On the shape of 0-dimensional paracompacta

by

G. Kozlowski and **J. Segal** (*) (Seattle, Washington)

Abstract. It is shown that if X and Y are 0-dimensional paracompacta, then they are of the same shape if and only if they are homeomorphic and that a 0-dimensional paracompactum which is not compact is not the shape of any compact space. These results are essentially consequences of the fact that any fundamental class from any space to a 0-dimensional paracompactum has a unique realization as a map.

M. Moszyńska raised the question about the relationship between the shape of a space and its compactification. Here we show

THEOREM 1. *If X and Y are 0-dimensional paracompacta, then X and Y have the same shape if and only if X and Y are homeomorphic.*

This is a generalization of the analogous result for compacta [4, Theorem 20]. Dimension here means covering dimension, and a paracompactum is a paracompact Hausdorff space. Since the Stone-Čech compactification βX of a normal space has the same covering dimension as X (see e.g., [3, Theorem 9.5]), Theorem 1 implies that no noncompact 0-dimensional paracompactum has the shape of its Stone-Čech compactification. In fact, a stronger result is true.

THEOREM 2. *No noncompact 0-dimensional paracompactum has the shape of any compact space. In fact, the shape of such a space is not dominated by the shape of any compact space* (**).

We shall discuss shape and then inverse systems involving certain open covers of a 0-dimensional paracompactum, after which we prove the following lemma which implies that a continuous map uniquely realizes any morphism in the shape category whose range is a 0-dimensional paracompactum.

LEMMA. *If Z is a 0-dimensional paracompactum and X is any space, then for any natural transformation F: $\pi_Z \to \pi_X$ there is a unique map f: $X \to Z$ such that $f^{\#} = F$.*

A polyhedron is the underlying space of a (not necessarily finite) simplicial complex. Let $\boldsymbol{P}$ be the category of polyhedra and homotopy

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(**) S. Mardešić has obtained an analogous result.

classes of continuous maps between them. If X is a (topological) space, then π_X is the functor from $\boldsymbol{P}$ to the category of sets and functions which assigns to a polyhedron P the set $\pi_X(P) = [X; P]$ of all homotopy classes of maps of X into P and which assigns to any homotopy class $h\colon P \to Q$ between polyhedra the induced function $h_{\#}\colon [X; P] \to [X; Q]$ which maps the homotopy class $\varphi\colon X \to P$ into the composition $h\varphi = \varphi_{\#}(h)$ of the homotopy classes of h and φ. A natural transformation G of the functor π_X into the functor π_Y assigns to each homotopy class $\varphi\colon X \to P$ a homotopy class $G(\varphi)\colon Y \to P$ in such a way that for all homotopy classes $\varphi\colon X \to P$, $\psi\colon X \to Q$, and $h\colon P \to Q$ such that $h\varphi = \psi$ we have $hG(\varphi) = G(\psi)$. If $f\colon X \to Y$ is a map, then there is a natural transformation $f^{\#}\colon \pi_Y \to \pi_X$ which assigns to the homotopy class $h\colon Y \to P$ the composition $h[f] = f^{\#}(h)$ of the homotopy class $[f]$ of f with h. (The natural transformations from π_Y to π_X correspond to the fundamental classes from X to Y in Borsuk's theory of shape.)

Given two spaces X and Y we say that the *shape of X dominates the shape of Y* if and only if there are natural transformations $F\colon \pi_Y \to \pi_X$ and $G\colon \pi_X \to \pi_Y$ such that $GF = 1_Y^{\#}$. If, in addition, $FG = 1_X^{\#}$, then X and Y are said to be of the *same shape*. In other words, X and Y have the same shape if and only if there is an invertible natural transformation (i.e., a natural equivalence) of the functors π_X and π_Y.

It will be shown elsewhere by the first-named author, that this notion of shape coincides with that of Borsuk for compacta, of Mardešić-Segal for compact Hausdorff spaces, and of Fox for metric spaces. (Similar results have been obtained independently by S. Mardešić [5] and [6].)

An open cover of a space Z is discrete, if its members are nonempty and pairwise disjoint. A space Z is 0-dimensional (in the sense of covering dimension) if and only if every finite open cover has a discrete open refinement; it is a standard theorem that a paracompactum is 0-dimensional if and only if every open cover has a discrete open refinement [3, Cor. 9-14]. A discrete open cover of the space Z is considered as a set of subsets of Z, and all the discrete open covers of Z comprise a set D.

