a topological group acting effectively on F. The fibre bundle is represented by a coordinate bundle $(X, \pi, B, F, G, V_a, h_a)$ where $\{V_a\}_{a \in a}$ is an open cover of B and h_a: $V_a \times F \to \pi^{-1}(\pi_a)$ are homeomorphisms satisfying $h_\beta^{-1} \circ h_a(x, \cdot)$ acts on F as some element of G. Further the assignment $g_{\beta a}$: $V_a \cap V_\beta \to G$ by $g_{\beta a}(x) = h_\beta^{-1} \circ h_a(x, \cdot)$ is continuous. In fact, to construct a coordinate bundle all that is needed is B, F, G, V_a, and $g_{\beta a}$: $V_a \cap V_\beta \to G$ satisfying:

(i) $g_{\beta a}$ is continuous;

(ii) $g_{aa} \equiv e \in G$;

(iii) for $x \in V_a \cap V_\beta \cap V_\gamma$, $g_{\gamma\beta}(x) g_{\beta a}(x) = g_{\gamma a}(x)$.

DEFINITION III.1. Let $(X, \pi, B, F, G, V_a, h_a)$ be a coordinate bundle, $\varphi \in \mathcal{M}(X)$, and $\mu \in \mathcal{M}(B)$.

(a) The measure, φ, is called a *local μ-product measure* if and only if there exists $\{V_i\}_{i=1}^\infty$, an open cover of B, $\{\nu_i\}_{i=1}^\infty$ with $\nu_i \in \mathcal{M}(F)$ so that $\varphi L\pi^{-1}(V_i) = h_{i\#}(\mu \times \nu_i)$.

(b) The measure, φ, is called an *almost μ-product measure* if and only if there exists $\{V_i\}_{i=1}^\infty$, an open cover of B, $\{\nu_i\}_{i=1}^\infty$ with $\nu_i \in \mathcal{M}(F)$; and $\{\varphi_i\}_{i=1}^\infty$, non-negative μ-measurable functions with $\varphi_i = 0$ on $B \setminus V_i$ so that

$$\varphi = \sum_{i=1}^\infty h_{i\#}(\mu_i \times \nu_i) \quad \text{where} \quad \mu_i = \mu L\varphi_j .$$

THEOREM III.1. *Let $(X, \pi, B, F, G, V_a, h_a)$ be a coordinate bundle. If* (i) *B is connected,* (ii) *φ and μ are given so that φ is a local μ-product measure,* (iii) *the $\{(V_i, h_i, \nu_i)\}_{i=1}^\infty$ are the associate structures in Definition* III.1 (a), (iv) *for $i \neq j$ with $V_i \cap V_j \neq 0$ then $V_i \cap V_j$ has σ-finite μ-measure and open subsets have positive measure,* (v) *with $H =$ smallest closed group generated by $\{g_{ij}(x) | \, x \in V_i \cap V_j\}$, then each ν_i is H-mobile, then*

(1) *there exists $\nu \in \mathcal{M}(F)$ and $g_i \in H$ so that $\nu_i = g_{i\#}\nu$, and*

(2) *with $K =$ smallest closed group generated by $\{g_i^{-1} g_{ij}(y) g_{ji}(x) g_i | \, x$ and $y \in V_i \cap V_j\}$ then ν is K invariant.*

Proof. Let $i \neq j$ and $V_i \cap V_j \neq \emptyset$ then

$$h_{i\#}(\mu_{ij} \times \nu_i) = h_{j\#}(\mu_{ij} \times \nu_j) \quad \text{where} \quad \mu_{ij} = \mu L V_i \cap V_j .$$

Thus $\mu_{ij} \times \nu_i = (h_i^{-1} \circ h_j)_\#(\mu_{ij} \times \nu_j)$. Select $E \in c^{2^F}$ with $\nu_i(E) < \infty$ and $A \in b^{2^{V_i \cap V_j}}$ with $\mu(A) < \infty$. Hence $\mu_{ij} \times \nu_i(A \times E) < \infty$ and

$$\mu_{ij}(A) \cdot \nu_i(E) = (h_i^{-1} \circ h_j)_\#(\mu_{ij} \times \nu_j)(A \times E) ,$$

so the Borel set $h_j^{-1} \circ h_i(A \times E)$ has finite $\mu_{ij} \times \nu_j$ measure and Fubini's theorem gives

$$\mu_{ij}(A) \nu_i(E) = \int_A \nu_j(g_{ji}(x)(E)) \, d\mu_{ij}(x) .$$

As $V_i \cap V_j$ has σ-finite μ-measure, it follows that $\nu_i(E) = g_{ij}(x)_\# \nu_j(E)$ for μ-almost all $x \in V_i \cap V_j$. As ν_j is mobile and open sets have positive μ-measure,

$$\nu_i(E) = g_{ij}(x)_\# \nu_j(E) \quad \text{for all } x \in V_i \cap V_j .$$

It follows that $\nu_i(S) = g_{ij}(x)_\# \nu_j(S)$ for all sets S of σ-finite ν_i measure. If S does not have σ-finite ν_i-measure then S does not have σ-finite $g_{ij}(x)_\# \nu_j$ measure for any $x \in V_i \cap V_j$. For if it did then $g_{ij}(x)(S) = S'$ has σ-finite ν_j-measure and $g_{ji}(x)_\# \nu_i(S')$ has σ-finite ν_i measure; however $g_{ji}(x)_\# \nu_i(S') = \nu_i(S)$. Thus $g_{ji}(x)_\# \nu_i = \nu_j$ for every $x \in V_i \cap V_j$.

