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Reçu par la Rédaction le 9. 9. 1972

The G_δ-topology on compact spaces

by

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Abstract. Is every G_δ-covering of a compact topological space reducible to a $2^{\aleph_0}$-subcovering? This question is answered in the affirmative for linearly ordered topological space. In fact, the authors' major result is that each G_δ-covering of a finite product of compact linearly ordered topological spaces has a $2^{\aleph_0}$-star-finite refinement consisting of G_δ sets.

1. Introduction. Nearly forty years ago Alexandrov and Urysohn [1] conjectured that a compact first countable Hausdorff space could contain no more than $2^{\aleph_0}$ points. Arhangel'skiĭ [2] proved that conjecture true several years ago, and thereby re-opened some related questions, one of which was related to W. Fleischman by Professor I. Juhasz. In the present paper we offer more than a solution to this question:

Each covering by G_δ-sets of a compact linearly ordered topological space is reducible to a subcovering of cardinality not exceeding $2^{\aleph_0}$.

2. Compact linearly ordered topological spaces. We will call compact linearly ordered spaces *CLOTS*.

Let (X, τ) be a topological space. The collection of τ-G_δ-sets can form a base for a finer topology δ on X, called the *G_δ-topology* on X induced by τ, or simply the G_δ-topology when no confusion results.

If for each Σ-open covering of the space (X, Σ) there exists a subcovering of cardinal not exceeding m, we say (X, Σ) is *m-lindelöf*. With this definition in mind, we observe that our problem may be formulated as follows:

If (X, τ) is a CLOTS, then (X, δ) is $2^{\aleph_0}$-lindelöf.

Of course if (X, τ) is first countable, (X, δ) is discrete, and therefore, (X, δ) is $|X|$-lindelöf. Hence, the unit interval with the Euclidean topology exemplifies a CLOTS whose G_δ-topology is not m-lindelöf for any $m < 2^{\aleph_0}$.

Our first proposition seems to follow from [3, p. 27]; nevertheless, we shall offer an alternative proof igniting the techniques used in this work.

PROPOSITION A. *If the linear ordering of the CLOTS (X, τ) is a well-ordering, then (X, δ) is lindelöf ($\aleph_0$-lindelöf).*

Proof. Certainly if X is an ordinal not exceeding ω_0+1, the theorem is true; hence, we use induction and suppose the theorem true for each ordinal $\beta+1$ satisfying $\beta+1<X$, where $X>\omega_0+1$. Let R be a δ-open cover of X with no countable subcover and let L denote the union of all intervals $[a, X-1]$ covered by some countable subcollection of R.

$L\neq\{X-1\}$, for otherwise $X-1$ is the supremum of a countable increasing sequence of ordinals $\{x_n\colon n\in\omega_0\}$. By induction we will allow for each n, R_n to be a countable subcollection of R covering $[0, x_n]$. One can now see that $0\in L$ which is a contradiction. Therefore, $\inf L<X-1$. So $(\inf L)+1\in L$ and subsequently $\inf L\in L$. Inductively, $[0, \inf L]$ and $[\inf L, X-1]$ are covered by countable subcollections of R.

We might point out that the aforementioned result from [3] actually illustrates the conclusion of Proposition A true for any compact Hausdorff dispersed space, where a dispersed space is one for which each subspace contains an isolated point.

Paracompactness may appear a bit out of phase here; however, we will see that its "strongest" form is most fundamental to our major result. Recall that a *strongly paracompact space* is one in which each open cover has an open refinement each of whose members intersect at most a finite number of the other members. The condition on the refinement is known as star-finite but may be weakened to star-countable in T_3-spaces.

LEMMA B. *Let (X, τ) be a CLOTS. Then (X, δ) is strongly paracompact.*

Proof. Let R be a δ-open cover of X by intervals of X. Fix $x\in X$ and let I_x be the collection and $\bigcup I_x$ the union of all intervals $[a, b]\subseteq X$ such that $x\in[a, b]$ and a countable subcollection may be extracted from R to cover $[a, b]$. Denote $\sup\bigcup I_x$ by b_x and $\inf\bigcup I_x$ by a_x. If $\{y\in X\colon y\geqslant b_x\}$ is δ-open, then b_x is the supremum of a countable non-decreasing sequence of values less than b_x and therefore, as in Proposition A, $b_x\in\bigcup I_x$. If $\{y\in X\colon y\geqslant b_x\}$ is not δ-open, each member of R meeting b_x has non-empty intersection with $\bigcup I_x$; hence $b_x\in\bigcup I_x$. Similarly, $a_x\in\bigcup I_x$.