We shall now show that if Z is a 0-dimensional paracompactum, then Z is the inverse limit of a family of discrete spaces. If $U \in D$, U will be also considered as a discrete topological space. There is a map $\varphi_U\colon Z \to U$ which assigns to each $z \in Z$ the unique member of U which contains z. If $U, V \in D$, and if V refines U, then there is a unique map $\varphi_{UV}\colon V \to U$ with the property that each member of V is contained in its image under φ_{UV}. Note that $\varphi_U = \varphi_{UV}\varphi_V$. If A is closed in Z and $z \in Z \setminus A$, then there is $U \in D$ such that $\varphi_U(z) \notin \varphi_U(A)$. It follows [1, 4.5 Embedding Lemma] that the map φ of Z defined by the family $\{\varphi_U \mid U \in D\}$ is a homeomorphism into the product $\Pi\{U \mid U \in D\}$. The image of φ is contained in the inverse limit L of the system $\{U, \varphi_{UV}; U \in D\}$ whose members are functions λ defined on D with the properties $\lambda(U) \in U$ and if V refines U, then $\lambda V \subset \lambda U$. If $\lambda \in L$ and $z \in \bigcap \{\lambda(U) \mid U \in D\}$, then $\varphi(z) = \lambda$. To see that φ maps onto L, let $\lambda \in L$, and suppose $\bigcap \{\lambda(U) \mid U \in D\} = \emptyset$. Then $\{Z \setminus \lambda(U) \mid U \in D\}$ is an open cover of Z and consequently has a refinement $V \in D$. This implies that $\lambda(V) \subset Z \setminus \lambda(U)$ for some $U \in D$. If $W \in D$ is a refinement of both U and V, then $\lambda(W) \subset \lambda(U) \cap \lambda(V) = \emptyset$, which is impossible. The result is that the natural map φ defines a homeomorphism of Z onto the inverse limit L.

Proof of lemma. For each $U \in D$ the space U is a 0-dimensional polyhedron; hence any homotopy class of a space into U consists of exactly one map. Let $f_U\colon X \to U$ be the map in the class $F[\varphi_U]$, and note that for a refinement $V \in D$ of U we have $f_U = \varphi_{UV} f_V$. Since Z is the inverse limit of the system $\{U, \varphi_{UV}; U \in D\}$, there is a unique map $f\colon X \to Z$ such that $\varphi_U f = f_V$ for every $U \in D$. These statements also give the uniqueness assertion of the lemma.

To establish that $f^{\#} = F$ consider any $\psi\colon Z \to P$ of Z into a polyhedron P. Let $U \in D$ refine the open cover of X consisting of the sets $\psi^{-1}(S)$, where S ranges over the open stars of all vertices of P. Let $g\colon U \to P$ be the map which assigns to each member of U a vertex of P whose open star contains the image under f of that member. The restrictions of ψ and of $g\varphi_U$ to any member of U map into the same open star, which is a contractible set. Since the restrictions of ψ and $g\varphi_U$ to each member of a discrete open cover are homotopic, $[\psi] = [g\varphi_U]$. Because F is a natural transformation, $F[\psi] = [g]F[\varphi_U] = [g\varphi_U f] = f^{\#}[g\varphi_U] = f^{\#}[\psi]$.

Proof of Theorem 1. If $f\colon X \to Y$ is a homeomorphism with inverse $f^{-1}\colon Y \to X$, then $f^{\#}$ is a natural transformation with inverse $(f^{-1})^{\#}$; hence X and Y have the same shape.

If there are natural transformations $F\colon \pi_Y \to \pi_X$ and $G\colon \pi_X \to \pi_Y$ whose compositions satisfy $GF = (1_Y)^{\#}$ and $FG = (1_X)^{\#}$, then by the Lemma there are maps $f\colon X \to Y$ and $g\colon Y \to X$ for which $f^{\#} = F$ and $g^{\#} = G$. Since $(gf)^{\#} = f^{\#}g^{\#} = (1_X)^{\#}$, the Lemma implies that $gf = 1_X$; similarly $fg = 1_Y$.

