Some properties of the remainder of Stone-Čech compactifications

by

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Abstract. It is the purpose of this note to investigate some properties of $\beta X - X$. In § 1, we give some relationships between μX and νX, for example, $\mu X \neq \nu X$ implies that νX is not a k-space and hence the local compactness of νX implies $\mu X = \nu X$ (Theorem 1.3). In §§ 3-4, introducing the notion of being "almost locally compact", we improve the results obtained by Fine and Gillmann [3], [4] and Robinson [14] (Theorems 3.3-3.6) and moreover generalize the results obtained by Rudin [15] and Plank [13] (Theorems 4.4-4.7).

Introduction. Fine and Gillman [3], [4] obtained several interesting properties for $X^* = \beta X - X$ under suitable conditions and Robinson [14] improved these results in the following forms replacing "realcompactness" by "topological completeness".

(FGR_1) *If X is locally compact and admit a complete uniform structure, then each $Z \in Z(X^*)$ is the closure of its interior.*

(FGR_2) *Let X admit a complete uniform structure and $H \subset X^*$. If $X \cup H$ is pseudocompact, then H is dense in X^*. Conversely, if H is dense in X^* and X is locally compact as well as complete, then $X \cup H$ is pseudocompact.*

From these theorems Robinson proved further the following.

(R_1) *If X admits a complete uniform structure, $\nu X - X$ has a void interior in X^*.*

(R_2) *If X is locally compact and complete, then $\nu X - X$ is nowhere dense in X^*.*

(R_3) [CH] *If X is locally compact and complete, then X^* contains a dense set of P-points.* (This is a generalization of Rudin's theorem [15]).

(R_4) *Every locally compact metric space X without isolated points contains a set of remorte points which is dense in X^*.* (This is a generalization of Plank's theorem 5.5 [13].

In this paper, introducing the notion of being "almost locally compact" and using the notion of relatively pseudocompactness, we shall make in § 3 Theorems (FGR_1), (FGR_2) and (R_2) in good order (in the sense of Theorems 3.3-3.6 below) and in § 4 generalize moreover Theorems (R_3), (R_4) and Plank's theorems 3.3 and 3.4 [13] (see Theorems 4.4-4.7 below). Before proving these main theorems, in §§ 1–2 we shall consider local compactness of νX and μX and investigate some properties of X^*. In § 5 we give some relationship between round subsets and P-points in νX. In general, there are delicate questions concerning the operation "ν" taking realcompactification of a given space X which depends heavily on either measurability or nonmeasurability of the cardinality of X. But it seems to me that there are no such questions when we consider a topological completion of a given space. Thus it is more convenient for a treating topological completion than realcompactification. From this point of view we will consider topological complete spaces as much as possible.

Throughout this paper by a space we shall mean a completely regular T_1-space and all functions are assumed to be continuous and use the terminology and conventions of [5]. For $f^\beta \in C(\beta X)$, we denote by f the function $f^\beta|X$ and by $Z(f)$ the zero set $Z(f^\beta|X)$. Conversely, for $g \in C^*(X)$ we denote by g^β the Stone-extension of g over βX. It is well known that i) for $g \in C^*(X)$, $g = g^\beta|X$ and ii) for each $f^\beta \in C(\beta X)$, $f^\beta = (f^\beta|X)^\beta$. For a point p in a given space Y, its neighborhood (= nbd) U in Y will be denoted by $U_Y(p)$. Similarly we denote by $\mathrm{cl}_Y A$ (or $\mathrm{int}_Y A$) the closure (or interior) of A as a subset of Y. For simplicity, in case $Y = \beta X$ or X^* we use especially "β" and "$*$" instead of βX and X^*. For instance, $\mathrm{cl}_\beta A = \mathrm{cl}_{\beta X} A$, $\mathrm{int}^* A = \mathrm{int}_{X^*} A$, $U^*(p) = U_{X^*}(p)$ and $U_\beta(p) = U_{\beta X}(p)$.

A space will be called *topologically complete* provided that it is complete with respect to its finest uniformity. We denote by μX the completion of a space X with respect to its finest uniformity. Thus X is topologically complete if and only if $\mu X = X$. Let us put $X_1 = \{x;\ x$ has a compact nbd$\}$ and $R(X) = X - X_1$ according to [6]. X is said to be *almost locally compact* if $\mathrm{int}_X R(X) = \emptyset$. In [6], it is proved that if both X and X^* are σ-compact, then X is almost locally compact. A subset E of X is a *relatively pseudocompact* (= rpc) subset (in X) if $f|E$ is bounded for every $f \in C(X)$. Let Ω be the first uncountable ordinal and N the set of natural numbers or its copy. $|A|$ denotes the cardinality of a set A and $\mathfrak{c} = |R|$ where R denote the set of real numbers. [CH] indicates the continuum hypothesis is being assumed.

§ 1. Relatively pseudocompact subsets and local compactness of νX and μX.

For future use the following is stated as a lemma.

LEMMA 1.1. i) *For each $p \in \beta X - \nu X$, there is $h^\beta \geqslant 0$ with $p \in Z(h^\beta) \subset \beta X - \nu X$* [5].

ii) *$Z(g^\beta) \subset \nu X$ implies $Z(g) \neq \emptyset$ for each $g^\beta \in C(\beta X)$* [5].

iii) $X \subset \mu X \subset \nu X \subset \beta X$ [5], [11].

iv) *For each $p \in \nu X - X$, $f^\beta(p) = 0$ implies $Z(f) \neq \emptyset$* [5].

v) *A relatively pseudocompact closed subset of a topologically complete space is compact* [2].

From the preceding lemma we summarize matters which are useful throughout this paper as the following theorem.

THEOREM 1.2. i) *For $A \subset X$, A is relatively pseudocompact if and only if $\mathrm{cl}_\beta A \subset \mu X$ if and only if $\mathrm{cl}_\beta A \subset \nu X$.*

ii) *$Z(f^\beta) \subset \nu X$ implies $Z(f^\beta) = \mathrm{cl}_\beta Z(f)$.*

iii) *$Z(g) \neq \emptyset$ $(g \in C^*(X))$ implies $\mathrm{cl}_\beta Z(g) \cap \nu X = Z(g^\beta) \cap \nu X$.*

iv) *Let $Z(f) \neq \emptyset$. Then $Z(f^\beta) \subset \mu X$ if and only if $Z(f^\beta) \subset \nu X$ if and only if $Z(f)$ is relatively pseudocompact and $\mathrm{cl}_\beta Z(f) = Z(f^\beta)$.*

v) *For each $p \in \beta X - \mu X$, $p \in Z(h^\beta)$ implies $Z(h^\beta) \cap (\beta X - \nu X) \neq \emptyset$.*

Proof. i) Suppose that A is rpc. For each $g \in C(\mu X)$, we have $g|X \in C(X)$ and $g|A$ is bounded, and hence $g|\mathrm{cl}_{\mu X} A$ is bounded which shows that $\mathrm{cl}_{\mu X} A$ is rpc. By (v) of Lemma 1.1, $\mathrm{cl}_{\mu X} A$ must be compact, i.e., $\mathrm{cl}_\beta A = \mathrm{cl}_{\nu X} A = \mathrm{cl}_{\mu X} A \subset \mu X$.

