which are not homotopic but which satisfy $\psi_1 f = \psi_2 f$; hence $F[\psi_1] = F[\psi_2]$. Now if there were a natural transformation $G\colon \pi_X \to \pi_Z$ satisfying $GF = 1_Z^{\#}$, then F would map $[Z, S^0]$ injectively to $[X, S^0]$. Hence the shape of Z is not dominated by the shape of X.

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On the hyperspaces of snake-like and circle-like continua

by

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Abstract. J. Segal has proved a theorem which says that the hyperspace of a snake-like continuum has the fixed point property. In this paper we give a shorter proof of this theorem and we prove also that the hyperspace of a circle-like continuum has this property. The structure of these hyperspaces is studied and it is shown that the Whitney maps induce interesting decompositions of these hyperspaces.

0. Introduction. By a *map* we mean a continuous function. The term *continuum* means a compact connected metric space. If X is a continuum, then $C(X)$ denotes the *hyperspace* of subcontinua of X with the Hausdorff metric: $\operatorname{dist}(A, B) = \inf\{\varepsilon > 0\colon\ B \subset K(A, \varepsilon)$ and $A \subset K(B, \varepsilon)\}$, with $K(A, \varepsilon)$ denoting the open ε-neighbourhood of A in X. A map $f\colon X \to Y$ into a continuum Y generates a map $\hat{f}\colon C(X) \to C(Y)$, usually called the *map induced by* f, given by the formula $\hat{f}(A) = f(A)$. We introduce a terminology connected with a given hyperspace $C(X)$. The continuum X is, in a sense, a maximal point of $C(X)$ and is called the *vertex of* $C(X)$. By $\hat{X}$ we denote the subspace of $C(X)$ consisting of all singletons. It is called the *base of* $C(X)$. The base of $C(X)$ is isometric to X. For every two continua $A, B \in C(X)$ such that $A \subset B$ there exists a maximal monotone collection of continua between A and B, which forms an arc in $C(X)$ provided $A \neq B$. This collection is denoted by AB and is called a *segment from A to B*. In the case where A is a continuum consisting of a single point and $B = X$ the segment AB is said to be *maximal*. In [10] Whitney described a map μ, from $C(X)$ (where X is nondegenerate) onto the unit interval I, having the following properties:

(i) $\mu(X) = 1$,

(ii) if $A \subset B$ and $A \neq B$ then $\mu(A) < \mu(B)$,

(iii) $\mu(\{x\}) = 0$, for $x \in X$.

In the sequel every map with these properties will be called a *Whitney map*. If X is nondegenerate, then any Whitney map restricted to a maximal segment of $C(X)$ is a homeomorphism onto I.

In what follows the letter D denotes the unit closed circular disc, $D=\{z\in E^2\colon |z|\leqslant 1\}$, and S denotes the unit circle, $S=\{z\in E^2\colon |z|=1\}$. Thus S is the boundary of D.

1. ε-maps and the fixed point property. Given metric spaces X, Y and a number $\varepsilon>0$, we say that a map $f\colon X\to Y$ is an *ε-map* provided $\operatorname{diam} f^{-1}(y)<\varepsilon$ for $y\in Y$. The map is called *inessential*, $f\sim 1$, provided it is homotopic to a constant map. Otherwise f is *essential*, $f\not\sim 1$. If Y is a closed n-dimensional ball Q, then f is *essential in the sense of Alexandroff-Hopff*, shortly: AH-*essential*, if $f|f^{-1}(Q)\colon f^{-1}(\dot{Q})\to\dot{Q}$ cannot be extended onto X, where $\dot{Q}$ denotes the boundary of Q. If every map $f\colon X\to S$ is inessential, then X is said to be *contractible relative to* S, $\operatorname{cr} S$. In [5] Lokuciewski proved the following

THEOREM. *If X is a compact space and for every number $\varepsilon>0$ there exists an* AH-*essential ε-map of X into a closed n-dimensional ball, then X has the fixed point property.*

As an easy consequence of this theorem we obtain the following proposition

1.1. *If a continuum X is* $\operatorname{cr} S$ *and for every $\varepsilon>0$ there exists an ε-map $f\colon X\to D$ and a closed subset $A\subset f^{-1}(S)$ such that $f|A\colon A\to S$ is essential, then X has the fixed point property.*

1.2. *If $A, B\subset X$ and $f(B)\subset f(A)$ for an ε-map $f\colon X\to Y$, then $B\subset K(A,\varepsilon)$.*

In fact let $b\in B$. Then there exists a point $a\in A$ such that $f(a)=f(b)$. Hence $a, b\in f^{-1}(f(b))$ and $\operatorname{diam} f^{-1}(f(b))<\varepsilon$. Thus $b\in K(A,\varepsilon)$, which completes the proof.

