$\dot{V}$ of V is a strong deformation retract of $\overline{V}-\{p\}$. Although our spaces have nonsymmetric distance we define, following [5] and [9], an r-space as a locally compact general metric space of constant topological dimension where each point possesses arbitrarily small canonical neighborhoods. Since the above theorem says that arbitrarily small positive balls are canonical neighborhoods, we see that our spaces are also r-spaces.

6.3. Therefore all the results contained in the theory of r-spaces (see [5], § 3), carry over. For example, the spheres $K^+(p, \varrho)$, $0<\varrho<\gamma(p)$, are not contractible and the property of domain invariance holds in our spaces. Many important results on conjugate points proved by Busemann [see [5], pp. 14–20) depend only on these two properties and hence they c rry over to our spaces. Since his methods generalize, we omit the details.

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Weakly smooth dendroids

by

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Abstract. Let X be a dendroid. For each point in X let η_p denote the function from X into 2^X given by $\eta_p(x) = [p, x]$, where 2^X is the space of all nonempty closed subsets of X with the Vietoris topology and $[p, x]$ is the unique irreducible continuum from p to x. Observe that X is smooth if for some p, η_p is a homeomorphism of X onto its image $\mathfrak{D}(X, p)$. The converse is also true. The space $\mathfrak{D}(X, p)$ is studied for non-smooth dendroids. Define X to be *weakly smooth* if there exists a point p such that $\mathfrak{D}(X, p)$ is a compact subset of 2^X. Order-theoretic characterizations of weakly smooth dendroids are obtained.

1. Introduction. Throughout this paper *continuum* will mean a compact connected metric space containing more than one point. A continuum is *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected. The *weak cut point order* on a hereditarily unicoherent continuum X with respect to p, $\leqslant_p$, is defined by $x \leqslant_p y$ if and only if $x \in [p, y]$, where $[p, y]$ denotes the intersection of all subcontinua of X containing p and y. *A dendroid* is an arcwise connected hereditarily unicoherent continuum. If X is a dendroid, then $\leqslant_p$ is a partial order and $[p, y]$ is an arc for all $y \in X$. For any point p in a dendroid X denote by $\mathfrak{D}(X, p)$ the set of all arcs in X of the form $[p, x]$. We view $\mathfrak{D}(X, p)$ as a subspace of 2^X, where 2^X denotes the space of nonempty closed subsets of X with the Vietoris topology [6].

Charatonik and Eberhart [1] investigate smooth dendroids (Definition 1). Here the more general notion of weakly smooth dendroids is introduced: A dendroid X is said to be *weakly smooth* if $\mathfrak{D}(X, p)$ is a compact subset of 2^X for some $p \in X$.

The work is divided into three sections. The first section deals with the structure of $\mathfrak{D}(X, p)$ and with two partial order characterizations of weakly smooth dendroids similar to those of smooth dendroids (Theorem 2). In the second section these results are applied to obtain necessary and sufficient conditions for a dendroid to be smooth and for a dendroid to be a dendrite ($\equiv$ locally connected dendroid). We discuss necessary and sufficient conditions for hereditarily unicoherent continua to be arcwise connected in the final section.

The author wishes to express his sincere gratitude to Professor L. E. Ward, Jr. for his advice and guidance throughout the period during which this research was conducted. The author also wishes to thank David Paulowich and William Linn for a number of helpful conversations.

2. Weakly smooth dendroids. In what follows assume X is a dendroid. For any point $p \in X$ denote by $\mathfrak{D}(X, p)$ the subspace of 2^X consisting of all subcontinua of the form $[p, x]$, where $x \in X$. Let η_p denote the natural function from X onto $\mathfrak{D}(X, p)$ defined by $\eta_p(x) = [p, x]$. Hereafter we will write η for η_p. Note that η is one-to-one.

Before we begin our discussion of $\mathfrak{D}(X, p)$ we need a result about smooth dendroids.

Definition. A dendroid X is *smooth* if there exists a point $p \in X$, called an *initial point of* X, such that given any sequence x_n in X with $\lim\limits_{n\to\infty} x_n = x_0$, it follows that $\operatorname*{Lim}\limits_{n\to\infty}[p, x_n] = [p, x_0]$ (for the definitions of Li, Ls, and Lim see [4], pp. 335–339).

