On the shape of 0-dimensional paracompacta

by

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Abstract. It is shown that if $X$ and $Y$ are 0-dimensional paracompacta, then they are of the same shape if and only if they are homeomorphic and that a 0-dimensional paracompactum which is not compact is not the shape of any compact space. These results are essentially consequences of the fact that any fundamental class from any space to a 0-dimensional paracompactum has a unique realization as a map.

M. Mazurkiewicz raised the question about the relationship between the shape of a space and its compactification. Here we show

**Theorem 1.** If $X$ and $Y$ are 0-dimensional paracompacta, then $X$ and $Y$ have the same shape if and only if $X$ and $Y$ are homeomorphic.

This is a generalization of the analogous result for compacta [4, Theorem 20]. Dimension here means covering dimension, and a paracompactum is a paracompact Hausdorff space. Since the Stone-Čech compactification $\beta X$ of a normal space has the same covering dimension as $X$ (see e.g., [3, Theorem 9.5]), Theorem 1 implies that no noncompact 0-dimensional paracompactum has the shape of its Stone-Čech compactification. In fact, a stronger result is true.

**Theorem 2.** No noncompact 0-dimensional paracompactum has the shape of any compact space. In fact, the shape of such a space is not dominated by the shape of any compact space (**).

We shall discuss shape and then inverse systems involving certain open covers of a 0-dimensional paracompactum, after which we prove the following lemma which implies that a continuous map uniquely realizes any morphism in the shape category whose range is a 0-dimensional paracompactum.

**Lemma.** If $Z$ is a 0-dimensional paracompactum and $X$ is any space, then for any natural transformation $F: \pi_Z \Rightarrow \pi_X$ there is a unique map $f: X \to Z$ such that $f \circ \pi = F$.

A polyhedron is the underlying space of a (not necessarily finite) simplicial complex. Let $P$ be the category of polyhedra and homotopy

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(**) S. Maciejewski has obtained an analogous result.
classes of continuous maps between them. If \( X \) is a (topological) space, then \( \pi_X \) is the functor from \( P \) to the category of sets and functions which assigns to a polyhedron \( P \) the set \( \pi_X(P) = \{ X \to P \} \) of all homotopy classes of maps of \( X \) into \( P \) and which assigns to any homotopy class \( \lambda : P \to Q \) between polyhedra the induced function \( h(\lambda): \{ X \to P \} \to \{ X \to Q \} \) which maps the homotopy class \( \phi : X \to P \) into the composition \( h(\lambda) \circ \phi \) of the homotopy classes of \( \lambda \) and \( \phi \). A natural transformation \( G \) of the functor \( \pi_X \) into the functor \( \pi_Y \) assigns to each homotopy class \( \phi : X \to P \) a homotopy class \( Y \to P \) in such a way that for all homotopy classes \( \phi : X \to P \), \( \psi : Y \to Q \), and \( \lambda : P \to Q \), we have \( G(\lambda) \circ \phi = \psi \circ h(\lambda) \). If \( f : X \to Y \) is a map, then there is a natural transformation \( f^* : \pi_Y \to \pi_X \); which assigns to the homotopy class \( \phi : Y \to P \) the composition \( k(f) \circ \phi \) of the homotopy class \( k(f) \) of \( f \) with \( \phi \). The natural transformations from \( \pi_Y \) to \( \pi_X \) correspond to the fundamental classes from \( X \) to \( Y \) in Borsuk's theory of shape.

Given two spaces \( X \) and \( Y \) we say that the \textit{shape of} \( X \) \textit{dominates the shape of} \( Y \) if and only if there are natural transformations \( F : \pi_Y \to \pi_X \) and \( G : \pi_X \to \pi_Y \) such that \( GE = 1_X \). If, in addition, \( FG = 1_Y \), then \( X \) and \( Y \) are said to be of the \textit{same shape}. In other words, \( X \) and \( Y \) have the same shape if and only if there is an invertible natural transformation \( \eta \) (i.e., a natural equivalence) of the functors \( \pi_Y \) and \( \pi_X \).

It will be shown elsewhere by the first-named author, that this notion of shape coincides with that of Borsuk for compacta, of Mardešić–Segal for compact Hausdorff spaces, and of Fox for metric spaces. (Similar results have been obtained independently by S. Mardešić [5] and [6].)