1.3. ([4], § 56, VI). *Let A, B be two continua and K, L two arcs such that $A\cap B=\{a, b\}$ and $K\cap L=\{p, q\}$, where p and q are the end-points of both K and L. If $f\colon A\cup B\to K\cup L$ is such that*

$$f(A)\subset K,\quad f(B)\subset L \tag{1}$$

and

$$f(a)=p\quad \text{and}\quad f(b)=q, \tag{2}$$

then f is essential.

2. Properties of the hyperspace of a continuum.

2.1. *$C(X)$ is* $\operatorname{cr} S$ (see [7] and also [3]). *In particular, $C(X)$ is unicoherent.*

2.2. *A Whitney map $\mu\colon C(X)\to I$ is monotone and open* (comp. [2]).

In fact, the sets $\mu^{-1}([0,t])$ and $\mu^{-1}([t,1])$ are continua and $\mu^{-1}(t)=\mu^{-1}([0,t])\cap\mu^{-1}([t,1])$. Since $C(X)=\mu^{-1}(I)=\mu^{-1}([0,t])\cup\mu^{-1}([t,1])$, then by 2.1 we infer that $\mu^{-1}(t)$ is a continuum. Thus μ is monotone. To prove that μ is open let M be an open subset of $C(X)$ and let $t\in\mu(M)$. Hence, for some $A\in M$ we have $\mu(A)=t$. Let L be a maximal segment containing A. Then $L\cap M$ is an open subset of L containing A; therefore $\mu(L\cap M)\subset\mu(M)$ is an open subset of I containing t. This completes the proof.

2.3. *Let $\mu\colon C(X)\to I$ be a Whitney map and let $t\in I$. Then for every number $\varepsilon>0$ there exists a number $\eta>0$ such that, for every pair $A, B\in\mu^{-1}(t)$, if $B\subset K(A,\eta)$, then $\operatorname{dist}(A,B)<\varepsilon$.*

Proof. Suppose, on the contrary, that for each natural number n there exist continua $A_n, B_n\in\mu^{-1}(t)$ such that $B_n\subset K(A_n, 1/n)$ and $\operatorname{dist}(A_n, B_n)\geqslant\varepsilon$. We may assume that A_n's and B_n's converge to A and B respectively. Then A and B are continua and, by the continuity of μ, we have $A, B\in\mu^{-1}(t)$. By the continuity of the Hausdorff metric we also have $\operatorname{dist}(A,B)\geqslant\varepsilon$ and $B\subset A$. Then B is a proper subcontinuum of A and, by (ii), we obtain $t=\mu(B)<\mu(A)=t$, a contradiction which completes the proof.

2.4. *Let X and Y be continua and let Z be a closed subset of $C(X)$ which does not contain the vertex of $C(X)$. Then there exists a number $\eta>0$ such that for every η-map $f\colon X\to Y$ the set $\hat{f}(Z)$ does not contain the vertex of $C(Y)$, i.e. $f(A)\neq Y$ for $A\in Z$.*

Proof. Suppose that for each natural number n there exist an $1/n$-map $f_n\colon X\to Y$ and a point $A_n\in Z$ such that $f_n(A_n)=Y$. We may assume that A_n's converge to a continuum A. Then $A\in Z$ and therefore $A\neq X$; hence there is a point $b\in X\setminus A$. It follows that $\varepsilon=\min\{\varrho(b,x)\colon x\in A\}$ is a positive real number. Let n be so large an integer that

$$\operatorname{dist}(A, A_n)<\varepsilon/2\quad\text{and}\quad f_n \text{ is an } \varepsilon/2\text{-map}. \tag{1}$$