By (iii) of Lemma 1.1, $\mathrm{cl}_\beta A \subset \mu X$ implies $\mathrm{cl}_\beta A \subset \nu X$.

Suppose that A is not rpc. Then there exists $f \in C(X)$ such that $f|A$ is unbounded. The extension of f over νX is unbounded on $\mathrm{cl}_{\nu X} A$ which is impossible because $\mathrm{cl}_{\nu X} A \subset \mathrm{cl}_\beta A \subset \nu X$.

ii) Being $\mathrm{cl}_\beta Z(f) \subset Z(f^\beta)$, suppose that there is a point $p \in Z(f^\beta) - \mathrm{cl}_\beta Z(f) \subset \nu X$. Since $\mathrm{cl}_\beta Z(f)$ is compact, there exists $h^\beta \geqslant 0$ such that $h^\beta(p) = 0$ and $h^\beta = 1$ on $\mathrm{cl}_\beta Z(f)$ which shows that $g^\beta = |f|^\beta + h^\beta > 0$ on X and $p \in Z(g^\beta)$. This contradicts (iv) of Lemma 1.1.

Similarly we obtain (iii).

iv) If $Z(f^\beta) \subset \nu X$, then by (ii) $\mathrm{cl}_\beta Z(f) = Z(f^\beta)$ and the relative pseudocompactness of $Z(f)$ follows directly from (i). Other implications follow immediately from (i) and (ii).

v) If $Z(h^\beta) \subset \nu X$, then by (iv) we have $Z(h^\beta) \subset \mu X$ which contradicts the assumption $p \notin \mu X$, i.e., $Z(h^\beta) \cap (\beta X - \nu X) \neq \emptyset$.

We notice that the above theorem does not assert that the compactness of $B \subset \nu X$ implies $B \subset \mu X$.

Local compactness or k-ness of any subspace Y, containing X properly, of νX was discussed in [8], [9], but several theorems in [9] follow immediately from the following theorem.

THEOREM 1.3. *Let Y be any subspace, containing X, of νX.*

i) *If X is a space having the property that every relatively pseudocompact closed subset of X is compact (for example, in case X is topologically complete), then Y is not a k-space.*

ii) *If $\mu X \neq \nu X$, then νX is not a k-space* (and hence not locally compact).

iii) *The local compactness of νX implies $\nu X = \mu X$.*

Proof. i) Let Y be a k-space, K a compact subset of Y and $F = X \cap K$. If F is not rpc, then by (i) of Theorem 1.2 we have $\mathrm{cl}_\beta F \cap (\beta X - \nu X) \neq \emptyset$ which is impossible because $\mathrm{cl}_\beta F \subset K \subset \nu X$. Thus F must be a rpc closed subset of X. From the assumption, F is compact. Since Y is a k-space, we conclude to be X closed in Y which contradicts the denseness of X in Y. ii) and iii) follow from i).

Comfort [1] established the following theorem concerning local compactness of νX: In order that νX be locally compact it is necessary and sufficient that for each $p \in \nu X$ there exist pseudocompact ($=$ pc) subsets A and B of X for which $p \in \mathrm{cl}_{\nu X} A$ and there exists $f \in C^*(X)$ such that $f = 0$ on A and $f = 1$ on $X - B$. In [7] we stated that this theorem remains true for μX in case X is an M'-space. But we have further the following

THEOREM 1.4. *For a given X, μX is locally compact if and only if for each $p \in \mu X$, there exist relatively pseudocompact subsets A and B of X and $f \in C^*(X)$ such that $p \in \mathrm{cl}_{\mu X} A$, $f = 0$ on A and $f = 1$ on $X - B$.*

Proof. Using (i) of Theorem 1.2, necessity follows from local compactness and complete regularity of μX. Conversely, $\mathrm{cl}_{\mu X} B$ is a compact subset contained in μX by (i) of Theorem 1.2. It is easy to see that $p \in \mathrm{int}_{\mu X} \mathrm{cl}_{\mu X} B$ from (iii) of Theorem 1.2 which leads local compactness of μX.

It seems to me that the above Comfort's theorem is dissatisfied with the formulation used points of $\nu X - X$. Hence we shall improve for form's sake using only X and $C(X)$, but its proof is essentially the same as one given by Comfort.

It is known that there is one to one correspondence between the set of points of νX and the set of real z-ultrafilters [5]. A z-filter $\mathfrak{F}$ on X will be called *round* if for each $Z \in \mathfrak{F}$, there are $W \in \mathfrak{F}$ and a cozero set S in X with $W \subset S \subset Z$. For any z-filter $\mathfrak{F}$ on X, we denote by $\mathfrak{F}^0$ the family of all zero sets Z in X such that there are a member W of $\mathfrak{F}$ and a cozero set S with $W \subset S \subset Z$. Then $\mathfrak{F}^0$ is a round filter [10].

THEOREM 1.5. *For a given space X, νX is locally compact if and only if for any real z-ultrafilter $\mathfrak{F}$, $\mathfrak{F}^0$ contains at least one member which is relatively pseudocompact in X.*

Proof. We notice that $\mathfrak{F}$ is the set of zero sets Z of X such that $p \in \mathrm{cl}_\beta Z$ for some fixed point $p \in \nu X$. Since necessity is obvious, we shall prove sufficiency. Let $\mathfrak{F} = \{Z;\ p \in \mathrm{cl}_\beta Z,\ Z \in Z(X)\}$ for a fixed point $p \in \nu X$ and let $Z \in \mathfrak{F}^0$ be rpc. By (i) of Theorem 1.2, $\mathrm{cl}_\beta Z$ is compact in μX. From the assumption, there are a zero set $W = Z(f)$ and a cozero set S $= X - Z(h)$ with $W \subset S \subset Z$. Since $Z(f) \cap Z(h) = \emptyset$, we have $\mathrm{cl}_\beta Z(f) \cap \mathrm{cl}_\beta Z(h) = \emptyset$. Thus $p \in \mathrm{cl}_\beta Z(f) \subset \beta X - \mathrm{cl}_\beta Z(h) \subset \mathrm{cl}_\beta S \subset \mathrm{cl}_\beta Z$, that is, $p \in \mathrm{int}_\beta \mathrm{cl}_\beta Z \subset \mu X$ which shows local compactness of μX.

§ 2. Properties of X^*.

In this section we shall investigate some elementary properties of subsets in X^*. Firstly we shall prove the following key lemma in the sequel.