An open cover of a space \( Z \) is discrete, if its members are nonempty and pairwise disjoint. A space \( Z \) is \( 0 \)-dimensional (in the sense of covering dimension) if and only if every finite open cover has a discrete open refinement; it is a standard theorem that a paracompact space is \( 0 \)-dimensional if and only if every open cover has a discrete open refinement [3, Cor. 9–14]. A discrete open cover of the space \( Z \) is considered as a set of subsets of \( Z \), and all the discrete open covers of \( Z \) comprise a set \( D \).

We shall now show that if \( Z \) is a \( 0 \)-dimensional paracompact space, then \( Z \) is the inverse limit of a family of discrete spaces. If \( U \subseteq D \), \( U \) will be also considered as a discrete topological space. There is a map \( \varphi_U : Z \to U \) which assigns to each \( z \in Z \) the unique member of \( U \) which contains \( z \). If \( U, V \subseteq D \), and if \( V \) refines \( U \), then there is a unique map \( \varphi_{UV} : V \to U \) with the property that each member of \( V \) is contained in its image under \( \varphi_{UV} \). Note that \( \varphi_U = \varphi_{UU} \).

If \( A \) is closed in \( Z \) and \( z \in Z - A \), then there is \( U \subseteq D \) such that \( \varphi_{UV}(z) \neq \varphi_{U}(A) \). It follows [3, 4.5 Embedding Lemma] that the map \( \varphi \) of \( Z \) defined by the family \( \{ \varphi_U : U \subseteq D \} \) is a homeomorphism into the product \( \Pi(U \subseteq D) \) of the system \( \{ \varphi_U : U \subseteq D \} \) whose members are functions \( \lambda \) defined on \( D \) with the properties \( \lambda(U \subseteq D) \) and if \( V \) refines \( U \), then \( \lambda(U \subseteq D) \subseteq \lambda(U \subseteq D) \). If \( z \in U \subseteq D \), then \( \varphi(z) = \lambda \). To see that \( \varphi \) maps onto \( Z \), let \( \lambda \in \Lambda \), and suppose \( \{ \lambda(U \subseteq D) \} \subseteq \Theta \). Then \( \{ \lambda(U \subseteq D) \} \subseteq \Theta \) is an open cover of \( Z \) and consequently has a refinement \( \lambda(U \subseteq D) \subseteq \Theta \). This implies that \( \lambda(V) \subseteq \Lambda \subseteq \lambda(U \subseteq D) \) for some \( U \subseteq D \). If \( W \subseteq D \) is a refinement of both \( U \) and \( V \), then \( \lambda(W) \subseteq \lambda(U \subseteq D) \subseteq \Theta \), which is impossible. The result is that the natural map \( \varphi \) defines a homeomorphism of \( Z \) onto the inverse limit \( \Lambda \).

Proof of Lemma. For each \( U \subseteq D \) the space \( U \) is a \( 0 \)-dimensional polyhedron, hence any homotopy class of a space into \( U \) consists of exactly one map. Let \( f_U : X \to U \) be the map in the class \( \pi_U(\varphi_U) \), and note that for a refinement \( \lambda \subseteq \Lambda \) of \( U \) we have \( f_U = \varphi_{UV} f_{\lambda(U \subseteq D)} \). Since \( Z \) is the inverse limit of the system \( \{ U \subseteq D \} \), there is a unique map \( f : X \to Z \) such that \( \varphi_{UV} f_{\lambda(U \subseteq D)} \). These statements also give the uniqueness assertion of the lemma.

To establish that \( f^* = f \) consider any \( \psi : Z \to P \) of \( Z \) into a polyhedron \( P \). Let \( U \subseteq D \) refine the open cover of \( X \) consisting of the sets \( p^{-1}(U) \), where \( S \) ranges over the open stars of all vertices of \( P \). Let \( g : U \subseteq D \) be the map which assigns to each member of \( U \) a vertex of \( P \) whose open star contains the image under \( f \) of that member. The restrictions of \( \psi \) and of \( \varphi_U \) to any member of \( U \) map into the same open star, which is a contractible set. Since the restrictions of \( \psi \) and \( \varphi_U \) to each member of a discrete open cover are homotopic, \( [\psi_U] = [\varphi_U] \). Because \( F \) is a natural transformation, \( F[\psi_U] = f^* [\varphi_U] = [\varphi_{UV}] = f^* [\varphi_{UV}] \).

Proof of Theorem 1. If \( f : X \to Y \) is a homoeomorphism with inverse \( f^{-1} : Y \to X \), then \( f^* = f^{-1} \) is a natural transformation with inverse \( f^{-1} f^* \), hence \( X \) and \( Y \) have the same shape.