Theorem 1. *A point $p \in X$ is an initial point if and only if the function η is continuous, and hence a homeomorphism.*

Proof. Since X is compact and η is one-to-one and onto, it follows that η is continuous if and only if η is a homeomorphism. Moreover, the statement that η is continuous is precisely the statement that p is an initial point of X.

Corollary 1.1. *A point $p \in X$ is an initial point of X if and only if η is upper semicontinuous.*

Proof. By Theorem 2 ([5], p. 62) η is always lower semicontinuous.

Theorem 2. *The space $\mathfrak{D}(X, p)$ is arcwise connected for any point $p \in X$. In particular, for any $x \in X$, the set $\eta([p, x])$ is an arc in $\mathfrak{D}(X, p)$ with endpoints $[p, p]$ and $[p, x]$.*

Proof. The result follows from Theorem 1 since p is an initial point of the smooth dendroid $[p, x]$.

The partial order, $\leqslant_p$, on X induces a partial order, $\leqslant$, on $\mathfrak{D}(X, p)$ defined by $[p, x] \leqslant [p, y]$ if and only if $x \leqslant_p y$. This partial order on $\mathfrak{D}(X, p)$ is always closed (Theorem 3) even though $\leqslant_p$ may not be a closed partial order on X.

Theorem 3. *For any point $p \in X$ the space $\mathfrak{D}(X, p)$ is metrizable and the induced partial order on $\mathfrak{D}(X, p)$ is closed.*

Proof. It is well known that 2^X is a metric space with closed partial order, τ, given by $A \tau B$ if and only if $A \subseteq B$. Thus, $\mathfrak{D}(X, p)$ inherits both the metric and the closed partial order. Clearly, the order on $\mathfrak{D}(X, p)$ induced by τ coincides with the order $\leqslant$.

It is not necessarily true, as we shall see, that $\mathfrak{D}(X, p)$ is a closed subset of 2^X. Since 2^X is a compact Hausdorff space, $\mathfrak{D}(X, p)$ is closed in 2^X if and only if $\mathfrak{D}(X, p)$ is compact. It follows from Theorem 1 that if X is a smooth dendroid, then $\mathfrak{D}(X, p)$ is compact for some $p \in X$. These remarks motivate the following definition.

Definition. A dendroid X is *weakly smooth* if there exists a point $p \in X$, called a *weak initial point of* X, such that $\mathfrak{D}(X, p)$ is compact.

In this terminology every smooth dendroid with initial point p is weakly smooth with weak initial point p. We construct a weakly smooth dendroid which is not smooth in Example 1. There exists a dendroid (Example 3) which is not weakly smooth.

Example 1. In the plane define $X_1 = T \cup \bigcup\limits_{n=-1}^{\infty} A_n$ and $Y_1 = T \cup \bigcup\limits_{n=-1}^{\infty} B_n$, where:

$$T = [0, 1] \times \{0\}, \quad A_{-1} = B_{-1} = \{0\} \times [0, 1],$$

and for $n = 0, 1, 2, \ldots$

$$A_n = \left\{\frac{1}{2^n}\right\} \times [0, 1], \quad \text{and} \quad B_n = \left\{\frac{1}{2^n}\right\} \times \left[0, \frac{1}{2^n}\right].$$

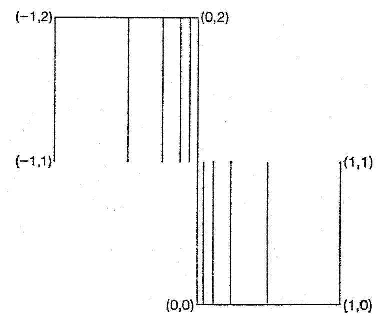

Fig. 1

Now let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$, where X_2 (Y_2) is the reflection of X_1 (Y_1) about the point $p = (0, 1)$ (see Figures 1 and 2, respectively). Clearly, X is a non-smooth dendroid. On the other hand, it is not hard to verify that $\mathfrak{D}(X, p)$ is homeomorphic to Y, which is compact.

The following lemma is a corollary to Theorem 1 ([3], p. 680).

Lemma 1. *A dendroid X is smooth with initial point $p \in X$ if and only if the partial order $\leqslant_p$ is closed.*

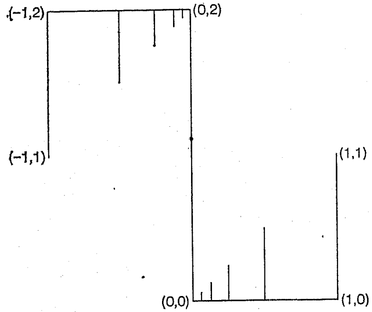

Fig. 2

The next theorem gives an analogous characterization of weakly smooth dendroids.