Since $f_n(b)$ is a point of $Y=f_n(A_n)$, there is a $c\in A_n$ such that $f_n(c)=f_n(b)$. By (1) we infer that $\varrho(b,c)<\varepsilon/2$ and there exists an $a\in A$ such that $\varrho(a,c)<\varepsilon/2$. Finally we obtain

$$\varepsilon=\min\{\varrho(b,x)\colon x\in A\}\leqslant\varrho(b,a)\leqslant\varrho(b,c)+\varrho(c,a)<\varepsilon,$$

a contradiction which finishes the proof.

2.5. *If $f\colon X\to Y$ is an ε-map, then the induced map $\hat{f}\colon C(X)\to C(Y)$ is also an ε-map.*

Proof. Let $A, B\in\hat{f}^{-1}(E)$, where $E\in C(Y)$. By the compactness of A and B and by symmetry it suffices to show that for a given point $b\in B$ there exists a point $a\in A$ such that $\varrho(a,b)<\varepsilon$. But $f(A)=f(B)$, and hence there exists a point $a\in A$ such that $f(a)=f(b)$. Thus $a, b\in f^{-1}(f(b))$ and $\operatorname{diam} f^{-1}(f(b))<\varepsilon$, which completes the proof.

3. Properties of snake-like and circle-like continua. A continuum X is *snake-like* provided that for every $\varepsilon > 0$ there exists an ε-map of X onto the unit interval I. Likewise, X is *circle-like* if for every $\varepsilon > 0$ there exists an ε-map of X onto S. A circle-like continuum which is not snake-like is called *proper circle-like*.

3.1. *If X is a circle-like continuum and for every number $\varepsilon > 0$ there exists an ε-map of X into S which is inessential, then X is snake-like.*

Proof. For a given $\varepsilon > 0$ we shall find an ε-map of X onto I. Without loss of generality we may assume that ε is so small that for every ε-map f of X into S the set $f(X)$ is nondegenerate. Let $f\colon X \to S$ be an inessential ε-map. By a well-known theorem of Eilenberg (see [4], p. 426) there is a real-valued map g of X such that

$$f(x) = e^{ig(x)} \quad \text{for} \quad x \in X .$$

It follows that $g(X)$ is a closed interval, and it is easy to see that $g\colon X \to g(X)$ is an ε-map. If h is a homeomorphism of $g(X)$ onto I, then $h \circ g$ is the desired ε-map of X onto I. This proves the theorem.

Theorem 3.1 implies the following one.

3.2. *If X is proper circle-like, then there exists a number $\varepsilon > 0$ such that every ε-map of X into S is essential.*

Easy examples show that there are continua which are simultaneously snake-like and circle-like. However, the class of such continua is limited by the following result.

3.3. *If X is simultaneously snake-like and circle-like, then X is either indecomposable or the union of two indecomposable continua.*

Since a snake-like continuum contains no triod, it is irreducible (see [6], p. 180). Hence 3.3 is contained in the following theorem, which results from [4], p. 422, but for which we shall give an independent proof.

3.4. *An irreducible circle-like continuum X is either indecomposable or the union of two indecomposable continua.*

Proof. Let X be irreducible between a and b. Suppose X is decomposable, i.e. there are continua A and B such that

$$X = A \cup B , \quad A \neq X \neq B , \quad a \in A \quad \text{and} \quad b \in B .$$

Moreover, we may assume that each of the two continua A and B is the closure of the complement of the other, i.e.

$$(1) \qquad A = \overline{X \setminus B} \quad \text{and} \quad B = \overline{X \setminus A} .$$

Indeed, otherwise the continua $\overline{X \setminus B}$ and $\overline{X \setminus \overline{X \setminus B}}$ constitute a decomposition of X with the above property (see [4], § 48).

By symmetry we need only to show that A is indecomposable. Suppose, on the contrary, that there are continua C and E such that

$$(2) \qquad A = C \cup E , \quad C \neq A \neq E \quad \text{and} \quad a \in C .$$

Then

$$(3) \qquad B \cap C = \emptyset ,$$

for otherwise, by the irreducibility of X, we have $X = B \cup C$. This, by (1), implies that $A \subset C$, which contradicts (2).