LEMMA 2.1. *Let $\{U_n\}$ be a locally finite disjoint family of open subsets of X. Then we have*

i) *There exists $f \in C^*(X)$ such that $0 \leqslant f \leqslant 1$, $f = 0$ on $X - \bigcup U_n$ and $f(x_n) = 1$ for each n and some point x_n in U_n.*

ii) *If $\mathrm{cl}_X U_n$ is compact and there are a sequence $\{\varepsilon_n\} \downarrow 0$ and $h^\beta \in C(\beta X)$ with $U_n \subset \{x;\ h(x) < \varepsilon_n\}$, then there exists g^β such that $0 \leqslant g^\beta \leqslant 1$, $P(g) \subset X_1$, $P(g^\beta) \cap X^* \neq \emptyset$ and $P(g^\beta) \cap X^* \subset \mathrm{cl}_\beta(\bigcup \{U_n\}) \cap \mathrm{cl}_\beta X_1 \cap (\beta X - \nu X) \cap Z(h^\beta)$ where $P(g^\beta) = \beta X - Z(g^\beta)$.*

Proof. i) is obtained form the usual method, that is, let f_n be a function such that $0 \leqslant f_n \leqslant 1$, $f = 0$ on $X - U_n$ and $f_n(x_n) = 1$ for some point $x_n \in U_n$. Then $f = \Sigma f_n$ is a desired function.

ii) Let us put $g^\beta = f^\beta$ where f is a function in (i). Obviously $0 \leqslant g^\beta \leqslant 1$. Let $x \in P(g) = P(g^\beta) \cap X$. By the method of construction of g^β, there is U_n with $x \in U_n$ and hence compactness of $\mathrm{cl}_X U_n$ implies $x \in X_1$.

Next let $p \in X^*$ and $g^\beta(p) > 0$. Any $U_\beta(p)$ intersects infinitely many U_n. The assumption $U_n \subset \{x;\ h(x) < \varepsilon_n\}$ implies $h^\beta(p) = 0$ which shows that $p \in Z(h^\beta)$. Let $V_\beta(p)$ be a compact nbd of p with $V_\beta(p) \subset U_\beta(p)$ and $U_n \supset \mathrm{cl}_X(U_n \cap V_\beta(p)) = W_n$. Then W_n is compact and $p \in \mathrm{cl}_\beta(\bigcup U_n)$ implies $p \in \mathrm{cl}_\beta(\bigcup W_n)$ which leads the fact that $p \in \mathrm{cl}_\beta(\bigcup U_n) \cap \mathrm{cl}_\beta X_1$. Next we shall show that $p \in \beta X - \nu X$. Since W_n is compact and $\{U_n\}$ is locally finite, there is $k \in C^*(X)$ with $k = 1/n$ on W_n for each n and $k = 1$ on $X - \bigcup U_n$ and $k > 0$ on X. Obviously $k^\beta(p) = 0$. This means $p \notin \nu X$ by (iv) of Lemma 1.1.

THEOREM 2.2. *Let $p \in \beta X - \mu X$. Then $p \in \mathrm{cl}_\beta X_1$ if and only if for any $p \in Z(f^\beta)$, there exists $g^\beta \geqslant 0$ such that $P(g) \subset X_1$ and $\emptyset \neq P(g^\beta) \cap X^* \subset Z(f^\beta) \cap (\beta X - \nu X) \cap \mathrm{cl}_\beta X_1$.*

Proof. Necessity. Since $p \in \mathrm{cl}_\beta X_1 - \mu X$, we have $Z(f^\beta) \cap (\beta X - \nu X) \neq \emptyset$ by (v) of Theorem 1.2. Using (i) of Lemma 1.1, we may select some $h_\beta \geqslant 0$ with $Z(h^\beta) \subset Z(f^\beta) \cap (\beta X - \nu X)$ and $Z(h^\beta) \neq \emptyset$. For a suitable sequence $\{\varepsilon_n\} \downarrow 0$, $V_n = h^{\beta^{-1}}(\varepsilon_{2n+1}, \varepsilon_{2n}) \cap X_1$ is not empty and $\{V_n\}$ is locally finite family of open sets of X. For each n there exists an open set U_n such that $\mathrm{cl}_X U_n \cap R(X) = \emptyset$, $\mathrm{cl}_X U_n \subset V_n$ and $\mathrm{cl}_X U_n$ is compact. Thus g^β in (ii) of Lemma 2.1 is a desired function.

Sufficiency. For any $U_\beta(p)$, there is $f^\beta \geqslant 0$ with $p \in Z(f^\beta) \subset U_\beta(p)$. Let g^β be a function in the assumption for this f^β. Since $P(g) \subset X_1$, $U_\beta(p) \cap$

$\cap P(g^\beta) \neq \emptyset$. On the other hand $P(g^\beta) \subset \mathrm{cl}_\beta P(g)$ and hence $P(g^\beta) \subset \mathrm{cl}_\beta X_1$. This shows that $U_\beta(p) \cap \mathrm{cl}_\beta X_1 \neq \emptyset$, that is, $p \in \mathrm{cl}_\beta X_1$.

2.3. *Let B be a subset of X^*.*

i) $\mathrm{Int}^* B \cap (\nu X - \mu X) \neq \emptyset$ *implies* $\mathrm{int}^* B \cap (\beta X - \nu X) \neq \emptyset$.

ii) $\mathrm{Int}^* B \cap (\beta X - \mu X) \neq \emptyset$ *if and only if* $\mathrm{int}^* B \cap (\beta X - \nu X) \neq \emptyset$.

iii) $\mathrm{Int}^*(\nu X - X) = \mathrm{int}^*(\mu X - X)$. $\mathrm{Int}^*(\nu X - \mu X) = \emptyset$.

iv) $\beta X - \nu X$ *is dense in* $\beta X - \mu X$.

v) $\mathrm{Int}^* B \cap (\nu X - X) \neq \emptyset$ *implies either* $\mathrm{int}^* B \cap (\nu X - \mu X) = \emptyset$ *or* $\mathrm{int}^* B \cap (\beta X - \nu X) \neq \emptyset$ *whenever* $\mathrm{int}^* B \cap (\nu X - \mu X) \neq \emptyset$.

Proof. i) For $p \in \mathrm{int}^* B \cap (\nu X - \mu X)$ there exists an open nbd $U^*(p)$ of p such that $V_\beta(p) \cap X^* = U^*(p) \subset B$ and $\mathrm{cl}_\beta U_\beta(p) \subset V_\beta(p)$ for some $V_\beta(p)$ and $U_\beta(p)$. Let us select a zero set $Z(f^\beta)$ with $p \in Z(f^\beta) \subset U_\beta(p)$. By (v) of Theorem 1.2, $Z(f^\beta) \cap (\beta X - \nu X) \neq \emptyset$, that is, $U^*(p) \cap (\beta X - \nu X) \neq \emptyset$. Since $U^*(p) \subset B$ and $U^*(p)$ is open in X^*, we have $\mathrm{int}^* B \cap (\beta X - \nu X) \neq \emptyset$.

ii) $\mathrm{Int}^* B \cap (\beta X - \nu X) = \emptyset$ implies $\mathrm{int}^* B \cap (\nu X - \mu X) = \emptyset$ by i). Since $\mathrm{int}^* B \cap (\beta X - \mu X) = (\mathrm{int}^* B \cap (\beta X - \nu X)) \cup (\mathrm{int}^* B \cap (\nu X - \mu X))$, it is easily seen that ii) is valid.

iii), iv) and v) follow directly from (i).