Unfix x. If $x, y\in X$ either $\bigcup I_x=\bigcup I_y$ or $\bigcup I_x\cap\bigcup I_y=\emptyset$. Observe also that each $\bigcup I_x$ is the union of a countable subcollection of R which we call R_x. Certainly

$$S=\bigcup\{R_x\colon x\in X\}$$

is a star-countable refinement of R. It is well-known [see 4, p. 173] that an open star-countable covering of a T_3-space has an open star-finite refinement.

At this point we request the reader to keep in mind the collection I_x and its construction, for it is most fundamental to each of theorems given in this paper.

LEMMA C. *If (X, τ) is a CLOTS, then (X, δ) is $2^{\aleph_0}$-lindelöf.*

Proof. Let R be a δ-open cover of X by intervals of X. The sets $\bigcup I_x$ in the above Lemma B partition X and since they are closed subsets of (X, δ), they generate a quotient space which inherits a linear ordering from the ordering of X in a most natural way. That is, let $I=\{I_x\colon x\in X\}$ and define

$$I_x=I_y \quad\text{iff}\quad I_x\cap I_y\neq\emptyset,$$
$$I_x<I_y \quad\text{iff}\quad I_x\cap I_y=\emptyset \text{ and } x<y.$$

Some very elementary calculations illustrate that the order topology $\mathcal{K}$ on I induced by $<$ is the very same as the quotient topology obtained from (X, τ) by collapsing the intervals $\bigcup I_x$ to single points.

We now show that $|I|\leqslant 2^{\aleph_0}$. Given $x\in X$, one sees that $\bigcup I_x$ is a δ-clopen subset of X, hence $b_x=\sup\bigcup I_x$ ($a_x=\inf\bigcup I_x$) is either an end-point of a separation of (X, τ) or the limit of a countable decreasing (increasing) sequence in X or an end-point of X. Therefore, I_x is either a right (left)-end-point of a separation of I or it is the limit of a countable decreasing (increasing) sequence in I or it is an end-point of I. In any case $(I, \mathcal{K})$ is a first countable CLOTS. By Arhangel'skiĭ's result mentioned above, I consists of at most $2^{\aleph_0}$-elements. Therefore, S, as constructed in Lemma B, consists of at most $2^{\aleph_0}$-elements.

Our major result is a proof by induction and using Lemmas B and C we have shown the first step.

THEOREM. *Let (Y, τ) be a finite product of compact linearly ordered spaces. Then each covering of Y by G_δ-sets has a star-countable G_δ-refinement consisting of at most $2^{\aleph_0}$-elements.*

Proof. We will suppose that

$$Y=X\times X_1\times\ldots\times X_n, \qquad n\geqslant 1,$$

and moreover, that the conclusion is true for $X_1\times\ldots\times X_n$; i.e., each covering of $X_1\times\ldots\times X_n$ by G_δ-sets has a star-countable G_δ-refinement consisting of at most $2^{\aleph_0}$-elements. Ordinarily when we speak of a refinement of a covering we mean for the refinement to also be a covering; however, throughout this proof we will have need to speak of refinements which cover various subsets of the set originally covered; hence, we use the notation "T is pro-Z by R" to mean

(i) R is a given δ-open covering of Y,

(ii) Z is a given subspace of Y, and

(iii) T is a star-countable G_δ-refinement of R covering Z and consisting of at most $2^{\aleph_0}$-elements.

Let R be a G_δ-covering of Y and without loss of generality we shall urther require each element of R to have an interval as its projection in each coordinate. Fix $x \in X$ and I_x to be the collection and $\bigcup I_x$ to be the union of all intervals $[a, b,] \subseteq X$ such that (i) $x \in [a, b]$ and (ii) there exists T such that T is

$$\text{pro-}[a, b] \times X_1 \times \ldots \times X_n \text{ by } R .$$

Denote $\sup \bigcup I_x$ by b_x and $\inf \bigcup I_x$ by a_x.