If there are natural transformations \( F : \pi_Y \to \pi_X \) and \( G : \pi_X \to \pi_Y \) whose compositions satisfy \( G F = 1_X \) and \( F G = 1_Y \), then by the Lemma there are maps \( f : X \to Y \) and \( g : Y \to X \) for which \( f^* = f \) and \( g^* = G \). Since \( G \) is a homoeomorphism, \( [f^*] = [g^*] = [1_X] \), the Lemma implies that \( g f = f^* g = 1_X \); similarly \( f = 1_Y \).

Proof of Theorem 2. Let \( Z \) be a noncompact \( 0 \)-dimensional paracompact space, and let \( X \) be a compact space. We shall show that for every natural transformation \( F : \pi_X \to \pi_Z \) there are distinct homotopy classes of \( [F] \in \pi_Z \) which are mapped by \( F \) to a single class in \( \pi_X \). By the Lemma there is a map \( f : X \to Z \) such that \( f \circ f = f \). Since \( Z \) is not compact, there are distinct points \( x_i \in X \), \( 1 \leq i \leq n \), such that \( f(x_i) = 1 \) for \( i = 1, 2 \). If \( y_i \) maps \( f(X_i) \) to \( 1 \) and \( Z \setminus f(X_i) \) to \( -1 \) for \( i = 1, 2 \), then \( y_1 \) and \( y_2 \) are maps from \( Z \to \pi = (1, -1) \).
which are not homotopic but which satisfy \( y_0 f = y_0 g \); hence \( F[y_0] = F[y_g] \). Now if there were a natural transformation \( G: \pi_1 \to \pi_2 \) satisfying \( GF = \mathbb{1}_F \), then \( F \) would map \([Z, S']\) injectively to \([X, S']\). Hence the shape of \( Z \) is not dominated by the shape of \( X \).

References


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On the hyperspaces of snake-like and circle-like continua

by

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Abstract. J. Segal has proved a theorem which says that the hyperspace of a snake-like continuum has the fixed point property. In this paper we give a shorter proof of this theorem and we prove also that the hyperspace of a circle-like continuum has this property. The structure of these hyperspaces is studied and it is shown that the Whitney maps induce interesting decompositions of these hyperspaces.

0. Introduction. By a map we mean a continuous function. The term \textit{continuum} means a compact connected metric space. If \( X \) is a continuum, then \( C(X) \) denotes the hyperspace of subcontinua of \( X \) with the Hausdorff metric: \( \text{dist}(A, B) = \inf \{ \varepsilon > 0 : B \subseteq K(A, \varepsilon) \text{ and } A \subseteq K(B, \varepsilon) \} \), with \( K(A, \varepsilon) \) denoting the open \( \varepsilon \)-neighbourhood of \( A \) in \( X \). A map \( f: X \to Y \) into a continuum \( Y \) generates a map \( \tilde{f}: C(X) \to C(Y) \), usually called the \textit{map induced by} \( f \), given by the formula \( \tilde{f}(A) = f(A) \).

We introduce a terminology connected with a given hyperspace \( C(X) \). The continuum \( X \) is, in a sense, a maximal point of \( C(X) \) and is called the \textit{vertex} of \( C(X) \). By \( \hat{X} \) we denote the subspace of \( C(X) \) consisting of all singletons. It is called the \textit{base} of \( C(X) \). The base of \( C(X) \) is isometric to \( X \). For every two continua \( A, B \subseteq C(X) \) such that \( A \cap B \neq \emptyset \), there exists a maximal monotone collection of continua between \( A \) and \( B \) which forms an arc in \( C(X) \) provided \( A \neq B \). This collection is denoted by \( AB \) and is called a \textit{segment} from \( A \) to \( B \). In the case where \( A \) is a continuum consisting of a single point and \( B = X \), the segment \( AB \) is said to be \textit{maximal}. In [10] Whitney described a map \( \mu \), from \( C(X) \) (where \( X \) is nondegenerate) onto the unit interval \( I \), having the following properties:

(i) \( \mu(X) = 1 \),
(ii) if \( A \subseteq C(B) \text{ and } A \neq B \text{ then } \mu(A) < \mu(B) \),
(iii) \( \mu(\emptyset) = 0 \), for \( x \in X \).

In the sequel every map with these properties will be called a \textit{Whitney map}. If \( X \) is nondegenerate, then any Whitney map restricted to a maximal segment of \( C(X) \) is a homeomorphism onto \( I \).