Let $\nu = \nu_1$. As B is connected there is a simple chain from $x_1 \in V_1$ to $x_j \in V_j$, $V_{i_1}, \ldots, V_{i_n}$, with $i_1 = 1$ and $i_n = j$. Further as $V_{i_k} \cap V_{i_{k+1}} \neq 0$, selecting $x_{i_k} \in V_{i_k} \cap V_{i_{k+1}}$ we see that

$$g_{i_{k+1} i_k}(x_{i_k})_\# \nu_{i_k} = \nu_{i_{k+1}} \quad \text{and} \quad \nu_j = [g_{i_n i_{n-1}}(x_{i_{n-1}}) \ldots g_{i_2 i_1}(x_{i_1})]_\# \nu .$$

Call this element of G, g_j. To see the invariance of ν under K, observe that for x and $y \in V_i \cap V_j$,

$$g_{ij}(y)_\# g_{ji}(x)_\# \nu_i = \nu_i$$

or

$$g_{ij}(y)_\# g_{ji}(x)_\# g_{i\#} \nu = g_{i\#} \nu .$$

Hence, $(g_i^{-1} g_{ij}(y) g_{ji}(x) g_i)_\# \nu = \nu$. As this is true for the generators of a dense subgroup of K, it follows for K from the mobility of ν.

LEMMA III.1. *Let $\mu \in \mathcal{M}(X)$ and $\nu \in \mathcal{M}(Y)$ where X and Y are topological spaces. Let φ be a μ-measurable non-negative μ-almost everywhere finite valued function. If*

(i) $\mu L\varphi(S) = \inf\{\int_B \varphi\, d\mu \mid B \in b^{2^X}$ *and* $B \supset S\}$, *and*

(ii) $(\mu\times\nu)L\varphi(T) = \inf\{\int_A \varphi\, d\mu\times\nu \mid A \in b^{2^{X\times Y}}, A \supset T\}$ *then* $(\mu L\varphi)\times\nu = (\mu\times\nu)L\varphi$.

Proof. It is easy to see that $(\mu L\varphi)\times\nu(A\times B) = (\mu\times\nu)L\varphi(A\times B)$ for $A\in b^{2^X}$ and $B \in b^{2^Y}$ as φ is μ-integrable. Thus $\mu L\varphi\times\nu \geqslant (\mu\times\nu)L\varphi$ from the definition of $\mu_\varphi\times\nu$. The result will follow if we can show that for $B\in b^{2^{X\times Y}}$ with $(\mu\times\nu)L\varphi(B) < \infty$ equality holds. Consider $S = \{x \mid \varphi(x) \neq 0\}$ then $S \in \mu^{2^X}$. Define $\hat{B} = B \cap (S\times Y)$ then $\hat{B}$ has σ-finite $\mu\times\nu$-measure. As φ is μ-almost everywhere finite valued it follows that $\hat{B}$ has σ-finite $(\mu L\varphi)\times\nu$ measure. Thus Fubini's theorem holds for both measures and

$$\int_B \varphi\, d\mu\times\nu = \int_Y\int_{B_y} \varphi\, d\mu\, d\nu = \int_Y \mu L\varphi(B_y)\, d\nu = (\mu L\varphi)\times\nu(B)\,.$$

THEOREM III.2. *Given $(X, \pi, B, F, G, V_i, h_i)$, a coordinate bunde, $\varphi \in \mathcal{M}(X)$, and $\mu\in\mathcal{M}(B)$. If φ is a local μ-product measure then φ is la almost μ-product measure.*

Proof. Let $\{(V_i, h_i, \nu_i)\}_{i=1}^\infty$ be the data from Definition III.1 (a). Select a partition of unity subordinate to $\{V_i\}_{i=1}^\infty$, say $\{\psi_i\}_{i=1}^\infty$. Define $\mu_i = \mu L\psi_i$. It suffices to show that for Borel sets

$$\varphi(A) = \sum_{k=1}^\infty h_{k\#}(\mu_k\times\nu_k)A\cap V_k\,.$$

Observing that $\varphi L\pi^{-1}(V_i) = h_{i\#}(\mu\times\nu_i)$ implies that

$$\varphi_i L\pi^{-1}(V_i) = h_{i\#}(\mu\times\nu_i)L\psi_i \quad \text{where} \quad \varphi_i = \int \psi_i\, d\varphi.$$

Lemma III.1 gives that $\varphi_j L\pi^{-1}(V_i) = h_{i\#}(\mu_j\times\nu_i)$. However $\psi_j \equiv 0$ outside $\pi^{-1}(V_i)$ thus $\varphi_i = h_{i\#}(\mu L\psi_i\times\nu_i)$ and

$$\sum_{i=1}^\infty \varphi_i = \sum_{i=1}^\infty h_{i\#}(\mu L\psi_i\times\nu_i)\,.$$

However $\sum_{i=1}^\infty \varphi_i = \varphi$.

Let $(V, \pi, B, E, G, V_a, h_a)$ be a coordinate vector bundle where E is a linear space normed by ν and B is a group of linear homeomorphism maps on E. If E^* is the dual space of continuous linear maps with the topology generated by ν^* then G acts effectively on E^* by $\langle z, ga\rangle = \langle g^{-1}z, a\rangle$. If G is a group of isometries on E and U^* is the subset of E^* with $\nu^*(a) \leqslant 1$ endowed with weak topology generated by $\{a \mid a < \langle z, a\rangle < b\}$, then G acts effectively on U^* by $\langle z, g(a)\rangle = \langle g^{-1}(z), a\rangle$. In either case, we can construct $(V^*, \pi^*, B, E^*, G, V_a, h_a^*)$ and $(V_1^*, \pi_1^*, B, U^*, G, V_a, h_a^*)$ so that there are continuous maps on $V^*\oplus V$ and $V_1^*\oplus V$ into the reals induced by the natural map on $E\times E^*\to R$ where $V^*\oplus V = \{(a, v) \mid \pi^*(a) = \pi(v)\} \subset V^*\times V$, the Whitney sum of the bundles [S]. The following theorem is a generalization of a major theorem of Federer [F; 2.5.12]. Our proof will depend heavily on Federer's techniques.