Proof of Theorem 2. Let Z be a noncompact 0-dimensional paracompactum, and let X be a compact space. We shall show that for every natural transformation $F\colon \pi_Z \to \pi_X$ there are distinct homotopy classes of $[Z; S^0] = \pi_Z(S^0)$ which are mapped by F to a single class in $[X; S^0] = \pi_X(S^0)$. By the Lemma there is a map $f\colon X \to Z$ such that $F = f^{\#}$. Since Z is not compact, there are distinct points $z_1, z_2 \in Z \setminus f(X)$; consequently, there is a discrete open cover U of Z such that $\varphi_U(z_1) \neq \varphi_U(z_2)$ and $\varphi_U(z_i) \cap \varphi_U(f(X)) = \emptyset$ for $i = 1, 2$. If ψ_i maps $\varphi_U(z_i)$ to 1 and $Z \setminus \varphi_U(z_i)$ to -1 for $i = 1, 2$, then ψ_1 and ψ_2 are maps from Z to $S^0 = (1, -1)$

which are not homotopic but which satisfy $\psi_1 f = \psi_2 f$; hence $F[\psi_1] = F[\psi_2]$. Now if there were a natural transformation $G\colon \pi_X \to \pi_Z$ satisfying $GF = 1^{\#}_Z$, then F would map $[Z, S^0]$ injectively to $[X, S^0]$. Hence the shape of Z is not dominated by the shape of X.

References

[1] J. Kelley, *General Topology*, New York 1955.
[2] G. Kozlowski, *Polyhedral limitations on shape* (to appear).
[3] K. Nagami, *Dimension Theory*, New York 1970.
[4] S. Mardešić and J. Segal, *Shape of compacta and* ANR-*systems*, Fund. Math. 72 (1971), pp. 41–59.
[5] — *Shapes for topological spaces*, Gen. Top. Appl. (to appear).
[6] — *Equivalence of two notions of shape for metric spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. (to appear).

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On the hyperspaces of snake-like and circle-like continua

by

J. Krasinkiewicz (Warszawa)

Abstract. J. Segal has proved a theorem which says that the hyperspace of a snake-like continuum has the fixed point property. In this paper we give a shorter proof of this theorem and we prove also that the hyperspace of a circle-like continuum has this property. The structure of these hyperspaces is studied and it is shown that the Whitney maps induce interesting decompositions of these hyperspaces.

0. Introduction. By a *map* we mean a continuous function. The term *continuum* means a compact connected metric space. If X is a continuum, then $C(X)$ denotes the *hyperspace* of subcontinua of X with the Hausdorff metric: $\operatorname{dist}(A, B) = \inf\{\varepsilon > 0\colon\ B \subset K(A, \varepsilon)$ and $A \subset K(B, \varepsilon)\}$, with $K(A, \varepsilon)$ denoting the open ε-neighbourhood of A in X. A map $f\colon X \to Y$ into a continuum Y generates a map $\hat{f}\colon C(X) \to C(Y)$, usually called the *map induced by* f, given by the formula $\hat{f}(A) = f(A)$. We introduce a terminology connected with a given hyperspace $C(X)$. The continuum X is, in a sense, a maximal point of $C(X)$ and is called the *vertex of* $C(X)$. By $\hat{X}$ we denote the subspace of $C(X)$ consisting of all singletons. It is called the *base of* $C(X)$. The base of $C(X)$ is isometric to X. For every two continua $A, B \in C(X)$ such that $A \subset B$ there exists a maximal monotone collection of continua between A and B, which forms an arc in $C(X)$ provided $A \neq B$. This collection is denoted by AB and is called a *segment from A to B*. In the case where A is a continuum consisting of a single point and $B = X$ the segment AB is said to be *maximal*. In [10] Whitney described a map μ, from $C(X)$ (where X is nondegenerate) onto the unit interval I, having the following properties:

(i) $\mu(X) = 1$,

(ii) if $A \subset B$ and $A \neq B$ then $\mu(A) < \mu(B)$,

(iii) $\mu(\{x\}) = 0$, for $x \in X$.

In the sequel every map with these properties will be called a *Whitney map*. If X is nondegenerate, then any Whitney map restricted to a maximal segment of $C(X)$ is a homeomorphism onto I.