We now show that $a_x, b_x \in \bigcup I_x$. Suppose $\{y \in X:\ y \geqslant b_x\}$ is a G_δ-set in X, then b_x is the supremum of a countable non-decreasing sequence $\{b_i:\ i \in N\} \subseteq \bigcup I_x$ such that $x \leqslant b_1$. For each $i \in N$, let R_i be

$$\text{pro-}[x, b_i] \times X_1 \times \ldots \times X_n \text{ by } R;$$

also, let R_0 be

$$\text{pro-}\{b_x\} \times X_1 \times \ldots \times X_n \text{ by } R .$$

Then

$$\bigcup \{R_i:\ i = 0, 1, 2, \ldots\} \text{ is pro-}[x, b_x] \times X_1 \times \ldots \times X_n \text{ by } R .$$

Hence, $b_x \in \bigcup I_x$. Suppose $\{y \in X:\ y \geqslant b_x\}$ is not a G_δ-set in X and let

$$R_0 \text{ be pro-}\{b_x\} \times X_1 \times \ldots \times X_n \text{ by } R$$

further, let $z_U = \inf\{\Pi_X(U)\}$ for $U \in R$.

We show that $\sup\{z_U:\ U \in R_0\} < b_x$. Suppose the contrary, then $\sup\{z_U:\ U \in R_0\} = b_x$. For each $U \in R_0$, choose one $k_U \in Y$ such that

(a) $$c_i < \Pi_i(k_U) < d_i$$

whenever there exists a non-degenerate interval $]c_i, d_i[\subseteq X_i$ satisfying

$$\Pi_i^{-1}(]c_i, d_i[) \cap \Pi_X^{-1}(z_U) \subseteq U^-;$$

otherwise

(b) $\Pi_i(k_U)$ may be any element of $\Pi_i\big(U^- \cap \Pi_X^{-1}(z_U)\big)$.

Let $k \in \Pi_X^{-1}(b_x)$ be a cluster point of $\{k_U:\ U \in R_0\}$. There is a $U_k \in R_0$ such that $k \in U_k$; however, k may not be an interior point of any $U \in R_0$ for this would contradict the star-countability of R_0. Thus, k is on the boundary of U_k. Since $\{y \in X:\ y \geqslant b_x\}$ is not a G_δ-set in X, since R_0 is star-countable, and since $U_k \cap (\{b_x\} \times X_1 \times \ldots \times X_n)$ is a G_δ-subset of $\{b_x\} \times X_1 \times \ldots \times X_n$, we may find a transfinite sequence $s = \{z_\alpha:\ \alpha < \omega_\mu\}$ such that

(c) s is increasing and cofinal with b_x,

(d) $\omega_\mu > \omega_0$,

(e) $s \subseteq \{z_u:\ U \in R_0\}$,

(f) $z_\alpha = z_U$ implies $k_U \in U_k^-$.

Statements (a) through (f), together, imply that $k \in U$ for each $z_U \in s$ which contradicts the star-countability of R_0. Therefore, $\sup\{z_U:\ U \in R_0\} < b_x$.

We now have

$$\inf I_{b_x} \leqslant \sup\{z_U:\ U \in R_0\} < b_x = \sup I_X .$$

It follows that $\bigcup I_x \cap \bigcup I_{b_x} \neq \emptyset$. Hence, $b_x \in \bigcup I_x$ and $x \in \bigcup I_{b_x}$. A similar argument shows $a_x \in \bigcup I_x$ and $x \in \bigcup I_{a_x}$.

To show that a_x has cofinality $\leqslant \aleph_0$ and b_x has coinitiality $\leqslant \aleph_0$, suppose, in the first case, that $\{y \in X:\ y \geqslant a_x\}$ is not a G_δ-set in X, then by applying the arguments illustrating $b_x \in \bigcup I_x$ to a_x from the preceeding two paragraphs, we find

$$\inf \bigcup I_{a_x} < \inf \bigcup I_x ,$$

a contradiction since $\bigcup I_{a_x} = \bigcup I_x$. Therefore, a_x has cofinality $\leqslant \aleph_0$. In a similar fashion it is possible to show that b_x has coinitiality $\leqslant \aleph_0$.