Since $a \in C$ and $b \in B$, we have

$$(4) \qquad F = X \setminus (B \cup C) \quad \text{is connected}$$

see [4], § 48, II, Th. 4). By the assumption on X and by (3), there exist an ε-map f of X onto S and two nonvoid disjoint and open sets $U, V \subset S$ such that

$$F \subset f^{-1}(U) \cup f^{-1}(V) \quad \text{and} \quad F \cap f^{-1}(U) \neq \emptyset \neq F \cap f^{-1}(V) ,$$

which contradicts (4). This completes the proof.

4. The fixed point property for the hyperspaces of snake-like and circle-like continua. In [9] Segal proved that the hyperspace of a snake-like continuum has the fixed point property. However, his argument was long and complicated. In this section we shall give a shorter proof of this theorem. We shall also prove that the hyperspaces of circle-like continua possess this property. These results were independently obtained by J. T. Rogers [8].

4.1. (Segal). *The hyperspace $C(X)$ of a snake-like continuum X has the fixed point property.*

Proof. Let $\varepsilon > 0$ be a given real number. To prove the theorem we need only, by 1.1, to find an ε-map g of $C(X)$ into D and a closed subset $A \subset g^{-1}(S)$ such that $g|A$ maps essentially A onto S. Let f be an ε-map of X onto I and let $a \in f^{-1}(0)$ and $b \in f^{-1}(1)$. Then, by 2.5, the induced map $\hat{f}\colon C(X) \to C(I)$ is an ε-map. Let $M = \{a\}X \cup \{b\}X$, where $\{a\}X$ and $\{b\}X$ are two maximal segments in $C(X)$, and let $N = \{E \in C(I)\colon 0 \in E$ or $1 \in E\}$. It is easy to see that N is an arc and $N \cup \hat{I}$ is the boundary of $C(I)$, where $\hat{I}$ is the base of $C(I)$. Let us set

$$g = h \circ \hat{f} \quad \text{and} \quad A = \hat{X} \cup M ,$$

where h is a homeomorphism of the pair $(C(I), N \cup \hat{I})$ onto (D, S), and $\hat{X}$ is the base of $C(X)$. Hence g is an ε-map. Since $\hat{X} \cap M = \{\{a\}, \{b\}\}$, $N \cap \hat{I} = \{\{0\}, \{1\}\}$, $\hat{f}(\{a\}) = \{0\}$, $\hat{f}(\{b\}) = \{1\}$, $\hat{f}(\hat{X}) = \hat{I}$ and $\hat{f}(M) = N$, then, using 1.3, we infer that $\hat{f}|A$ is essential. It follows that $g|A$ is an essential map, which completes the proof.

4.2. *If X is a circle-like continuum, then $C(X)$ has the fixed point property.*

Proof. By 4.1 we may assume that X is proper circle-like. For a given number $\varepsilon > 0$ there exists, by 3.2, an essential ε-map f of X onto S. It follows from 2.5 that the induced map $\hat{f}$: $C(X) \to C(S)$ is an ε-map. Moreover, the map $\hat{f}|\hat{X}$: $\hat{X} \to \hat{S}$ is essential. There exists a homeomorphism h of the pair $(C(S), \hat{S})$ onto (D, S). Let us set $g = h \circ \hat{f}$ and $A = \hat{X}$. Then g is an ε-map of $C(X)$ into D and its restriction to A is an essential map of A onto S. Hence our theorem follows from 1.1.

5. A separation theorem. In this section we shall prove the following result.

5.1. *If X is a proper circle-like continuum and A is a closed subset of $C(X)$ which separates $C(X)$ and is such that $A \cap \hat{X}$ is connected, then there exists an essential map of A onto S.*

Proof. Let $C(X) \setminus A = M \cup N$, where M and N are nonvoid and separated. Since X is proper circle-like, then the set $\hat{X} \setminus A$ is connected. Hence we may assume that