THEOREM III.3. *Let $(V, \pi, B, E, G, V_a, h_a)$ be a coordinate vector bundle with E a separable normed linear space with norm ν and continuously varying norm n: $V\to R$. In case G acts as a group of isometries assume $n = \nu$. Let n^* be the dual norm on V^* or V_1^* defined as above. Assume further that Ω is a vector space of continuous sections of V with compact support so that:*

(i) *for $f\in K(B)$ and $\omega\in\Omega$ implies $f\cdot\omega\in\Omega$;*

(ii) *there exists a cover of B by local coordinates (V_i, h_i) so that with $y\in E$ and $f\in K(B)$ with* $\operatorname{spt} f\subset V_i$ *then*

$$\omega_{i,y}(x) = \begin{cases} f(x)\cdot h_i(x, y) & \text{for} \quad x\in V_i\,,\\ 0 & \text{for} \quad x\notin V_i\end{cases}$$

is in Ω; and,

(iii) *if f and $g\in K(B)$ and $\omega\in\Omega$ so that $f\leqslant n\circ\omega \leqslant g$ then*

$$\xi(x) = \begin{cases} \dfrac{f(x)}{g(x)}\,\omega(x) & \text{for} \quad g(x)\neq 0\,,\\ 0 & \text{for} \quad g(x) = 0\end{cases}$$

is in Ω.

Finally assume T: $\Omega\to R$ is linear with:

(i) $\lambda(f) = \sup\{T(\omega) \mid n\circ\omega\leqslant f,\ \omega\in\Omega\} < \infty$ *for* $f\in K(B)^+$;

and

(ii) *if $\xi_i\in\Omega$ and $n\circ\xi_i\downarrow 0$ then $T(\xi_i)\to 0$.*

Then

(1) *λ is a monotone Daniell integral on $K(B)^+$ with associated Radon measure φ;*

(2) *there exists a φ-unique section k of V^* or V_1^* so that for every $\omega\in\Omega$, $\langle\omega, k^*\rangle$ is φ-measurable, and $T(\omega) = \int\langle\omega(x), k(x)\rangle d\varphi$;*

(3) *if G is an isometry group then k is φ-measurable and $\nu^*(k) = 1$ φ-almost everywhere;*

(4) *if E^* is separable then k is φ-measurable and, in any case, $\nu^*(k)$ is bounded above and away from zero or compact sets.*

Proof. As in [F; 2.5.12], for $f \in K(B)^+$, $\lambda(f)$ is a monotone Daniell integral on $K(B)^+$ with associated Radon measure φ. Refine the cover $\{V_i\}_{i=1}^{\infty}$ by a locally finite cover $\{U_j\}$ of open sets with compact closures each contained in some V_i and restrict h_i to U_j. Thus we have (U_j, h_j) with $h_j\colon U_j \times E \to \pi^{-1}(U_j)$. As $\mathrm{Cl}\, U_j$ is compact and h_j is defined on $\mathrm{Cl}\, U_j$, there exist positive real numbers m_j and M_j so that $m_i \nu(y) \leqslant n \circ h_j(x, y) \leqslant M_i \nu(y)$. Define L_j as the set of $f \in K(B)$ with $\mathrm{spt} f \subset V_j$ and

$$\Omega_j = \{\pi_E \circ h_i^{-1}(\omega) |\ \omega \in \Omega \text{ and } \mathrm{spt}\,\omega \subset V_j\}$$

with $\pi_E\colon U_j \times E \to E$, the projection onto E.

We will verify the hypothesis of [F; 2.5.12] on U_j with L_j and Ω_j as the associated structures.

Observe there are a countable collection of functions in L_j so that $\Sigma k_i \geqslant 1$ in U_j. Further, if $f \in L_j$ and $y \in E$, $f \cdot y \in \Omega_j$ from hypothesis (ii). Both $\nu \circ w$ and $\langle w, \alpha \rangle \in L_j$ if $w \in \Omega_j$ by continuity. If $f \in L_j^+$ and $w \in \Omega_j$ so that $f \leqslant \nu \circ w$ then with $\omega \in \Omega$ corresponding to $w \in \Omega_j$

$$f \leqslant \frac{1}{m_j} n \circ \omega \leqslant \frac{M_j}{m_j} \nu \circ w$$

and by (iii),

$$\xi = \begin{cases} \dfrac{m_j f(x)}{M_j \nu \circ w(x)} \omega(x) & \text{for} \quad \nu \circ w(x) \neq 0\,, \\ 0 & \text{for} \quad \nu \circ w(x) = 0 \end{cases}$$

is in Ω so is $\dfrac{M_j}{m_j}\xi$. Thus $\pi_E \circ h_j^{-1}(\xi) \in \Omega_j$ and satisfies the requirement of [F; 2.5.12] that

$$\xi' = \begin{cases} \dfrac{f(x) w(x)}{\nu(w(x))} & \text{for} \quad \nu(w(x)) \neq 0\,, \\ 0 & \text{for} \quad \nu(w(x)) = 0 \end{cases}$$

be in Ω_j.

Let $T_j(w) = T(\omega)$ where $\omega \in \Omega$ corresponds to $w \in \Omega_j$ and observe that for $f \in L_j^+$

$$\begin{aligned} \lambda_j(f) &= \sup\{T_j(w) |\ \nu(w) \leqslant f,\ w \in \Omega_j\} \\ &\leqslant \sup\{T(w) |\ n \circ \omega \leqslant M_j f,\ \omega \in \Omega\} \\ &= \lambda(M_j f) = M_j \lambda(f) < \infty\,. \end{aligned}$$

In a similar fashion, we can show

$$m_i \lambda(f) \leqslant \lambda_j(f) \qquad \text{whenever} \qquad f \in L_j^+\,.$$