Remark 1. (iii) gives another proof of (ii) and non-local compactness of νX in Theorem 1.3 respectively.

Remark 2. (iii) gives a generalization of Theorem R_1 at the beginning, that is, if $X = \mu X$, then $\nu X - \mu X = \nu X - X$ has a void interior in X^*.

2.4. We shall list up properties concerning $R(X)$ and X_1 which are summarized in the implication diagram as follows.

$$\mathrm{Int}^*(\mathrm{cl}_\beta R(X) - R(X)) = \emptyset \Leftrightarrow \begin{cases} \mathrm{int}^*(\mathrm{cl}_\beta R(X) - R(X)) \cap (\nu X - X) = \emptyset \overset{(1)}{\searrow} \nu X - X \subset \mathrm{cl}_\beta X_1 \cap X^* \\ \mathrm{int}^*(\mathrm{cl}_\beta R(X) - R(X)) \cap (\beta X - \nu X) = \emptyset \end{cases}$$

$$\Updownarrow (2)$$

$$\mathrm{int}^*(\mathrm{cl}_\beta R(X) - R(X)) \cap (\beta X - \mu X) = \emptyset \overset{(3)}{\Leftrightarrow} \beta X - \nu X \subset \mathrm{cl}_\beta X_1 \cap X^* \overset{(4)}{\Leftrightarrow} \beta X - \mu X \subset \mathrm{cl}_\beta X_1 \cap X^*$$

$$X \textit{ is almost locally compact} \Leftrightarrow$$

$$\Updownarrow (5)$$

$$\beta X = \mathrm{cl}_\beta X_1 \Leftrightarrow X^* = \mathrm{cl}_\beta X_1 \cap X^*.$$

Proof. 1) $p \in \nu X - X - \mathrm{cl}_\beta X_1$ implies $p \in \mathrm{int}^*(X^* \cap \mathrm{cl}_\beta R(X))$ because $\mathrm{cl}_\beta R(X) \cup \mathrm{cl}_\beta X_1 = \beta X$. This contradicts our assumption (see Example 2.5).

2) Let $B = \mathrm{cl}_\beta R(X) - R(X)$ in (ii) of 2.3.

3) If $p \in \beta X - \nu X - \mathrm{cl}_\beta X_1$, then p is an inner point (in X^*) of $\mathrm{cl}_\beta R(X) - R(X)$ which contradicts the fact that $\mathrm{int}^*(\mathrm{cl}_\beta R(X) - R(X)) \cap (\beta X - \nu X) = \emptyset$.

Conversely, suppose that there exists a point $p \in \mathrm{int}^*(\mathrm{cl} R_\beta(X) - R(X)) \cap (\beta X - \nu X)$. Then $p \notin \nu X$ and there is $U_\beta(p)$ satisfying $U^*(p) = U_\beta(p) \cap X^* \subset \mathrm{cl}_\beta R(X) - R(X)$. Supposing that $\beta X - \nu X \subset \mathrm{cl}_\beta X_1$, we will lead the contradiction. Using Theorem 2.2 and $p \in \mathrm{cl}_\beta X_1 - \nu X$, there exists $Z(f^\beta)$ such that $Z(f^\beta) \subset U_\beta(p)$ and $X^* \cap Z(f^\beta) \subset U^*(p) \cap (\beta X - \nu X)$. Moreover there is $g^\beta \geqslant 0$ with $P(g) \subset X_1$ and $P(g^\beta) \cap X^* \subset Z(f^\beta) \cap X^*$ by Theorem 2.2. Let us put $U_{R(X)} = U_\beta(p) \cap R(X)$. If $U_{R(X)} = \emptyset$, then $p \notin \mathrm{cl}_\beta R(X) = \emptyset$, and hence we have $U_{R(X)} \neq \emptyset$. Since $R(X)$ is dense in $\mathrm{cl}_\beta R(X)$ and $U^*(p) \subset \mathrm{cl}_\beta R(X) - R(X)$, $U^*(p)$ is contained in $\mathrm{cl}_\beta R(X) \cap X^*$. On the other hand, since $P(g^\beta) \cap X^* \subset U^*(p)$ and $P(g) \subset X_1$ and $P(g^\beta) \subset \mathrm{cl}_\beta(P(g^\beta) \cap X)$, $U^*(p)$ contains a point q with $g^\beta(q) = k > 0$. $P(g) \subset X_1$ implies $g = 0$ on $R(X)$ and $U^*(p) \subset \mathrm{cl}_\beta R(X)$ implies $g^\beta(q) = 0$. This is impossible.

4) Let $\beta X - \nu X \subset \mathrm{cl}_\beta X_1 \cap X^*$ and $p \in \beta X - \mu X - \mathrm{cl}_\beta X_1$. If $p \in \nu X$, then $p \in \nu X - \mu X$ and there exists $Z(h^\beta)$ such that $p \in Z(h^\beta) \cap X^* \subset \mathrm{int}^*(\nu X - X)$. On the other hand we have $Z(h^\beta) \cap (\beta X - \nu X) \neq \emptyset$ by (v) of Theorem 1.2 which is a contradiction.

5) Let $p \in X^* - \mathrm{cl}_\beta X_1$. There is a nbd $U_\beta(p)$ with $U_\beta(p) \cap \mathrm{cl}_\beta X_1 = \emptyset$. This implies $\mathrm{int}_\beta R(X) \neq \emptyset$. Now suppose that $x \in \mathrm{int}_\beta \mathrm{cl}_\beta R(X)$. There exists a nbd $U_\beta(p) \subset \mathrm{cl}_\beta R(X)$. Then $U_\beta(p) \cap X = U_X(p)$ is disjoint from X_1, so $p \notin \mathrm{cl}_\beta X_1$ and hence $\beta X \neq \mathrm{cl}_\beta X_1$.

2.5. Example. The reverse implication in (1) of 2.4 does not hold in general. Let $Y = [0, \Omega] \times [0, 1] - \{(\Omega, \alpha);\ \alpha \text{ is irrational}\}$ and $X = Y \cup R$ (top. sum). Then it is easy to see that $X_1 = [0, \Omega) \times [0, 1] \cup R$ (top. sum) and $\nu X = [0, \Omega] \times [0, 1] \cup R$ (top. sum) and $R(X) = \{(\Omega, \alpha);\ \alpha$ is irrational$\}$ and $\mathrm{cl}_\beta X_1 = \beta X$. On the other hand the set $\mathrm{cl}_\beta R(X) - R(X) = \{(\Omega, \gamma);\ \gamma$ is rational$\}$ is open and closed in X^* which coincides with $\nu X - X$.