(1) $$\hat{X} \subset M \cup A .$$

Since $M \cup A$ is a proper closed subset of $C(X)$, then, by 3.2 and 2.5, there exists an essential map f of X onto S such that $\hat{f}(M \cup A)$ is a proper subset of $C(S)$. Let h be a homeomorphism of the pair $(C(S), \hat{S})$ onto (D, S) and let $F = h \circ \hat{f}(M \cup A)$. Then F is a proper subset of D and therefore there is a point p of D in the complement of $F \cup S$ and a retraction

$$r\colon D \setminus \{p\} \to S .$$

We assert that the map g: $A \to S$ defined by the formula

$$g = r \circ h \circ \hat{f}|A$$

maps essentially A onto S. Suppose, on the contrary, that $g \sim 1$. Using the Borsuk homotopy extension theorem (see [4], p. 365), we extend g to g_1: $N \cup A \to S$. Then g_1 and $r \circ h \circ \hat{f}|M \cup A$ form together a map k of $C(X)$ into S. By 2.1 we have $k \sim 1$, and therefore, by (1), we obtain $\hat{f}|\hat{X} = k|\hat{X} \sim 1$. It follows that $f \sim 1$, a contradiction. This completes the proof.

6. Decompositions of hyperspaces by means of a Whitney map. Let X be a nondegenerate continuum and let μ be a Whitney map on $C(X)$. For each $t \in I$ let X_t denote the set $\mu^{-1}(t)$. Using 2.2, we obtain the following theorem.

6.1. *The collection $\mathcal{E} = \{X_t\colon t \in I\}$ is a monotone and continuous decomposition of $C(X)$ with the quotient space I_μ being an arc which has as its end-points X_0 (the vertex of $C(X)$) and X_1 (the base of $C(X)$). In particular, each element of $\mathcal{E}$ distinct from X_0 and X_1 separates $C(X)$ between the vertex and the base of $C(X)$.*

Moreover, it turns out that the following theorem holds.

6.2. (a) *If X is a snake-like continuum and $0 \leqslant t < 1$, then X_t is a snake-like continuum.*

(b) *If X is a proper circle-like continuum and $0 \leqslant t < 1$, then X_t is a proper circle-like continuum.*

Proof. The proof will be given in the case (b) only, because in the case (a) the proof is similar.

Let $\varepsilon = 2\varepsilon'$ be a given positive real number. We shall construct an essential ε-map g of X_t onto S. Let η_0 be a real satisfying 2.4, where we substitute X_t for Z. According to 2.3 there exists a positive number $\eta < \eta_0$ such that, for each $A, B \in X_t$,

(1) $$B \subset K(A, \eta) \Rightarrow \operatorname{dist}(A, B) < \varepsilon'.$$

By 3.2 there is an essential η-map f: $X \to S$. By the definition of η_0 we infer that $f(A) \neq S$ for every $A \in X_t$; hence $f(A)$ is an arc (or a single point) and therefore the centre of $f(A)$ is well defined. The map g: $X_t \to S$ is defined as follows: $g(A) =$ the centre of $f(A)$. By the geometry of S, g is continuous. We shall show that g is an ε-map.

Let $s \in S$. Let $H = \hat{f}(g^{-1}(s))$, where $\hat{f}$ is the map induced by f. Then H is a closed subset of $C(S)$ and each element of H is a point or an arc with the centre s. Therefore we have

(2) $$C, F \in H \Rightarrow C \subset F \text{ or } F \subset C .$$

Let μ' be a Whitney map on $C(S)$. Since $\mu'(H)$ is closed, there is a maximal number in $\mu'(H)$, say r. Let $A' \in H$ be such that $\mu'(A') = r$. By (2) we obtain

(3) $$F \in H \Rightarrow F \subset A'.$$

There is an $A \in g^{-1}(s)$ such that $f(A) = A'$. Let $B \in g^{-1}(s)$. Then $\hat{f}(B) \in H$ and by (3) we obtain $f(B) \subset A' = f(A)$. Since f is an η-map, by 1.2 we have $B \subset K(A, \eta)$. It follows from (1) that $\operatorname{dist}(A, B) < \varepsilon'$. Thus we have shown that $g^{-1}(s) \subset K(A, \varepsilon')$, where $K(A, \varepsilon')$ denotes an open ball in $C(X)$ with centre at A. Therefore $\operatorname{diam} g^{-1}(s) < 2\varepsilon' = \varepsilon$, which shows that g is an ε-map.