THEOREM 4. *If p is a point in the dendroid X, then the following statements are equivalent:*

(i) *Given any two sequences a_n and b_n in X with $\lim\limits_{n\to\infty} a_n = a_0$, $\lim\limits_{n\to\infty} b_n = b_0$, and $a_n \leqslant_p b_n$ for each n, it follows that $a_0 \leqslant_p b_0$ or $b_0 \leqslant_p a_0$.*

(ii) *Given any convergent sequence x_n in X, it follows that $\operatorname{Li}\limits_{n\to\infty}[p, x_n] = [p, x]$ for some $x \in X$.*

(iii) *The point $p \in X$ is a weak initial point of X.*

Proof. (i) implies (ii). Let x_n be a convergent sequence in X. It is known that $\operatorname{Li}\limits_{n\to\infty}[p, x_n] = A$ is a non-degenerate subcontinuum of X. It follows from (i) that A is a $\leqslant_p$-totally ordered subset of X. Since $p \in A$ and A is a totally ordered subcontinuum of X, A must be of the form $[p, x]$, for some $x \in X$.

(ii) implies (iii). Let $[p, x_n]$ be a convergent sequence of points in $\mathfrak{D}(X, p)$. Then $\operatorname{Lim}\limits_{n\to\infty}[p, x_n] = A$ is a non-degenerate subcontinuum of X. Choose in X a convergent subsequence, x_{n_k}, of the sequence x_n. By (ii) there exists a point $x \in X$ such that $\operatorname{Li}\limits_{k\to\infty}[p, x_{n_k}] = [p, x]$. Thus,

$$A = \operatorname{Lim}_{n\to\infty}[p, x_n] = \operatorname{Li}_{k\to\infty}[p, x_{n_k}] = [p, x].$$

That is, $\mathfrak{D}(X, p)$ is a closed subset of 2^X.

(iii) implies (i). Let a_n and b_n be two sequences in X satisfying the hypotheses of (i). Let $[p, b_{n_k}]$ be a convergent subsequence of the sequence $[p, b_n]$ in $\mathfrak{D}(X, p)$ (see [9], Theorem 7.1, p. 11). Since $\mathfrak{D}(X, p)$ is a closed subset of 2^X, $\operatorname{Lim}\limits_{k\to\infty}[p, b_{n_k}] = [p, x]$, for some $x \in X$. It follows immediately that $a_0 \leqslant_p b_0$ or $b_0 \leqslant_p a_0$.

Charatonik and Eberhart ([1], Corollary 6, p. 299) have shown the property of being a smooth dendroid is hereditary to subcontinua. This result generalizes to weakly smooth dendroids.

THEOREM 5. *Each subcontinuum of a weakly smooth dendroid is a weakly smooth dendroid.*

Proof. Let Y be a subcontinuum of the dendroid X with weak initial point p. It is well known that Y is a dendroid. Let $q \in Y$ be the zero of Y with respect to the partial order $\leqslant_p$ on X (see [8], Lemma 2, p. 923). We show q is a weak initial point of Y. Let x_n be a convergent sequence of points in Y. Since p is a weak initial point of X there exists a point $x \in X$ such that $\operatorname{Li}\limits_{n\to\infty}[p, x_n] = [p, x]$. Since q is the zero of Y and $x_n \in Y$, it follows that

$$[p, x_n] = [p, q] \cup [q, x_n]$$

and $[q, x_n] \subset Y$, for each n. Whence

$$[p, x] = [p, q] \cup \operatorname{Li}_{n\to\infty}[q, x_n]$$

and $\operatorname{Li}\limits_{n\to\infty}[q, x_n] \subset Y$. Now since

$$[p, q] \cap \operatorname{Li}_{n\to\infty}[q, x_n] = \{q\}$$

it is straightforward to verify that $x \in Y$ and

$$\operatorname{Li}_{n\to\infty}[q, x_n] = [q, x].$$

That is, q is a weak initial point of Y.