Construct the linearly ordered space I "in the sense of Lemma C" and we see that

$$|\{\bigcup I_x:\ x \in X\}| \leqslant 2^{\aleph_0} .$$

Moreover, since a_x has cofinality $\leqslant \aleph_0$ and b_x has coinitiality $\leqslant \aleph_0$ there exists for each distinct $\bigcup I_x$ a collection S_x such that

$$S_x \text{ is pro-}[a_x, b_x] \times X_1 \times \ldots \times X_n \text{ by } R$$

and such that

$$\bigcup \{V:\ V \in S_x\} = [a_x, b_x] \times X_1 \times \ldots \times X_n .$$

Let $S = \bigcup \{S_x:\ x \in X\}$, where $S_x = S_y$ if $\bigcup I_x = \bigcup I_y$. Then S is pro-Y by R.

3. Other spaces and remarks. The original problem may be answered in the affirmative for many non-compact linearly ordered topological spaces by our techniques with a few adaptations; nevertheless, the theorem is not extendable to the totality of linearly ordered topological spaces. The entire question seems to revolve around the type and the plurality of the gaps of the space. For a lengthy discussion of gaps we refer the reader to [5]; we give a short discussion here.

Let X be a linearly ordered topological space, A and B be intervals of X (either A or B may be empty) such that

(i) $A \cup B = X$,

(ii) $A \cap B = \emptyset$,

(iii) $a \in A$ and $b \in B$ imply $a < b$,

(iv) $\sup A, \inf B \notin X$.

Then the pair (A, B) is said to be a *gap* of X. The *Dedekind completion* X^+ of X is the union of X together with its gaps suitably linearly ordered such that X retains its original order and topology as a subspace of X^+. It is to be noted that

(i) X^+ is always compact,

(ii) X is compact if and only if it has no gaps,

(iii) if X is lindelöf, each element of $X^+ - X$ has a countable local base in X^+.

PROPOSITION D. *Let (X, τ) be a lindelöf linearly ordered topological space. Then (X, δ) is $2^{\aleph_0}$-lindelöf.*

Feasting off the constructions of Lemmas B and C and using (iii) from above, the proof of Proposition D becomes immediate, and via a mimicry of our theorem the following corollary is easily seen.

COROLLARY E. *Let (Y, τ) be a finite product of lindelöf linearly ordered topological spaces. Then each covering of Y by G_δ-sets has a star-countable G_δ-refinement consisting of at most $2^{\aleph_0}$ elements.*

To find a linearly ordered topological space not satisfying the conclusion of Proposition D, we need only a space X whose gaps have no $2^{\aleph_0}$ local base in X^+; in fact, a space of all ordinals less than some suitably large limit ordinal should do.

To find a compact Hausdorff space not satisfying the conclusion of Proposition D appears a bit more difficult; however, J. Isbell and S. Mrówka have informed the authors of the following example:

Consider βD, where D is a discrete space of non-measurable cardinal greater than $2^{\aleph_0}$. Since $\beta D - D$ is a Q-space, it may be covered by G_δ subsets of itself. Take such a covering together with singleton points of D and one obtains the desired covering.

"The algebra of Baire functions on the compact space (X, τ) is defined as the smallest class of bounded functions containing the subalgebra of all τ-continuous real-valued functions and closed under the operation of taking pointwise limits of sequences. The weak topology on X generated by the Baire functions is called the ι-topology and has a basis the zero-sets of the τ-continuous real-valued function on X".

The ι-topology lies between τ and δ and is equal to δ for normal spaces. This topology has been an object of study in both [3] and [6]; it therefore seems necessary to compile a list of spaces for which we know the G_δ-topology to be $2^{\aleph_0}$-lindelöf.

(1) Finite products of lindelöf linearly ordered topological spaces,

(2) Hausdorff compact dispersed (scattered) spaces [3],

(3) one-point compactifications of topological spaces,

(4) $\boldsymbol{R}^{\boldsymbol{R}}$, where $\boldsymbol{R}$ is the real line and hence βN,

(5) 1st countable compact Hausdorff spaces [2].

Professor Juhasz in [7] has obtained a closely related result, namely, let (X, τ) be a compact Hausdorff space, then every δ-covering is reducible to a subfamily of cardinal $2^{\aleph_0}$ whose union is δ-dense in (X, δ), whenever (X, τ) satisfies the Suslin condition.

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Reçu par la Rédaction le 6. 10. 1972