2.6. i) $\mathrm{Int}^*(\mathrm{cl}_\beta X_1 \cap \mathrm{cl}_\beta R(X) \cap X^*) \cap (\beta X - \nu X)$
$$= \mathrm{int}^*(\mathrm{cl}_\beta X_1 \cap \mathrm{cl}_\beta R(X) \cap X^*) \cap (\beta X - \mu X) = \emptyset.$$

ii) $\mathrm{Int}^*(\mathrm{cl}_\beta X_1 \cap \mathrm{cl}_\beta R(X) \cap X^*) \subset \mu X - X$.

Proof. i) Since the first equality is obtained from ii) of 2.3 by setting $B = \mathrm{cl}_\beta X_1 \cap \mathrm{cl}_\beta R(X) \cap X^*$, we shall show that these subsets are empty. Now suppose that $p \in \mathrm{int}^* B \cap (\beta X - \nu X)$. Then there is $U^*(p)$ contained in B. On the other hand, since $p \in \mathrm{cl}_\beta R(X) - \nu X$, using Theorem 2.2 there is a cozero set $P(g^\beta)$ such that $P(g) \subset X_1$ and $P(g^\beta) \cap X^* \subset U^*(p) \cap (\beta X - \nu X)$. This implies that there is a point q in $U^*(p)$ with $g^\beta(q)$

$= k > 0$. But $U^*(p) \subset \mathrm{cl}_\beta R(X)$ and $g = 0$ on $R(X)$ which shows $g^\beta(q) = 0$. This is a contradiction.

2.7. *Let $X_1 \neq \emptyset$. Then the following are equivalent:*

i) *X_1 is relatively pseudocompact.*

ii) $\mathrm{Int}^*(\mathrm{cl}_\beta X_1 - X_1) \cap (\beta X - \mu X) = \emptyset$.

iii) $\mathrm{Int}^*(\mathrm{cl}_\beta X_1 - X_1) \cap (\beta X - \nu X) = \emptyset$.

iv) $(\mathrm{cl}_\beta X_1 - X_1) \cap (\beta X - \nu X) = \emptyset$.

v) $(\mathrm{cl}_\beta X_1 - X_1) \cap (\beta X - \mu X) = \emptyset$.

Proof. v) $\Leftrightarrow$ i) (Theorem 1.2). v) $\to$ iv) $\to$ iii) (Obvious). ii) $\Leftrightarrow$ iii) (Put $B = \mathrm{cl}_\beta X_1 - X_1$ in (ii) of 2.3). Suppose that X_1 is not rpc. There is a point $p \in \mathrm{cl}_\beta X_1 - \mu X$ by (i) of Theorem 1.2. $\mathrm{Cl}_\beta X_1 - X_1$ contains at least one inner point (in X^*) which belongs to $\beta X - \nu X$ by Theorem 2.2. This shows ii) $\to$ i).

The followings are obvious.

2.8. *$\mu X - X$ ($\nu X - X$, resp.) is dense in X^* if and only if X_1 is relatively pseudocompact and $R(X) \subset \mathrm{cl}_\beta(\mu X - X)$ ($\subset \mathrm{cl}_\beta(\nu X - X)$, resp.).*

§ 3. Main theorems. In this section we shall consider improvements and generalizations Theorems, except with (R_1), listed up at the beginning in this paper. Summarizing Theorems 2.2 and 2.4, we restate as the following theorem.

THEOREM 3.1. *If X is not pseudocompact, then the following are equivalent:*

i) $\mathrm{Int}^*(\mathrm{cl}_\beta R(X) - R(X)) \cap (\beta X - \mu X) = \emptyset$.

ii) $\beta X - \mu X \subset \mathrm{cl}_\beta X_1$.

iii) *For any $p \in \beta X - \mu X$ and any zero set $Z(h^\beta)$, $p \in Z(h^\beta)$, $h^\beta \geq 0$, there exists $g^\beta \geq 0$ such that $P(g) \subset X_1$ and $P(g^\beta) \cap X^* \subset Z(h^\beta) \cap (\beta X - \nu X) \cap \mathrm{cl}_\beta X_1$.* (In these statements, we can replace μX by νX as shown in 2.4, 2.6 and (v) of Theorem 1.2).

THEOREM 3.2. *If $R(X)$ is topologically complete (especially, X is topologically complete), then X is almost locally compact if and only if $\mathrm{int}^*(\mathrm{cl}_\beta R(X) - R(X)) \cap (\beta X - \nu X) = \emptyset$.*

Proof. Necessity follows from 2.4. Conversely suppose that $\mathrm{int}_X R(X) \neq \emptyset$. If $\mathrm{cl}_X \mathrm{int}_X R(X)$ is rpc, then $\mathrm{cl}_X \mathrm{int}_X R(X)$ must be compact by (v) of Lemma 1.1. This contradicts the definition of $R(X)$ and hence it is not rpc. There is a point p and a nbd $U_X(p) \subset R(X)$. Without loss of generality, we can assume that $U_\beta(p) = \mathrm{cl}_\beta U_X(p) \subset \mathrm{int}_\beta \mathrm{cl}_\beta R(X)$. On the other hand, $\mathrm{int}^*(\mathrm{cl}_\beta R(X) - R(X)) \cap (\beta X - \nu X) = \emptyset$ implies $U_\beta(p) \cap (\beta X - \nu X) = \emptyset$. Then $U_X(p)$ is rpc by (i) of Theorem 1.2. Since μX is topologically complete, $\mathrm{cl}_X U_X(p)$ must be compact which contradicts the definition of $R(X)$.

The following two theorems are improvements of Theorems FGR_1 and FGR_2.

THEOREM 3.3. *X is not pseudocompact. Then $\mathrm{int}^*(\mathrm{cl}_\beta R(X) - R(X)) \cap (\beta X - \nu X) = \emptyset$ if and only if for any $Z(h^\beta) \subset \beta X - \nu X$, $\mathrm{cl}^* \mathrm{int}^* Z(h^\beta) = Z(h^\beta)$.*

Proof. Necessity. For $p \in Z(h^\beta)$ and for any nbd $U^*(p)$, there is $h_0^\beta \geq 0$ with $Z(h_0^\beta) \subset U^*(p) \cap Z(h^\beta)$ by (i) of Lemma 1.1. Using the assumption (take $h^\beta = h_0^\beta$ in (iii) of the preceding theorem), we have an open set $P(g^\beta) \cap X^*$ of X^* which is contained in $Z(h_0^\beta)$ and hence $\mathrm{int}^* Z(h_0^\beta) \neq \emptyset$. Since $U^*(p)$ is an arbitrary nbd of p, $p \in \mathrm{cl}^* \mathrm{int}^* Z(h^\beta)$, that is, $\mathrm{cl}^* \mathrm{int}^* Z(h^\beta) = Z(h^\beta)$.

Sufficiency. We shall show that $p \in \mathrm{int}^*(\mathrm{cl}_\beta R(X) - R(X)) \cap (\beta X - \nu X)$ leads the contradiction. From (i) of Lemma 1.1, there is $h^\beta \geq 0$ such that $Z(h^\beta) \subset (\mathrm{cl}_\beta R(X) - R(X)) \cap (\beta X - \nu X)$. Let $G^* = \mathrm{int}^* Z(h^\beta)$. There exists an open set G_β of βX with $G^* = G_\beta \cap X^*$. For G^*, there is $g^\beta \geq 0$ with $P(g^\beta) \subset G_\beta$ and $P(g^\beta) \cap X^* \subset G^*$. Let $g(x) = a > 0$ for some $x \in P(g)$. $E = \{y;\ g^\beta(y) > a/3,\ y \in \beta X\}$ is open in βX. Since $E - X^* = E - Z(h^\beta)$ is locally compact and open in βX, $E \cap X$ is a locally compact open subset of X, i.e., $E \cap X \subset X_1$ and $P(g) \subset X_1$. Thus we have $P(g^\beta) \cap X^* \subset \mathrm{int}^*(\mathrm{cl}_\beta R(X) \cap \mathrm{cl}_\beta X_1 \cap X^*) \cap (\beta X - \nu X)$ which contradicts 2.6.

From the preceding two theorems we have

THEOREM 3.4. *If X is not a pseudocompact space such that $R(X)$ is topologically complete (especially, X is topologically complete), then X is almost locally compact if and only if for any $Z(h^\beta) \subset \beta X - \nu X$, $\mathrm{cl}^* \mathrm{int}^* Z(h^\beta) = Z(h^\beta)$.*

Theorem R_1 is generalized in Remark 2 of 2.3. For Theorem R_2 we have.

THEOREM 3.5. *Let X be topologically complete and not pseudo-compact. If X is almost locally compact, then $\nu X - X$ is nowhere dense in X^*.*

Proof. Let us put $G = \bigcup\{\mathrm{int}^* Z(h^\beta);\ Z(h^\beta) \subset \beta X - \nu X\}$. From Theorem 3.4, $\bigcup\{Z(h^\beta);\ Z(h^\beta) \subset \beta X - \nu X\} \subset \mathrm{cl}^* G$ and hence by (i) of Lemma 1.1 we have $\beta X - \nu X \subset \mathrm{cl}^* G$. Since X is topologically complete, $\beta X - \nu X$ is dense in $\beta X - \mu X = \beta X - X = X^*$ by (iv) of 2.3. Thus $\mathrm{cl}^* G = X^*$. On the other hand, G is open in X^* and $G \cap (\nu X - X) = \emptyset$ implies $G \cap \mathrm{cl}^*(\nu X - X) = \emptyset$. Hence $\mathrm{int}^* \mathrm{cl}^*(\nu X - X) = \emptyset$ follows from directly the fact that $G \cap \mathrm{cl}^*(\nu X - X) = \emptyset$ and $\mathrm{cl}^* G = X^*$.

The following theorem is an improvement of Theorem FGR_2 as shown in 3.7.

THEOREM 3.6. *If X is not pseudocompact and $H \subset \beta X - \nu X$, then*

i) *The pseudocompactness of $X \cup H$ implies the denseness (in $\beta X - \nu X$) of H.*

ii) *The converse of* i) *is valid whenever* $R(X)$ *is relatively pseudocompact.*

Proof. i) Let $p \in \beta X - \nu X - \mathrm{cl}^* H$. Since $p \notin \mathrm{cl}_\beta H$, there is $f^\beta \geqslant 0$ such that $f^\beta(p) = 0$ and $f^\beta = 1$ on $\mathrm{cl}_\beta H$ by (i) of Lemma 1.1 and $f^\beta > 0$ on X. But this contradicts the pseudocompactness of $X \cup H$ because $1/(f^\beta|(X \cup H))$ is an unbounded member of $C(X \cup H)$.

ii) Suppose that $X \cup H$ is not pc. Then there exists $g^\beta \geqslant 0$ with $Z(g^\beta) \subset \beta X - (X \cup H)$. If $p \in Z(g^\beta) \cap (\beta X - \mu X)$, then $Z(g^\beta)$ contains an open set (in X^*) contained in $\beta X - \nu X$ (notice that $\mathrm{cl}_\beta R(X) \subset \mu X$). But this contradict the denseness (in $\beta X - \nu X$) of H. Thus $Z(g^\beta) \cap (\beta X - \mu X) = \emptyset$ i.e., $Z(g^\beta) \subset \mu X - X$ which is impossible by (ii) of Lemma 1.1.

3.7. We shall show that Theorem FGR_2 is obtained as a corollary of Theorem 3.6. To do this, it is sufficient to prove the following i) and ii) (notice that $\mu X = X$ and $\beta X - \nu X$ is dense in X^* by 2.3).

i) *If* $H \subset X^*$ *and* $X \cup H$ *is pseudocompact, then so is* $K = X \cup (H \cap (\beta X - \nu X))$.

Proof. If K is not pc, then there exists $g^\beta \geqslant 0$ such that $g^\beta > 0$ on K and $Z(g^\beta) \subset \beta X - K$. Since $X \cup H$ is pc, $H \cap (\nu X - X) \cap Z(g^\beta) \neq \emptyset$ and hence $X \cap Z(g^\beta) \neq \emptyset$ by (i) of Lemma 1.1 which contradicts $Z(g^\beta) \cap X = \emptyset$.

ii) *If* H $(\subset X^*)$ *is dense in* X^*, *then* $H_1 = H \cap (\beta X - \nu X)$ *is dense in* $\beta X - \nu X$.

Proof. Without loss of generality we can assume that $H_1 \neq \emptyset$ and let $(\beta X - \nu X) - \mathrm{cl}^* H_1 \neq \emptyset$. $X^* - \mathrm{cl}^* H_1$ is open in X^* and $\mathrm{int}^* \mathrm{cl}^* (\nu X - X) = \emptyset$ by Theorem 3.5. Thus $\mathrm{cl}^*(H_1 \cup (\nu X - X)) = \mathrm{cl}^* H_1 \cup \mathrm{cl}^*(\nu X - X) \neq \beta X - \nu X$. On the other hand $H \subset H_1 \cup (\nu X - X)$ and $\mathrm{cl}^* H = X^*$, and we have $\mathrm{cl}^*(H_1 \cup (\nu X - X)) = X^*$ which is a contradiction.

§ 4. Generalizations of Rudin's and Plank's theorems. In this section we generalize firstly Theorems R_3 and R_4 whose proofs are modifications of ones used in [13] and [14].

A point p of Y is called a *P-point* of Y if any G_δ subset of Y containing p is a nbd of p. Y is said to have the *G_δ-property* if every nonvoid G_δ subsets of Y has a nonvoid interior.

LEMMA 4.1. *For any point* $p \in \beta X - (\mathrm{cl}_\beta R(X) \cup \mu X)$, *any neighborhood* $U^*(p)$ *of* p *contains a copy of* $\beta N - N$.

Proof. By Theorem 2.2 there are $h^\beta \geqslant 0$ and $g^\beta > 0$ such that $Z(h^\beta) \subset U^*(p) \cap (\beta X - \nu X)$, $P(g) \subset X_1$ and $P(g^\beta) \cap X^* \subset Z(h^\beta)$. From the method of construction of g^β, there is a subset $N \subset P(g^\beta) \cap X$ which is C-embedded in X (cf. Theorem 2.2). Thus $\beta N - N \subset (\beta X - \nu X) \cap P(g^\beta) \subset U^*(p)$.