THEOREM 6. *Let X be a dendroid and let $p \in X$. If Y is a subcontinuum of X and $q \in Y$ is the zero of Y with respect to $\leqslant_p$, then $\mathfrak{D}(Y, q)$ is homeomorphic to $\eta(Y) \subset \mathfrak{D}(X, p)$.*

Proof. Define the function f from $2^X \times 2^X$ onto 2^X by $f\big((K, L)\big) = K \cup L$. By Corollary 4a ([4], p. 116) f is continuous. If h denotes the restriction of f to the subspace $\mathfrak{D}(Y, q) \times \{[p, q]\}$, then h is a homeomorphism into $\mathfrak{D}(X, p)$.

COROLLARY 6.1. *In the notation of Theorem 6 if p is a weak initial point of X, then $\eta(Y)$ is a subcontinuum of $\mathfrak{D}(X, p)$.*

3. Applications to smooth dendroids and dendrites. The first application gives an affirmative answer to a question posed by Sam B. Nadler, Jr. He asked: If a dendroid X is homeomorphic to $\mathfrak{D}(X, p)$ for some $p \in X$, is X a smooth dendroid?

THEOREM 7. *A dendroid X is smooth if and only if there exists a point $p \in X$ such that X is homeomorphic to $\mathfrak{D}(X, p)$.*

Proof. If X is homeomorphic to $\mathfrak{D}(X, p)$, then $\mathfrak{D}(X, p)$ is a smooth dendroid by Theorem 3 and Lemma 1. But then X is a smooth dendroid. The converse is Theorem 1.

In Theorem 7 one might conjecture if X is homeomorphic to $\mathfrak{D}(X, p)$, then p is an initial point of X. This is not necessarily true as the example below shows.

EXAMPLE 2. Let X_1 and Y_1 be as in Example 1. For $n = 0, 1, 2, \ldots$ define homeomorphisms h_n (k_n) from X_1 (Y_1) into the plane by:

$$h_n((x, y)) = \left(\frac{2^n-1}{2^{n-1}} + \frac{x}{2^n}, y\right) \quad \text{and} \quad k_n((x, y)) = \left(\frac{1-2^{n+1}}{2^n} + \frac{x}{2^n}, y\right).$$

Let X denote the closure in the plane of $\bigcup_{n=0}^{\infty} \big(h_n(X_1) \cup k_n(Y_1)\big)$ (see Figure 3). It can be shown that X is a dendroid which is homeomorphic to $\mathfrak{D}(X, p)$, where $p = (0, 1)$, but p is not an initial point of X. Note that the set of initial points of X consists of all points except those of the form (x, y) where $x = \dfrac{2^n-1}{2^{n-1}}$ $(n = 0, 1, 2, \ldots)$ and $y > 0$.

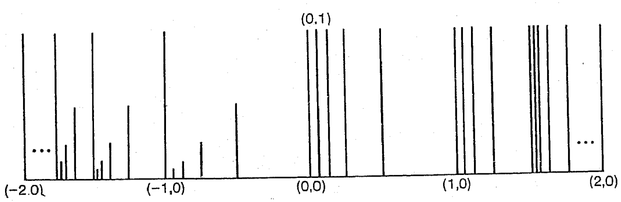

Fig. 3

Koch and Krule [3] have shown that a dendroid is locally connected at each of its initial points. Hence if every point of X is an initial point, then X is a dendrite (see [1], pp. 298–299). Although the first statement does not hold if "initial point" is replaced by "weak initial point" (in Example 1, X is not locally connected at $p = (0, 1)$), the latter does hold (Theorem 8).

THEOREM 8. *If X is a dendroid, then X is a dendrite if and only if every point of X is a weak initial point.*

Proof. If X fails to be locally connected, then there exists a pairwise disjoint sequence, Y_n $(n = 0, 1, 2, \ldots)$, of non-degenerate subcontinua of X satisfying $\operatorname{Lim}_{n\to\infty} Y_n = Y_0$ ([9], Theorem 12.1, p. 18). Let q and r be distinct points in Y_0, then $[q, r] \subset Y_0$. Since $\operatorname{Lim}_{n\to\infty} Y_n = Y_0$ there are points q_n and r_n, for $n = 1, 2, \ldots$, such that

$$\lim_{n\to\infty} q_n = q \quad \text{and} \quad \lim_{n\to\infty} r_n = r.$$

Let p_0 be a cutpoint of $[q, r]$. It is easy to see that

$$p_0 \in [q, q_n] \quad \text{or} \quad p_0 \in [r, r_n],$$

for each n (otherwise X contains a simple closed curve). Whence, for infinitely many n,