Finally, we need to know that if $\xi_i \in \Omega_j$ with $\nu(\xi_i) \downarrow 0$ then $T_j(\xi_i) \to 0$. The $\nu(\xi_i) \downarrow 0$, however, does not imply that $n \circ (h_j^{-1}(x, \xi(x)) \downarrow 0$. A careful analysis of the proof of [F; 2.5.12] reveals that all that is needed is that if $\xi_i(x)$ and $\xi_j(x)$ are linearly dependent for each x, i and j then $\nu(\xi_i) \downarrow 0$ implies $T_j(\xi_i) \to 0$. Such a condition insures that $n \circ h_j^{-1}(x, \xi_i(x)) \downarrow 0$ and $T_j(\xi_i) \to 0$. Thus from [F; 2.5.12], each $w \in \Omega_j$ is φ_j-measurable, $k_j\colon U_j \to E^*$ with k_j φ_j-almost unique and φ_j-measurable with $\nu^*(k_j) = 1$ φ_j almost everywhere so that $T_j(w) = \int \langle w, k_j \rangle d\varphi_j$. As $\varphi L U_j$ and φ_j are mutually absolutely continuous there exists an p_j with $m_i \leqslant \mathrm{ess\,inf}\, p_j \leqslant \mathrm{ess\,sup}\, p_j \leqslant M_i$ so that $\varphi_j = \int p_j d\varphi$. Thus $p_j \cdot k_j$ satisfies the hypothesis with φ_j replaced by φ. Further in $U_j \cap U_l$, $p_j k_j = p_l k_l$ φ-almost everywhere from φ-uniqueness. Thus we may define $k\colon B \to V^*$ or $k\colon B \to V_1^*$, a φ-measurable section. The measurability questions are clear, as in the case of V^*, E^* has a countable base. To verify the integral equality, select a partition of unity $\{\eta_j\}_{j=1}^{\infty}$ subordinate to U_j and observe that for $\omega \in \Omega$, $\omega = \sum_{j=1}^{n} \eta_j \omega$ as the $\mathrm{spt}\,\omega$ is compact. Thus

$$T(\omega) = \sum_{j=1}^{n} T(\eta_j \omega) = \sum_{j=1}^{n} \int \langle \eta_j \omega, k \rangle d\varphi = \int \Big\langle \sum_{j=1}^{n} \eta_j \omega, k \Big\rangle d\varphi = \int \langle \omega, k \rangle d\varphi\,.$$

In case the section is of V^* then $n^* k$ is essentially bounded above and away from zero on compact sets. In case the section is of V_1^*, then φ almost everywhere, $n^* k = 1$. In case E^* is not separable, the section k still exists but is not necessarily φ-measurable. This fact in some measure justifies the constructions in the next section where topological considerations are ignored.

As an application of the preceeding theorem we will characterize all Radon measures on fibre bundles satisfying a minimal boundedness condition with both fibre and base locally compact metric spaces. This is much like [F; 2.5.20]. In fact, if the fibre is compact the boundedness condition is always satisfied. Let $(X, \pi, B, F, G, V_a, h_a)$ be a coordinate bundle. Observing that $K(F)$ is a separable normed linear space with the sup norm topology, ν, it follows that G acts effectively on $K(F)$ by $g(f) = f \circ g^{-1}$ and each g is an isometry. Constructing the associated bundle $(K, \pi^*, B, K(F), G, V_a, h_a^*)$, observe that there is a continuous real map on $K \oplus X$ given locally by $\langle f, y \rangle = h(y')$ with $f \oplus y \in K \oplus X$ where $h_a^{*-1}(f) = (\pi^*(f), h)$ and $h_a^{-1}(y) = (\pi(y), y')$. This is easily seen to be coordinate invariant. As K has a continuously varying norm n induced by ν, let Ω be the vector space of continuous sections of K with compact support, and satisfy the further requirement that $\{y |\ y \in X$ so that $\langle \omega(\pi(y)), y \rangle \neq 0\}$ has compact closure. If Ω is normed by $N(\omega) = \sup n \circ \omega$ then $\Theta\colon \Omega \to K(X)$ by $\Theta(\omega)(y) = \langle \omega(\pi(y)), y \rangle$ is an isometric isomorphism using the sup norm on $K(X)$. We will denote the section $\Theta^{-1}(f)(x)$ by f_x.

THEOREM III.4. *Let $(X, \pi, B, F, G, V_a, h_a)$ be a fibre bundle and $(K_1^*, \pi^*, B, K_1^*(F), G, V_a, h_a^*)$ be the bundle of Radon measures on F with total variation less or equal to 1 with the weak topology. If $T\colon K(X) \to R$ is linear and*

(i) $\sup\{T(g)|\ 0 \leqslant g \leqslant f\} < \infty$ *for* $f \in K(X)^+$; *and,*

(ii) $\sup\{T(g)|\ 0 \leqslant n \circ g \leqslant f\} < \infty$ *for* $f \in K(B)^+$;

then there exists a Daniell integral on B, λ, with associated Radon measure φ and a φ-unique φ-measurable section of K_1^* with $n^*(k) = 1$ φ-almost everywhere so that

$$T(f) = \int \langle f_x, k(x)\rangle d\varphi(x)$$

for every $f \in K(X)$. Further if ψ and ν_x are the Radon measures associated with T and $k(x)$ and $A \subset X$ is ψ measurable and $\mathrm{Ch}A$ is ψ-summable

$$\psi(A) = \int \nu_x(A_x)\, d\varphi \quad \text{where} \quad A_x = A \cap \pi^{-1}(x)\,.$$

Proof. It follows from Riesz's representation theorem [F; 2.5.13] that T is a Daniell integral and, hence, by Lebesgue's dominated convergence theorem if $\xi_i \in K(X)$ with $\nu(\xi_i) \downarrow 0$ that $T(\zeta_i) \to 0$. Define $S = T \circ \Theta$. It is elementary to verify the hypothesis of Theorem III.3 for Ω and S. Thus the first part follows.

To verify the final conclusion, see Corollary IV.1 of Section IV.

In [G1] and [G2], Gelbaum describes a method of integrating continuous sections of a vector bundle with isometric group actions and a tame measure on the base. The values of the integral are in some fixed copy of the fibre. As another application of Theorem III.3, we will relate this integral to a section of a certain bundle and usual Lebesgue integration.