LEMMA 4.2. *Let* $N^* = \beta N - N$ *be the set in* 4.1. *If* $R(X)$ *is relatively pseudocompact, then every* P*-point of* N^* *is a* P*-point of* X^*.

Proof. Let p be a P-point of N^* and $f^* \in C^*(X^*)$, $0 \leqslant f^* \leqslant 1$ and $f^*(p) = 0$. To prove lemma it is sufficient to show that $p \in \mathrm{int}^* Z(f^*)$. Without loss of generality, $f^* = 1$ on $\mathrm{cl}_\beta R(X) \cap X^*$ because $\beta N \cap \mathrm{cl}_\beta R(X) = \emptyset$. Put $k^* = f^* - 1 \in C^*(X^*)$. $k^*(p) = -1$ and $k^* = 0$ on $X^* \cap \mathrm{cl}_\beta R(X)$. There is h^β with $0 \leqslant h^\beta \leqslant 1$, $h^\beta = 1$ on βN and $h^\beta = 0$ on some nbd of $\mathrm{cl}_\beta R(X)$. Then $k^* \cdot (h^\beta | X^*) \in C^*(X^*)$ and we may extend this function to a member of $C^*(X^* \cup R(X))$ by setting its value 0 on $R(X)$. Since $X^* \cup R(X)$ is compact, this extended function has a continuous extension f_1^β over βX. Put $g_1^\beta = f_1^\beta + 1$. Obviously $g_1^\beta(p) = 0$, $g_1^\beta = 1$ on $\mathrm{cl}_\beta R(X)$ and $Z(g_1^\beta) \cap X^* \subset Z(f^*)$. Since $p \in N^*$, $p \in \beta X - \nu X$ and as in the proof of Lemma 2.1, there is $g^\beta \geqslant 0$ such that $P(g^\beta) \cap X^* \subset Z(g_1^\beta) \cap X^* \subset Z(f^*)$ and $p \in P(g^\beta)$. Thus $p \in \mathrm{int}^* Z(f^*)$.

LEMMA 4.3. *Every point* p *in* $\beta X - \mu X$ (*and hence in* $X^* - (\mathrm{cl}_\beta R(X) \cup \mu X)$) *is not isolated in* X^*.

Proof. Let $p \in \nu X - \mu X$. Since $\beta X - \nu X$ is dense in $\beta X - \mu X$ by 2.3, p is not isolated in X^*. For $p \in \beta X - \nu X$, there is $Z(f^\beta)$ with $p \in Z(f^\beta) \subset \beta X - \nu X$. If p is isolated, then we may select f^β with $Z(f^\beta) = \{p\}$ which contradict 9.6 in [5] (every point p of X^* is not G_δ in βX).

Remark. A point in $\mu X - X$ may be isolated in X^*. For example, let $X = ([0, \Omega] \times [0, \Omega] - (\Omega, \Omega)) \cup R$ (top. sum). Then $\nu X = \mu X$ and $\mu X - X$ contains the point (Ω, Ω) which is an isolated point of X^*.

From above lemmas we have a generalization of Theorem R_3.

THEOREM 4.4 [CH]. *If* $R(X)$ *is relatively pseudocompact, then* $\beta X - \mu X$ *has a subset of* P*-points of* X^* *which is dense in* $\beta X - \mu X$.

Proof. Let p be an arbitrary point in $\beta X - \mu X$. Since $R(X)$ is rpc, we have $\mathrm{cl}_\beta R(X) \subset \mu X$ by (i) of Theorem 1.2 which assures $p \notin \beta X - (\mathrm{cl}_\beta R(X) \cup \mu X)$ and p is not isolated in X^* by Lemma 4.3. By Lemma 4.1 any nbd $U^*(p)$ contains a copy of N^*. Thus Lemma 4.2 and Rudin's result [15], it follows that $U^*(p)$ contains a P-point of X^* which concludes the proof.

THEOREM 4.5 [CH]. *Let* X *be an almost locally compact metric space without isolated points. Then* X^* *contains a set of, at least,* 2^c *remote points which is dense in* $X^* - (\mathrm{cl}_\beta R(X) \cup \nu X)$.

Proof. Let p be an arbitrary point in $X^* - (\mathrm{cl}_\beta R(X) \cup \nu X)$. p is not isolated in X^* by Lemma 4.3. Since $p \in \mathrm{cl}_\beta X_1$, any nbd $U_\beta(p)$ contains a cozero set $P(g^\beta)$ such that $g^\beta \geqslant 0$, $P(g^\beta) \cap \mathrm{cl}_\beta R(X) = \emptyset$ and $P(g) \neq \emptyset$ by Theorem 2.2. Then $P(g)$ contains a locally finite disjoint family $\{U_n\}$ of open sets of X such that $\mathrm{cl}_X U_n$ is compact and $S = \bigcup \{\mathrm{cl}_X U_n\}$ is C-embedded and $\mathrm{cl}_\beta S \subset P(g^\beta)$. Let us put $W = \bigcup U_n$. Then $S = \mathrm{cl}_X W$

$= \bigcup \{\mathrm{cl}_X U_n\}$ is σ-compact. By Plank's theorem 5.5 [CH] [13], βS has a collection of $2^{\mathfrak{c}}$ remote points of βS which forms a dense subset of $\beta S - S$. On the other hand, by Lemma (in § 3 of [14]) every remote point of βS is a remote point of βX. Thus $U^*(p) = U_\beta(p) \cap X^*$ contains $2^{\mathfrak{c}}$ remote points which completes the proof.

Next we generalized Plank's theorem 3.3 and 3.4 whose proof are the same as in [13] except using Theorems 3.2, 3.3 and above lemmas.

THEOREM 4.6 [CH]. *Let X be an almost locally compact space which is not pseudocompact. If A is a β-subalgebra of $C(X)$ with $|A| = \mathfrak{c}$, then X^* contains a subset of $2^{\mathfrak{c}}$ A-points which is dense in $\beta X - (\mu X \cup \mathrm{cl}_\beta R(X))$.*

Proof. Let $p \in X^* - (\mu X \cup \mathrm{cl}_\beta R(X))$. Similary to the proof of Theorem 4.5, there exists $g^\beta \geqslant 0$ with $\mathrm{cl}_\beta P(g^\beta) \cap \mathrm{cl}_\beta R(X) = \emptyset$ by Theorem 2.2. Thus we can consider that $P(g^\beta) \cap X^*$ is locally compact. On the other hand, by Theorems 3.2 and 3.3, it is easy to see that $P(g^\beta) \cap X^*$ has the property G_δ. From the same method of proofs of Theorems 3.2 and 3.3 in [13] we conclude the proof of our theorem.

A modification of the proof of above theorem gives the following generalization of Theorem 3.4 in [13].