$$p_0 \in [q, q_n]$$

or, for infinitely many n,

$$p_0 \in [r, r_n].$$

Without loss of generality we assume the latter. Passing to a subsequence, if necessary, we can further assume $p_0 \in [r, r_n]$, for each n. Let p be a cutpoint of $[p_0, r]$. It is easy to verify that for each n,

$$p_0 \in [p, r_n],$$

and hence,

$$[p_0, r] \subset \operatorname{Li}_{n\to\infty} [p, r_n].$$

Now by hypothesis p is a weak initial point of X, so there exists a point $r_0 \in X$ such that

$$\operatorname{Li}_{n\to\infty} [p, r_n] = [p, r_0].$$

But then,

$$[p_0, r] \subset [p, r_0]$$

which contradicts the definition of p. Therefore X is locally connected and hence X is a dendrite. The converse is easy.

COROLLARY 8.1. *Let X be a weakly smooth dendroid with weak initial point p. If $\mathfrak{D}(X, p)$ is homeomorphic to $\mathfrak{D}(X, q)$ for all $q \in X$, then X is a dendrite.*

In Corollary 8.1 it is necessary that X be a weakly smooth dendroid.

EXAMPLE 3. Let X be the dendroid illustrated in the Figure 4. It is not difficult to see that X contains no weak initial points and $\mathfrak{D}(X, p)$ is homeomorphic to $\mathfrak{D}(X, q)$ for all $p, q \in X$. (See also Figure 1 ([1], p. 305)). Example 3 is due to D. Paulowich.

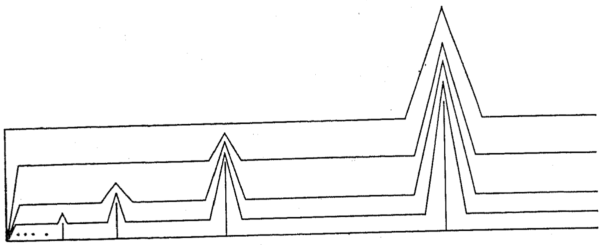

Fig. 4

4. Arcs in hereditarily unicoherent continua. Throughout this section X will denote a hereditarily unicoherent continuum (not necessarily arcwise connected). For each point $p \in X$ define η_p and $\mathfrak{D}(X, p)$ as in Section 1. In this setting $[p, x]$ need not be an arc, η_p need not be one-to-one, and the weak cutpoint order, $\leqslant_p$, may not be a partial order. In fact, η_p is one-to-one if and only if $\leqslant_p$ is a partial order.

THEOREM 9. *A hereditarily unicoherent continuum X is arcwise connected (i.e., X is a dendroid) if and only if η_p is one-to-one for all $p \in X$.*

Proof. If X is a dendroid, then $\leqslant_p$ is a partial order for all $p \in X$. For the converse, fix a point $q \in X$. It suffices to show $[p, q]$ is a continuum with exactly two non-cutpoints. To that end, choose any $x \in [p, q]$ and note that

$$[p, x] \cup [x, q] = [p, q].$$

Using the fact that η_p and η_q are one-to-one it is straightforward to verify that

$$[p, x] \cap [x, q] = \{x\}.$$

Now since

$$([p, q]-[x, q]) \cup ([p, q]-[p, x]) = [p, q]-\{x\},$$

x is a cutpoint of $[p, q]$. Together with the fact that $[p, q]$ contains at least two non-cutpoints, the above shows that $[p, q]$ has exactly two non-cutpoints.

Theorem 9 is not true if we assume only η_p is one-to-one for some p.

EXAMPLE 4. Let X be the $\sin\frac{1}{x}$ curve:

$$X = \left\{\left(x, \sin\frac{1}{x}\right) \middle|\ 0 < x \leqslant 1\right\} \cup \{(0, y) \mid\ -1 \leqslant y \leqslant 1\}.$$

It is apparent that X is a hereditarily unicoherent continuum which is not a dendroid. Note that the points for which η_p is one-to-one are those of the form $(0, y)$ for $-1 \leqslant y \leqslant 1$.

If $Y = X \cup Z$, where Z is the reflection of X about the line $x = 1$, then Y is a hereditarily unicoherent continuum for which η_p is one-to-one for no p.

The following is a consequence of Corollary 1 ([2], p. 726). See also ([7], p. 375).