In particular, let (E, ν) and (F, σ) be normed linear spaces. The dual spaces are (E^*, ν^*) and (F^*, σ^*). With $L(F^*, E^*)$ denoting the linear maps from F^* to E^*, then $L(F^*, E^*)$ is naturally isomorphic to $L(E, F^{**})$. Further G acts on $L(E, F^{**})$ by $g(T) = T \circ g^{-1}$. If $L_b(E, F^{**})$ denotes the bounded linear function then G acts effectively on $L_b(E, F^{**})$ endowed with the sup norm or weak topology. Construct the bundles with fibre $L(E, F^{**})$ (untopologized) and fibre $L_b(E, F^{**})$ (topologized) with the data given by $(V, \pi, B, E, G, V_a, h_a)$, a coordinate vector bundle and call them $(L, \pi', B, L(E, F^{**}), G, V_a, h'_a)$ and $(L_b, \pi', B, L_b(E, F^{**}), G, V_a, h'_a)$. Observe that there is a map $[\ ,\ ]\colon V \oplus L \to F^{**}$ given by $[\nu, T] = T(\nu)$ in each coordinate system.

THEOREM III.5. *Let $(V, \pi, B, E, G, V_a, h_a)$ be a coordinate vector bundle (E, ν) a separable normed linear space and continuously varying norm, n, on V. Let μ be a Radon measure on B and let Ω be the vector space of continuous sections of V with compact support. If (F, σ) is another normed linear space and $T\colon \Omega \to F$ is a linear map satisfying $\sigma(T(\omega)) \leqslant \int n \circ \omega\, d\mu$ then there exists a section of L so that for $a \in F^*$,*

$$\langle T(\omega), a\rangle = \int \langle a, [\omega(x), k(x)]\rangle d\mu$$

with $\langle a, [\omega(x), k(x)]\rangle$ μ-measurable and μ-unique for fixed a. If, in addition, F^* is separable then k is a μ-unique section of L_b. If in addition G acts as a group of isometries on E or E^* is separable then $[\omega(x), k(x)]$ is μ-measurable for every $\omega \in \Omega$.

Proof. Select $a \in F^*$ and define T_a by $T_a(\omega) = \langle T(\omega), a\rangle$ then

$$|T_a(\omega)| \leqslant \sigma[T(\omega)]\nu^*(a) \leqslant \nu^*(a) \int n \circ \omega\, d\mu\,.$$

So $\lambda_a(f) \leqslant \nu^*(a)\lambda(f)$ and if $n \circ \omega_i \downarrow 0$ then $T_a(\omega_i) \to 0$ by Lebesgue's dominated convergence theorem. Thus by Theorem III.3 there exists a λ_a, φ_a, and k_a so that

$$\langle T(\omega), a\rangle = \int \langle \omega, k_a\rangle d\varphi_a$$

with $k_a\varphi_a$-unique and $n^*(k_a)$ is essentially bounded on compact sets. However φ_a is absolutely continuous with respect to μ so there exists a p_a μ-measurable and essentially bounded by 1 so that

$$\langle T(\omega), a\rangle = \int \langle \omega, h_a\rangle d\mu$$

where $k_a = p_a \tilde{h}_a$, h_a is μ-unique, and $n^*(h_a)$ is essentially bounded on compact sets. To construct $k\colon B \to L$, select a Hamel basis for F^* and for each a select ak_a. Extending $k\colon B \times F^* \to V$ by linearity and using the duality there is a unique $k\colon B \to L$ so that

$$\langle T(\omega), a\rangle = \int \langle a, [k(x), \omega(x)]\rangle d\mu$$

with $\langle a, [k(x), \omega(x)]\rangle$ is μ-measurable and μ-unique. In case F^* is separable, then defining k_a for a countable dense vector space over the rationals, for almost all x, k_a is linear and bounded and this has a μ-unique extension $\hat{k}\colon B \times F^* \to V$. If G acts as a group of isometries or E^* is separable then $\hat{k}\colon B \times F^* \to V_1^*$ or V^* and it follows that $[k(x), \omega(x)]$ is μ-measurable with respect to the weak topology or sup norm topology respectively.

IV. A bundle of measures and Fubini's Theorem. In light of Theorem III.4, it is every natural to assign a measure on F that varies from point to point in a measurable fashion. In this section, the appropriate machinery will be developed to accomplish such constructions.

DEFINITION IV.1. Let $(X, \pi, B, F, G, V_a, h_a)$ be a coordinate bundle, $\varphi \in \mathcal{M}(B)$, and $\mathcal{M}(F)$ the set of Borel measures on F.

(a) Construct the bundle $(\mathcal{M}, \hat{\pi}, B, \mathcal{M}(F), G, V_a, \hat{h}_a)$ from $\mathcal{M}(F)$ with G acting on $\mathcal{M}(F)$ by $\tau(g_{\#}\mu) = g_{\#}\mu$ and $\hat{g}_{\beta a}(x) = g_{\beta a}$. This bundle (without a topology) is called the *bundle of measures on F over B*.

(b) Any section $\sigma\colon B \to \mathcal{M}$ is called a *Borel* (*φ-measurable*) *section with respect to* $\mathcal{A}$ where $\mathcal{A}$ is a collection of Borel sets so that every open set is a countable union of elements of $\mathcal{A}$ (and $\varphi \in \mathcal{M}(B)$) if and only if $\sigma(x)\big(A \cap \pi^{-1}(x)\big)$ is Borel (φ) measurable for every $A \in \mathcal{A}$.

(c) The measure $\sigma\times\varphi(C) = \inf\{\sum_{i=1}^{\infty}\int\sigma(x)\big(A_i\cap\pi^{-1}(x)\big)d\varphi|\ \bigcup_{i=1}^{\infty}A_i \supset C$ and $A_i \in \mathcal{A}\}$ is called the *$\sigma\times\varphi$-$\mathcal{A}$ measure*.

The following theorem is a version of Fubini's theorem.

THEOREM IV.1. *Let $\varphi \in \mathcal{M}(B)$, σ be a φ-measurable section with respect to $\mathcal{A}$, and $\mathcal{A}$ satisfy*:

(i) *$S\setminus T = \bigcup_{i=1}^{\infty} U_i$ when $S, T, U_i \in \mathcal{A}$ and the U_i's are pairwise disjoint; and*

(ii) *$S \cap T \in \mathcal{A}$ whenever S and $T \in \mathcal{A}$; then,*

(a) *$\sigma\times\varphi$ is a Borel regular measure;*

(b) *if $A \in \mathcal{A}$, $\sigma\times\varphi(A) = \int\sigma(x)\big(\pi^{-1}(x)\cap A\big)d\varphi$;*

(c) *if S is countably $\sigma\times\varphi$-measurable then $\sigma(x)\big(S\cap\pi^{-1}(x)\big)$ is φ-measurable and $\sigma\times\varphi(S) = \int\sigma(x)\big(S\cap\pi^{-1}(x)\big)d\varphi$; and*

(d) *if f is countably $\sigma\times\varphi$-integrable then*

$$\int f d\sigma\times\varphi = \int\int_{\pi^{-1}(x)} f(x, y)\, d\sigma(x)(y)\, d\varphi(x)\,.$$

Proof. Given $A \in \mathcal{A}$ then $\sigma\times\varphi(A) \leqslant \int\sigma(x)\big(A\cap\pi^{-1}(x)\big)d\varphi$. However, $A_i \in \mathcal{A}$ with $\bigcup_{i=1}^{\infty} A_i \supset A$ then

$$\sigma(x)\big(A\cap\pi^{-1}(x)\big) \leqslant \sigma(x)\big(\bigcup_{i=1}^{\infty}A_i\cap\pi^{-1}(x)\big) \leqslant \sum_{i=1}^{\infty}\sigma(x)\big(A_i\cap\pi^{-1}(x)\big)$$

and (b) follows.

Following [F; 2.6.2], let $F = \{S|\ \sigma(x)\big(\pi^{-1}(x)\cap S\big)$ is φ-measurable$\}$ and define

$$\varrho(S) = \int\sigma(x)\big(\pi^{-1}(x)\cap S\big)d\varphi\,.$$

Clearly $\mathcal{A}\subset F$, F_∞ is closed under countable disjoint unions, and if $S_i \in F$, $S_i \supset S_{i+1}$ and $\varrho(S_i) < \infty$ then $\bigcap_{i=1}^{\infty} S_i = C$. Define $\mathcal{A}_\sigma = \{\bigcup_{i=1}^{\infty} A_i|\ A_i \in \mathcal{A}\}$ then following [F; 2.6.2], $\mathcal{A}_\sigma \subset F$ from (i). Further defining $\mathcal{A}_{\sigma\delta} = \{\bigcup_{i=1}^{\infty} B_i|\ B_i \in \mathcal{A}_\sigma\}$ then [F; 2.6.2] shows that $C \in \mathcal{A}_{\sigma\delta}$ is the decreasing limit of elements of $\mathcal{A}_\sigma$ from (i) and (ii).

Notice that if $C \in \mathcal{A}_\sigma$ then C can be written as the disjoint union of elements of $\mathcal{A}$, hence $\varrho(C) = \sigma\times\varphi(C)$. Thus, $\sigma\times\varphi(S) = \inf\{\varrho(C)|\ C\supset S,\ C \in \mathcal{A}_\sigma\}$. Further, there exists an element D of $\mathcal{A}_{\sigma\delta}$ containing S with $\sigma\times\varphi(S) = \sigma\times\varphi(D) = \varrho(D)$ (in case $\sigma\times\varphi(S) < \infty$ this is easy, otherwise use the union of the countable cover in $\mathcal{A}$).

To see that $A \in \mathcal{A}$ is $\sigma\times\varphi$-measurable, select $T \in 2^X$ with $\sigma\times\varphi(T) < \infty$ and select $U \in \mathcal{A}_\sigma$ with $T\subset U$. As $U\cap A$ and $U\setminus A$ are disjoint numbers of $\mathcal{A}_\sigma$,

$$\sigma\times\varphi(T\cap A) + \sigma\times\varphi(T\setminus A) \leqslant \varrho(U\cap A) + \varrho(U\setminus A) = \varrho(U)\,.$$

Thus $\sigma\times\varphi(T\cap A) + \sigma\times\varphi(T\setminus A) \leqslant \sigma\times\varphi(T)$ and A is $\sigma\times\varphi$-measurable.

Further, every member of $\mathcal{A}_{\sigma\delta}$ is $\sigma\times\varphi$-measurable, the open sets are $\sigma\times\varphi$-measurable and every set is contained in a Borel set of equal $\sigma\times\varphi$-measure. Thus $\sigma\times\varphi$ is a Borel regular measure.

Now given W with $\sigma\times\varphi$-measure zero, then there exists a $C \in \mathcal{A}_{\sigma\delta}$ so that $W\subset C$ and with $\varrho(C) = 0$ thus $\varrho(W) = 0$. So given any $\sigma\times\varphi$-measurable set S, with $\sigma\times\varphi(S) < \infty$, there exist $D \in \mathcal{A}_{\sigma\delta}$ with $D\supset S$ and $\varrho(D) = \sigma\times\varphi(S)$. Hence $\sigma\times\varphi(D\setminus S) = 0$ and there exists a $C \in \mathcal{A}_{\sigma\delta}$ with $C \supset D\setminus S$ and $\varrho(C) = 0$. Now $\sigma\times\varphi(D\setminus C) = \varrho(D) - \varrho(C)$ and $D\setminus C \in F$ thus so is S. Hence (c) and (d) follow.

Associated with a cover of B by (V_i, h_i) is a natural family $\mathcal{A} = \{h_i(A\times C)|\ A \in b^{2^{V_i}}$ and $C \in b^{2^F}\}$. It is not surprising that if σ is a φ-measurable section with respect to $\mathcal{A}$, $\sigma\times\varphi$ satisfies the conclusions of Theorem IV.1. Thus the following Corollary.

COROLLARY IV.1. *Let $\varphi \in \mathcal{M}(B)$, $\{(V_i, h_i)\}_{i=1}^{\infty}$ a cover of B by coordinate systems, as above, and σ a φ-measurable section with respect to $\mathcal{A}$ then the conclusions of Theorem* IV.1 *hold.*

Proof. Let $\mu_i = LV_i\times\varphi LV_i$ be the measure generated with respect to $\mathcal{A}_i = \{h_i(A\times C)|\ A \in b^{2^{V_i}}$ and $C \in b^{2^F}\}$. Then μ_i satisfies the conclusions of Theorem IV.1. Clearly, for $S\subset\pi^{-1}(V_i)$, $\sigma\times\varphi(S) \leqslant \mu_i(S)$ and for any $T\subset X$,

$$\sigma\times\varphi(T) = \inf\Big\{\sum_{i=1}^{\infty}\mu_i(C_i)|\ C_i\subset V_i,\ C_i \in b^{2^X} \text{ and } \bigcup_{i=1}^{\infty} C_i\supset T\Big\}\,.$$

Thus $\sigma\times\varphi$ is a regular Borel measure, for given $T\subset X$ with $\sigma\times\varphi(T) < \infty$ and $A \in b^{2^X}$ then there exists for every $\varepsilon > 0$ a countable disjoint cover of T by Borel sets C_i so that $\sum_{i=1}^{\infty}\mu_i(C_i) \leqslant \sigma\times\varphi(T)+\varepsilon$, thus

$$\sigma\times\varphi(T\setminus A) + \sigma\times\varphi(T\cap A) \leqslant \sum_{i=1}^{\infty}\mu_i(C_i\setminus A) + \sum_{i=1}^{\infty}\mu_i(C_i\cap A)$$

$$= \sum_{i=1}^{\infty}\mu_i(C_i) \leqslant \sigma\times\varphi(T)+\varepsilon\,.$$

To verify the integral representation, let S be a Borel set with finite $\sigma\times\varphi$-measure. Select a cover of S by $\{C_i\}_{i=1}^{\infty}$, pairwise disjoint Borel subsets of V_i with $\mu_i(C_i)<\infty$, then $C_i\cap S$ has finite μ_i-measure and $\sigma(x)\big(S\cap C_i\cap\pi^{-1}(x)\big)$ is φ-measurable so

$$\sigma(x)\big(S\cap\pi^{-1}(x)\big)=\sum_{i=1}^{\infty}\sigma(x)\big(S\cap C_i\cap\pi^{-1}(x)\big)$$

is φ-measurable. As

$$\int\sigma(x)\big(S\cap\pi^{-1}(x)\big)d\varphi\leqslant\sigma\times\varphi(S)\leqslant\sum_{i=1}^{\infty}\mu_i(S\cap C_i)$$
$$=\sum_{i=1}^{\infty}\int\sigma(x)\big(S\cap C_i\cap\pi^{-1}(x)\big),$$

the result follows from Lebesgue's monotone convergence theorem.

The results follow for $\sigma\times\varphi$-measurable set and φ countably integrable function as in Theorem IV.1.

The natural question at this point is, what measures on X arise in this fashion?

THEOREM IV.2. *Let $\varphi\in\mathcal{M}(X)$ and $\mu\in\mathcal{M}(B)$ be given so that φ is an almost product measure with respect to μ with $\{(V_i,h_i,\nu_i,\psi_i)\}_{i=1}^{\infty}$ the associated data. If ν_i and μ are all σ-finite then there exists a μ-measurable section, σ, with respect to b^{2^X} so that*

$$\varphi=\sigma\times\dot{\mu}.$$

Proof. Denoting by $h_i(x)_{\#}$: $\mathcal{M}(F)\to\hat{\pi}^{-1}(x)$, define

$$\sigma(x)=\sum_{i=1}^{\infty}\psi_i(x)h_i(x)_{\#}(\nu_i)$$

with the convention that $\psi_i(x)=0$ and $h_i(x)_{\#}$ is undefined gives the zero measure in $\pi^{-1}(x)$. It follows from Fubini's theorem that in $\pi^{-1}(V_i)$, with $\sigma_i(x)=\psi_i(x)h_i(x)_{\#}\nu_i$,

$$h_{i\#}(\mu_i\times\nu_i)=\sigma_i\times\mu.$$

As $\sum_{i=1}^{\infty}\sigma_i\times\mu=\sigma\times\mu$, $\varphi=\sigma\times\mu$.

COROLLARY IV.2. *If $\varphi\in\mathcal{M}(X)$ and $\mu\in\mathcal{M}(B)$ then there exists a μ-measurable section, σ, with respect to b^{2^X} so that $\sigma\times\mu=\varphi$ provided φ is a σ-finite local product measure.*

Proof. This follows immediately from Theorem III.2 and Theorem IV.2 as φ being σ-finite implies μ and ν_i are σ-finite in V_i or $\nu_i\equiv 0$.

Without the σ-finiteness assumption the preceding theorems and corollary are false, even under the assumption that the ν_i's are G-quasi-invariant.

In concluding this paper, several methods of constructing μ-measurable sections are given and a local representation theorem is given when each $\sigma(x)$ is G-quasi-invariant and relatively well behaved in x.

THEOREM IV.3. *Let $\varphi\in\mathcal{M}(B)$ and $\{(V_i,h_i)\}_{i=1}^{\infty}$ be a localy finite cover of B by coordinate systems. If $\{\nu_i\}_{i=1}^{\infty}$ are given so that*

(i) *$\nu_i\in\mathcal{M}(F)$ for all i; and,*

(ii) *each ν_i is quasi-mobile; then,*

$$\sigma(x)=\sum_{\mathcal{I}(x)}h_i(x)_{\#}\nu_i\quad\textit{with}\quad\mathcal{I}(x)=\{i|\ x\in V_i\}$$

is a φ-measurable section with respect to

$$\mathcal{A}=\{h_i(A\times C)|\ A\in b^{2^{V_i}}\ \textit{and}\ C\in b^{2^F}\}.$$

Proof. If

$$\sigma_i(x)=\begin{cases}h_i(x)_{\#}\nu_i & \text{for}\quad x\in V_i,\\ 0 & \text{for}\quad x\notin V_i\end{cases}$$

is a φ-measurable section with respect to $\mathcal{A}$ then so is $\sum_{i=1}^{\infty}\sigma_i=\sigma$. To see that σ_i is φ-measurable, observe that $\sigma_i(x)\big(h_i(A\times C)\big)=\mathrm{Ch}_A(x)\cdot\nu_i(A)$ is φ-measurable. If $A\subset V_i\cap V_j$ then $\sigma_i(x)\big(h_j(A\times C)\big)=\mathrm{Ch}_A(x)\cdot g_{ji}(x)_{\#}\nu_i(C)$. However, $g_{\#}\nu_i(C)$ is Borel measurable and g_{ji}: $V_i\cap V_j\to G$ is continuous so $g_{ji}(x)_{\#}\nu_i(C)$ is φ-measurable. The result follows.