Suppose that g is an inessential map. Using the Borsuk homotopy extension theorem, [4], p. 365, we may extend g to a map

$$g_1\colon \mu^{-1}([t, 1]) \to S .$$

On the other hand, f may be regarded as a map of $\hat{X}$ (the base of $C(X)$), and it follows from the definition of g that it can be extended to a map

$$g_2\colon \mu^{-1}([0, t]) \to S$$

such that

$$g_2 | \hat{X} = f .$$

The maps g_1 and g_2 form together a map h of $C(X)$ into S. But by 2.1 we obtain $h \sim 1$, and therefore $f = h|\hat{X} \sim 1$, contrary to the choice of f. Hence g is essential, which completes the proof.

If X is a hereditarily indecomposable continuum, then for each $t \in I$ there exists a monotone map of X onto X_t [2]. It follows that, in this case, X_t is a hereditarily indecomposable continuum. Bing [1] proved that a hereditarily indecomposable snake-like continuum is a pseudoarc. Thus the above results imply the following corollary.

6.3. *If X is a pseudoarc, then $\mathcal{E} = \{X_t\colon t \in I\}$ is a monotone and continuous decomposition of $C(X)$ such that each element X_t for $t < 1$ is a pseudoarc which separates $C(X)$ between the vertex and the base of $C(X)$ provided $t > 0$* (comp. [2]).

Remark. In [8] J. T. Rogers showed that if X is a circle-like (snake-like) continuum and U is an open subset of $C(X)$ which contains the vertex of $C(X)$, then there exists a circle-like (snake-like) continuum in U which separates $C(X)$ between the vertex and the base of $C(X)$. Hence, except the case where X is simultaneously snake-like and circle-like, this theorem is a corollary of 6.1 and 6.2.

Using 6.2 we shall show that in the case where X is a circle or an arc the above decomposition has further interesting properties. Namely we have the following result.

6.4. (a) *If X is the unit interval I and $0 \leqslant r < 1$, then X_t is an arc.*

(b) *If X is the unit circle S and $0 \leqslant t < 1$, then X_t is a simple closed curve.*

Proof. Case (a). By 6.2 the continuum X_t is snake-like. Since every snake-like continuum is irreducible (see § 3), we need only to prove that X_t is locally connected. We identify $\hat{X}$ with X. Let us assign to a proper closed interval $[u_1, u_2]$, $u_1 \leqslant u_2$, of X a point $r([u_1, u_2])$ given by the formula:

$$r([u_1, u_2]) = \frac{u_1}{1-(u_2-u_1)} .$$

Then

$$r\colon C(X) \setminus \{X\} \to X$$

is a continuous retraction such that $r(A) \in A$. An easy computation shows that for every $u \in X$ the set $L_u = r^{-1}(u) \cup \{X\}$ is a maximal segment in $C(X)$. This can easily be deduced from the following observation:

$$r^{-1}(u) = \{[u(1-s), s+u(1-s)] \in C(X)\colon 0 \leqslant s < 1\} .$$

Assuming that X_t is not locally connected, we infer that there exist a nondegenerate continuum D and a sequence of continua $D_1, D_2, \ldots$ (in X_t) converging to D each of which is disjoint with D. For every $u \in X$ the set $L_u \cap X_t$ consists of a single point, and hence D intersects two distinct segments L_{u_1} and L_{u_2} with $u_1 < u_2$. Let $u_1 < u < u_2$. Since $t < 1$, the segment L_u separates $C(X)$ between two distinct point of D, and therefore D intersects L_u. On the other hand, since $\lim_n D_n = D$, there exists a positive integer n such that L_u separates $C(X)$ between two points of D_n. In particular, D_n intersects L_u. Since D and D_n are disjoint subsets of X_t, it follows that L_u has at least two distinct points in common with X_t, a contradiction which completes the proof.