THEOREM 4.7 [CH]. *Let X be an almost locally compact space which is not pseudocompact. If $\{A_\alpha;\ \alpha \in \Lambda\}$ is a family of β-subalgebras of $C(X)$ with $|A_\alpha| = \mathfrak{c}$ for each $\alpha \in \Lambda$ and $|\Lambda| = \mathfrak{c}$. Then X^* contains a subset of $2^{\mathfrak{c}}$ points which are simultaneously A-points for all $\alpha \in \Lambda$ and dense in $\beta X - (\mu X \cup \mathrm{cl}_\beta R(X))$.*

§ 5. Round subsets and P-points. A subset A of βX is said to be *round* if $\mathrm{cl}_\beta Z(f)$ is a nbd of A whenever $A \subset \mathrm{cl}_\beta Z(f)$ [10], and Mandelker proved the following interesting theorems: $\beta X - X$ is a round subset of βX if and only if the intersection of all free maximal ideals in $C(X)$ is precisely the family $C_K(X)$ of all functions with compact support. Moreover the following results was obtained by him: i) For any space X, $\beta X - \nu X$ is a round subset of βX. ii) X is a P-space if and only if every subset of βX is round.

THEOREM 5.1. *For any space X, $\beta X - \mu X$ is a round subset of βX.*

Proof. Let $\beta X - \mu X \subset \mathrm{cl}_\beta Z(f)$ and $f \geqslant 0$. If $P(f)$ is not rpc, then $P(f)$ contains a C-embedded subset N and $\mathrm{cl}_\beta N - N \subset \beta X - \nu X$. On the other hand, $Z(f) \cap N = \emptyset$ and by 1.20 of [5], $Z(f)$ and N are completely separated. This shows $\mathrm{cl}_\beta N \cap \mathrm{cl}_\beta Z(f) = \emptyset$, which contradicts the fact that $\beta X - \mu X \subset \mathrm{cl}_\beta Z(f)$. Thus $P(f)$ must be rpc. Since by Theorem 1.2, $\mathrm{cl}_\beta P(f) \subset \mu X$ and $\mathrm{cl}_\beta P(f) \cup \mathrm{cl}_\beta Z(f) = \beta X$, we have $\beta X - \mu X \subset \beta X - \mathrm{cl}_\beta P(f) \subset \mathrm{cl}_\beta Z(f)$ which leads $\beta X - \mu X \subset \mathrm{int}_\beta \mathrm{cl}_\beta Z(f)$.

THEOREM 5.2. *Let A be a compact round subset of βX contained in $\nu X - X$.*

i) *Every nonvoid G_δ subset of X^* containing A has a nonvoid interior containing A.*

ii) *For any subset $B \subset \beta X$, $|B| = \aleph_0$, $A \cap B = \emptyset$, we have $\mathrm{cl}_\beta B \cap A = \emptyset$.*

iii) *A point p of $\nu X - X$ is a round subset of βX, then p is a P-point of X^*.*

Proof. i) Suppose that there are open sets W_n in βX $(n \in N)$ such that $\mathrm{cl}_\beta W_n \subset W_{n-1}$, $U_n = X^* \cap W_n$ is a nbd of A and $\bigcap U_n$ is not a nbd (in X^*) of A. For each n, let f_n^β be a function such that $0 \leqslant f_n^\beta \leqslant 1$, $f_n^\beta = 1$ on $\beta X - W_n$ and $f_n^\beta = 0$ on A. Obviously $f^\beta = \sum (1/2^n) \cdot f_n^\beta \in C(\beta X)$ and $f^\beta = 0$ on A. Since $A \subset \nu X - X$ and $A \subset Z(f^\beta)$, we have $Z(f) \neq \emptyset$ and $A \subset \nu X \cap Z(f^\beta) = \nu X \cap \mathrm{cl}_\beta Z(f)$ by (iii) of Theorem 1.2, i.e., $A \subset \mathrm{cl}_\beta Z(f)$. But from the method of construction of f^β, $A \not\subset \mathrm{int}_\beta \mathrm{cl}_\beta Z(f)$ which shows that A is not a round subset of βX.

ii) and iii) follow from i).

5.3. EXAMPLE. i) In Theorem 5.2, we can not drop the condition $p \in \nu X - X$. Let $X = N$, $\beta X - \nu X = X^*$. Since X is a P-space, every subset of βX is a round subset of βX, but there is a point in X^* which is not a P-point.

ii) There is a space that every point of $\nu X - X$ is a round subset of βX and hence a P-point of X^*. Such a space is given in [5], i.e., there is a P-space which is not realcompact.

iii) The converse of iii) in Theorem 5.2 is not valid. Let X be a space given in Negrepontis [12]. If we put $X = N \cup E$, then X is locally compact and $\nu X = X \cup \{p\}$ where p is a P-point of X^*. Obviously $p \in \mathrm{cl}_\beta N$ and $\{p\}$ is not a round subset of βX by Theorem 5.2.

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The G_δ-topology on compact spaces

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Abstract. Is every G_δ-covering of a compact topological space reducible to a $2^{\aleph_0}$-subcovering? This question is answered in the affirmative for linearly ordered topological space. In fact, the authors' major result is that each G_δ-covering of a finite product of compact linearly ordered topological spaces has a $2^{\aleph_0}$-star-finite refinement consisting of G_δ sets.

1. Introduction. Nearly forty years ago Alexandrov and Urysohn [1] conjectured that a compact first countable Hausdorff space could contain no more than $2^{\aleph_0}$ points. Arhangel'skiĭ [2] proved that conjecture true several years ago, and thereby re-opened some related questions, one of which was related to W. Fleischman by Professor I. Juhasz. In the present paper we offer more than a solution to this question:

Each covering by G_δ-sets of a compact linearly ordered topological space is reducible to a subcovering of cardinality not exceeding $2^{\aleph_0}$.

2. Compact linearly ordered topological spaces. We will call compact linearly ordered spaces *CLOTS*.

Let (X, τ) be a topological space. The collection of τ-G_δ-sets can form a base for a finer topology δ on X, called the *G_δ-topology* on X induced by τ, or simply the G_δ-topology when no confusion results.

If for each Σ-open covering of the space (X, Σ) there exists a subcovering of cardinal not exceeding m, we say (X, Σ) is *m-lindelöf*. With this definition in mind, we observe that our problem may be formulated as follows:

If (X, τ) is a CLOTS, then (X, δ) is $2^{\aleph_0}$-lindelöf.

Of course if (X, τ) is first countable, (X, δ) is discrete, and therefore, (X, δ) is $|X|$-lindelöf. Hence, the unit interval with the Euclidean topology exemplifies a CLOTS whose G_δ-topology is not m-lindelöf for any $m < 2^{\aleph_0}$.

Our first proposition seems to follow from [3, p. 27]; nevertheless, we shall offer an alternative proof igniting the techniques used in this work.

PROPOSITION A. *If the linear ordering of the CLOTS (X, τ) is a well-ordering, then (X, δ) is lindelöf ($\aleph_0$-lindelöf).*