COROLLARY IV.3. *If $\varphi\in\mathcal{M}(B)$, $\{(V_i,h_i)\}_{i=1}^{\infty}$ is a localy finite cover of B by coordinate systems, $\nu_i\in\mathcal{M}(F)$ are quasi-mobile measures, and $\{\psi_i\}_{i=1}^{\infty}$ are φ-measurable functions with $\operatorname{spt}\psi_i\subset V_i$ then $\sigma(x)=\sum_{i=1}^{\infty}\psi_i(x)h_i(x)_{\#}\nu_i$ is a φ-measurable section with respect to $\mathcal{A}$.*

Proof. This follows as in Theorem IV.3.

Recall that G acts on G by left multiplication so there exists an associated bundle with fibre G called the principle bundle, (P,π,B,G,G,V_a,h'_a).

THEOREM IV.4. *Let s be a Borel measurable section of the principle bundle and $\nu\in\mathcal{M}(G)$ be a quasi-mobile measure. Then with s_i: $V_i\to V_i\times G$ a coordinate map, $\sigma(x)=\hat{h}_i(x)s_i(x)_{\#}\nu$ is a φ-measurable section with respect to $\mathcal{A}=\{h_a(A\times C)|\ A\in b^{2^{V_a}}$ and $C\in b^{2^F}\}$ for any $\varphi\in\mathcal{M}(B)$.*

Proof. Observe that $s_j(x)=g_{ji}(x)\cdot s_i(x)$ whenever $x\in V_i\cap V_j$ thus

$$\big(\hat{h}_j(x)s_j(x)\big)_{\#}\nu=\big(\hat{h}_j(x)g_{ji}(x)s_i(x)\big)_{\#}\nu=\big(\hat{h}_i(x)s_i(x)\big)_{\#}\nu$$

and σ is well defined. To see that σ is Borel measurable on $h_a(A\times B)$, observe that

$$\sigma\big(h_a(A\times B)\cap\pi^{-1}(x)\big)=\hat{h}_a(x)_{\#}s_a(x)_{\#}\nu\big(h_a(A\times B)\cap\pi^{-1}(x)\big)$$
$$=s_a(x)_{\#}\nu(B)\cdot \mathrm{Ch}\,A(x)\,.$$

As $s_a(x)$ is Borel measurable and ν is quasi-mobile, $s_a(x)_{\#}\nu(B)$ is Borel measurable as is $\mathrm{Ch}\,A(x)$ and the result follows.

The following theorem gives a method of computing $\sigma\times\mu$ in a coordinate system provided σ has a fairly reasonable local representation.

THEOREM IV.5. *Given* (X,π,B,F,G,V_a,h_a), $\mu\in\mathcal{M}(B)$. *If* $\nu\in\mathcal{M}(F)$ *is* G-*quasi-invariant and* σ *is section over* V_a *so that* $\sigma(x)=h_a(x)_{\#}\big(\psi(x)\times\times\gamma(x)_{\#}\nu LA\big)$ *where*:

(i) ψ *is a non-negative Borel measurable function on* V_a;

(ii) γ *is a Borel measurable map from* V_a *to* G;

and

(iii) $A\in b^{2^F}$ *has* σ-*finite* ν-*measure*; *then*:

(a) σ *is a Borel measurable section over* V_a *with respect to*

$$\mathcal{A}=\{h_a(C\times D)\mid A\in b^{2^{V_a}} \text{ and } D\in b^{2^F}\};$$

and

(b) *given any set* $S\subset V_a\times F$ *with* σ-*finite* $\mu\times\nu$ LA-measure,

$$\sigma\times\mu\big(h_a(S)\big)=\int_S\psi(x)\gamma'(x)(y,A)\,d\mu\times\nu$$
$$=\int_{V_a}\int_F \mathrm{Ch}\,S_x(y)\gamma'(x)(y,A)\psi(x)\,d\nu\,d\mu\,.$$

Proof. Observe that as $\gamma\colon V_a\to G$ is Borel $\gamma'(x)(y,A)$ is Borel measurable in $V_a\times F$ and, hence, so is $\psi(x)\gamma'(x)(y,A)$. Now, if $C\in b^{2^{V_a}}$ and $D\in b^{2^F}$ then $\sigma(x)\big(h_a(C\times D)\cap\pi^{-1}(x)\big)=\gamma(x)_{\#}\nu\,LA(D)\cdot\psi(x)\cdot\mathrm{Ch}_C(x)$ which is Borel measurable as $\nu\,LA$ is quasi-mobile and γ and ψ are Borel measurable. Thus

$$\sigma\times\mu\big(h_a(C\times D)\big)=\int_C\gamma(x)_{\#}\big(\nu\,LA(D)\big)\psi(x)\,d\mu=\int_C\int_F\gamma'(x)(y,A)\varphi(x)\,d\nu\,d\mu$$

from the Borel measurability of $\gamma'(x)(y,A)\cdot\psi(x)$. Following the proof of Fubini's theorem gives

$$\sigma\times\mu\big(h_a(S)\big)=\int_S\psi(x)\gamma'(x)(y,A)\,d\mu\times\nu=\int_{V_a}\int_F\mathrm{Ch}_{S_x}(y)\psi(x)\gamma'(x)(y,A)\,d\nu\,d\mu$$

whenever S has σ-finite $\mu\times\nu$ LA-measure. The result follows.

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