Case (b). Again we identify $\hat{X}$ with X. For every $A \in C(X) \setminus \{X\}$ let $r(A)$ be the centre of A (if A is a point then $r(A) = A$). Then

$$r\colon C(X) \setminus \{X\} \to X$$

is a continuous retraction. For each $x \in X$ let $L_x = r^{-1}(x) \cup \{X\}$. It is easy to see that each L_x is a maximal segment. Now we shall prove that

$$X_t \text{ is locally connected .} \tag{1}$$

Suppose that this is not true. Then there exist in X_t a nondegenerate continuum D and a sequence of continua $D_1, D_2, \ldots$ with the same properties as in the proof above. For every $x \in X$ the set $L_x \cap X_t$ consists of a single point; hence there exist two distinct points x_1 and x_2 in X and two continua A and B such that $D \cap L_{x_1} = \{A\}$ and $D \cap L_{x_2} = \{B\}$. Let y_1 and y_2 be two points of X which separate X between x_1 and x_2. Then $L = L_{y_1} \cup L_{y_2}$ separates $C(X)$ between A and B, i.e. there are in $C(X)$ two disjoint open sets U and V such that

$$C(X) \setminus L = U \cup V, \quad A \in U \quad \text{and} \quad B \in V. \tag{2}$$

Since $A, B \in D$, $D = \lim_n D_n$ and $D \cap D_n = \emptyset$, there exist three pairwise disjoint continua $D_{n_1}, D_{n_2}, D_{n_3}$ such that

$$D_{n_i} \cap U \neq \emptyset \neq D_{n_i} \cap V$$

for $i = 1, 2, 3$. It follows from (2) that D_{n_i} intersects L for $i = 1, 2, 3$. Hence either L_{y_1} or L_{y_2} intersects two continua D_{n_i}. But D_{n_i} is a subset of X_t, and hence one of the L_{y_i}'s has at least two distinct points in common with X_t, a contradiction which proves (1).

According to 6.2 (b) there is an essential map $f\colon X_t \to S$. However, for every closed subset F of X_t distinct from X_t we have $f|F \sim 1$. Indeed, if $f|F \nsim 1$, then, by [4], p. 425, there is a continuum $C \subset F$ such that $f|C \nsim 1$. Since C is distinct from X_t, C is a snake-like continuum. It follows that $f|C \sim 1$, a contradiction. Thus f irr $\nsim 1$. Therefore, by (1) and by [4], p. 421, the continuum X_t is a simple closed curve. This completes the proof of (b).

PROBLEM. Let X be a plane circle-like continuum and let μ be a Whitney map on $C(X)$. Can $\mu^{-1}(t)$ be embedded in the plane for each $t \in I$?

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Measures on bundles and bundles of measures

by

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Abstract. Various measures on bundles are characterised as a measure on the base space and a measurable section of an associated bundle of measures. In addition, machinery is developed to permit very general constructions of measures in this fashion. Various applications are given, including a new characterization of a method of integration sections of vector bundles.

I. Introduction. Measures on fibre bundles and associated structures have been studied in a variety of settings. For example, Goetz [G] constructed a product measure on a bundle, given Baire measures on the base and fibre with the measure on the fibre being translation invariant under the group of the bundle. He showed that this product measure deserved its name by proving a form of Fubini's theorem. In a similar fashion, Brothers [B; 3.3] makes a construction that lifts a current on the base of a bundle with prescribed fibre and group and which satisfies the conditions that on a product bundle the lifted current is the product of the current and the fibre and that the construction be natural with respect to bundle maps. It follows that such a lifting is unique. If one asks for such a lifting of measures with respect to a preassigned measure on the fibre satisfying Brother's conditions then, provided the base space of the bundle contains a measurable set with positive and finite measure, a necessary and sufficient condition is group invariance of the measure. To accomplish such constructions without group invariance of the measure on the fibre is still possible provided there is some compatibility between the group and measure. However, these liftings are not unique.

In another setting, Allard [A] studies the variational properties of a varifold. A varifold, introduced by Almgren, is defined as a Radon measure on a fibre bundle over a manifold with compact fibre. In Section III, we will characterize Radon measures on bundles with compact fibre as a measure on the base and a measurably varying Radon measure on each fibre. A form of this result for varifolds is in [A]. However, the proof these utilizes the intrinsic geometry of the varifold and various differentiation techniques in [F].