THEOREM 10. *If X is a hereditarily unicoherent continuum and $p \in X$ is such that $\leqslant_p$ is a closed partial order, then X is arcwise connected and hence X is a smooth dendroid.*

Theorem 10 enables us to generalize Theorems 1 and 7.

THEOREM 11. *If X is a hereditarily unicoherent continuum, then the following statements are equivalent:*

(i) *X is a smooth dendroid.*

(ii) *There exists a point $p \in X$ such that η_p is one-to-one and upper semicontinuous.*

(iii) *There exists a point $p \in X$ such that η_p is one-to-one and X is homeomorphic to $\mathfrak{D}(X, p)$.*

Proof. (i) implies (ii) follows from Theorem 1. From Theorem 2 ([5], p. 62) we infer (ii) implies (iii). Now assume there exists a point $p \in X$ such that η_p is one-to-one and X is homeomorphic to $\mathfrak{D}(X, p)$. Then $\leqslant_p$ is a partial order on X and $\mathfrak{D}(X, p)$ is a hereditarily unicoherent continuum. It follows that the induced partial order on $\mathfrak{D}(X, p)$ coincides with the weak cutpoint order, $\leqslant_{[p,p]}$, with respect to $[p, p]$. Moreover, since Theorem 3 does not depend on X being arcwise connected $\leqslant_{[p,p]}$ is a closed partial order on $\mathfrak{D}(X, p)$. We infer from Theorem 10 that $\mathfrak{D}(X, p)$ is a smooth dendroid. Consequently, (iii) implies X is a smooth dendroid.

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Theorien abelscher Gruppen mit einem einstelligen Prädikat

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Zusammenfassung. Mit Hilfe der Methode der Modellinterpretierbarkeit wird folgendes Theorem bewiesen:

Theorem. *Sei K eine Klasse abelscher Gruppen, deren Mächtigkeiten nicht endlich beschränkt sind (z. B. kann K aus einer einzigen unendlichen abelschen Gruppe bestehen), ferner sei K_P die Klasse aller abelschen Gruppen $\langle A, P\rangle$ mit einstelligem Prädikat P, $A \in K$, sowie K_P^* die Klasse aller abelschen Gruppen $\langle A, P\rangle$, $A \in K$, wobei P ein endliches einstelliges Prädikat ist. Dann sind die elementaren Theorien* $\mathrm{Th} K_P$ *und* $\mathrm{Th} K_P^*$ *rekursiv unentscheidbar.*

1. Das Resultat. Es hat sich herausgestellt, daß die in der Sprache $\langle + \rangle$ formalisierten elementaren Theorien der wichtigsten Klassen abelscher Gruppen rekursiv entscheidbar sind, speziell die Theorie der Klasse aller abelschen Gruppen. Die grundlegende Arbeit auf diesem Gebiet wurde 1954 von W. Szmielew veröffentlicht [7]. Die Situation verändert sich radikal, wenn die zugrundegelegte Sprache erweitert wird. In dieser Arbeit wird die um ein einstelliges Prädikatensymbol P erweiterte Sprache der abelschen Gruppen betrachtet. Ist K eine Klasse abelscher Gruppen, so daß die Mächtigkeiten der Gruppen aus K endlich beschränkt sind, so ist nicht nur die elementare Theorie $\mathrm{Th} K$, sondern auch $\mathrm{Th} K_P$ entscheidbar, wobei K_P die Klasse aller Strukturen $\langle A, P\rangle$ ist, in denen A eine abelsche Gruppe aus K und P ein einstelliges Prädikat in A ist. Das Hauptresultat ist nun, daß diese Fälle gewissermaßen die einzigen hinsichtlich der rekursiven Entscheidbarkeit sind, d.h., es gilt das folgende

THEOREM. *Sei K eine Klasse abelscher Gruppen, deren Mächtigkeiten nicht endlich beschränkt sind (z.B. kann K aus einer einzigen unendlichen abelschen Gruppe bestehen), ferner sei K_P die Klasse aller abelschen Gruppen $\langle A, P\rangle$ mit einstelligem Prädikat P, $A \in K$, sowie K_P^* die Klasse aller abelschen Gruppen $\langle A, P\rangle$, $A \in K$, wobei P ein endliches einstelliges Prädikat ist. Dann sind die elementaren Theorien* $\mathrm{Th} K_P$ *und* $\mathrm{Th} K_P^*$ *rekursiv unentscheidbar.*

Hieraus folgt unmittelbar das folgende Korollar: