Uncountable \( \beta \)-models with countable height

by

Wojciech Guzicki (Warszawa)

Abstract. A model of ZFC\(^-\)+V=H is constructed so that its Cohen generic extensions are elementary. This construction is used to prove that there exist uncountable \( \beta \)-models of second order arithmetic with countable height. In presence of Martin's Axiom they can have power \( 2^\omega \).

1. Introduction. In the present paper we prove the existence of uncountable standard models with countable height of the theory ZFC\(^-\) (ZF-set theory without the power set axiom and with the axiom scheme of choice):

\[
(x)(\exists z)(\exists f)(\forall x)(\exists \Phi)(\forall \Phi)(\exists \phi)(\forall \phi)(\exists \psi)(\forall \psi)(\forall \alpha)(\exists \beta)(\forall \gamma)(\exists \delta)(\forall \delta)

Hence we conclude that there exist uncountable \( \beta \)-models for second order arithmetics \( A_\beta \) with only countably many non-similar well-orderings. Assuming Martin's Axiom, we can strengthen these results by replacing the words “uncountable” by the words “of the power of the continuum”.

A. Mostowski in [1] raised the following problem: are there two non-isomorphic \( \omega \)-models of \( A_\beta \) of the same power \( \geq \omega_1 \)?

Our paper gives a positive answer for every power \( \leq 2^\kappa \), but with the aid of Martin's Axiom. The result in the case of \( 2^\kappa = \kappa \) is well known.

Gerald E. Sacks has kindly informed me that he proved the theorem on the existence of \( \beta \)-models of power \( 2^\kappa \) without any additional assumptions. His proof, which he sketched during the Logical Semester in Warsaw, was much more involved than the proof given below.

In the proof we use the method of forcing.

2. Forcing with proper classes. In the present section we quote some facts about forcing with proper classes in ZFC\(^-\), introduced by A. Zazh in [3]. We omit the proofs of facts which are proved there.

Let \( \mathfrak{M} \) be a countable standard model of ZFC\(^-\). We call a class \( P \subseteq \mathfrak{M} \) a notion of forcing iff there is a partial ordering \( \preceq_P \) of \( P \) (denoted further as \( \preceq \)) with the greatest element \( 1_P \), definable in \( \mathfrak{M} \) (possibly with parameters). The elements of \( P \) will then be called conditions. A condition \( p \) is stronger than \( q \) (or \( p \) is an extension of \( q \)) iff \( p \preceq q \).

--- Fundamentals Mathematicae, T. LXXXII
DEFINITION 1. A set \( G \subseteq P \) is \( P \)-generic over \( \mathcal{M} \) iff \( G \) satisfies the following three conditions:

1. \( p \in G \) and \( p \leq q \Rightarrow q \in G \).
2. \( p \in G \) and \( q \in G \Rightarrow \forall r \in \mathbb{R} [r \in G \land r \leq p \land r \leq q] \).
3. If \( S \subseteq P \) is a class of \( \mathcal{M} \) and \( S \) is dense in \( P \), then \( G \cap S \neq \emptyset \).

LEMMA 2. The condition of density in (3) can be replaced by one of the following conditions:

(a) \( S \) is predense, i.e.

\[
(p) (\mathcal{E} q) (\mathcal{E} r) [s \in S \land r \leq p \land r \leq q],
\]

(b) \( S \) is a maximal antichain, i.e. \( S \) is a maximal subclass of \( P \) with the property

\[
(p) (\mathcal{E} q) (\mathcal{E} r) [p \in S \land q \in S \land p \neq q \land r \leq p \land \neg (r \leq q)].
\]

An easy proof is left to the reader. Note that an antichain is maximal iff it is predense.

DEFINITION 3. Let us suppose that for every \( \alpha \in \text{On} \cap \mathcal{M} \), we can define (uniformly in \( \alpha \)) the set \( P_\alpha \subseteq P \), \( P_\alpha \in \mathcal{M} \), the class \( (\mathcal{M}, P) \subseteq \mathcal{M} \) and the mapping \( F_\alpha^+: P_\alpha \times P_\alpha \rightarrow P \) such that

\[
1_{\mathcal{M}} \in P_\alpha, P_\alpha \subseteq P_\alpha, P_\alpha \supseteq \bigcup_{\alpha \in \text{On} \cap \mathcal{M}} P_\alpha.
\]

We consider the standard product ordering and moreover, for \( p, q, r \in P_\alpha \), if \( p = F_\alpha^+(q, r) \) then \( p \) is the weakest common extension of \( q \) and \( r \). Then we call \( P \) a coherent notion of forcing.

DEFINITION 4. A coherent notion of forcing is continuous iff for every limit ordinal \( \alpha \), \( P_\alpha = \bigcup_{\beta < \alpha} P_\beta \). A notion of forcing \( \mathcal{P} \) satisfies in \( \mathcal{M} \) the set condition \( (s, c, o) \) iff every antichain of \( P \) definable in \( \mathcal{M} \) is an element of \( \mathcal{M} \).

In [3] it is proved that a coherent and continuous notion of forcing satisfies the set-\( c, o \)-condition, and if \( \mathcal{P} \subseteq P \) is \( P \)-generic over \( \mathcal{M} \) then \( \mathcal{M}(\mathcal{P}) \cong \mathbf{ZF^-} \), and \( \mathcal{M}(\mathcal{P}) \) is ZFC-countable containing \( \mathcal{M} \) (as subset) and all \( G \subseteq G \cong \mathcal{M} \) (as elements). In this case we have \( \mathcal{M}(\mathcal{P}) = \bigcup_{\alpha \in \text{On} \cap \mathcal{M}} \mathcal{M}(G_\alpha) \)

and \( G_\alpha \subseteq P_\alpha \)-generic over \( \mathcal{M} \). In [3] A. Zarach used Shoenfield’s definition of \( \mathcal{M}(\mathcal{P}) \):

\[
\mathcal{M}(\mathcal{P}) = \{K_\mathcal{P}(a): a \in \mathcal{M}\},
\]

where

\[
K_\mathcal{P}(a) = \{K_\mathcal{P}(b): (\mathcal{E} p) [p \in G \land (b, p) \in a]\}.
\]

3. The product lemma. Let us consider in \( \mathcal{M} \) two coherent and continuous notions of forcing \( \mathcal{P} \) and \( \mathcal{Q} \). Next assume that the formulae defining in \( \mathcal{M} \) the sets \( P, Q \), their decompositions and isomorphisms \( F_\alpha^+ \) and \( F_\alpha^+ \), define the same notions in each standard model of ZFC- of the same height as \( \mathcal{M} \) and containing \( \mathcal{M} \). For instance, we can satisfy this condition in the case where \( \mathcal{M} = \mathcal{L} \) (restrict the definitions to the class \( L \)); absoluteness of \( L \) in the models of the same height gives uniform definitions of these notions). The second case is where \( P \) and \( Q \) are classes of finite subsets of \( \text{On} \cap \mathcal{M} \), satisfying some absolute conditions, e.g. \( P \) and \( Q \) may be equal to the class of functions from finite subsets of \( \text{On} \cap \mathcal{M} \) to \( \mathbb{R} \) or to \( \mathbb{R} \) or to one of them is this class and the second is a set of \( \mathcal{M} \). This second case will be of special importance in our construction, though product lemmas are proved under the general assumptions stated above.

Let us define the notion of forcing \( \mathcal{R} = P \times Q \) with the product ordering. It can easily be shown that \( \mathcal{R} \) is a coherent notion of forcing:

\[
\mathcal{R}_\alpha = (P_\alpha) \times (Q_\alpha), \quad \mathcal{R}^+ = (P^+) \times (Q^+),
\]

\[
F_\alpha^+((p_1, q_1), (p_2, q_2)) = (F_\alpha^+(p_1, q_2), F_\alpha^+(p_2, q_1)).
\]

If \( P \) and \( Q \) are continuous, then so is \( \mathcal{R} \). Next observe that if \( P \) and \( Q \) have absolute definitions, then so does \( \mathcal{R} \).

LEMMA 5. Let \( P, Q \) and \( \mathcal{R} \) be as above. Let \( \mathcal{G} \subseteq \mathcal{R} \) be \( P \)-generic over \( \mathcal{M} \).

Then there exist \( \mathcal{G}_1 \subseteq P \) and \( \mathcal{G}_2 \subseteq Q \) such that \( \mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \), \( \mathcal{G}_1 \) is \( P \)-generic over \( \mathcal{M} \) and \( \mathcal{G}_2 \) is \( Q \)-generic over \( \mathcal{M}(\mathcal{G}_1) \). Moreover \( \mathcal{M}(\mathcal{G}) = \mathcal{M}(\mathcal{G}_1)(\mathcal{G}_2) \).

Let \( \mathcal{G}_1 \subseteq P \) be a maximal antichain parametrically definable in \( \mathcal{M}(\mathcal{G}_1) \). Since \( Q \) is a coherent and continuous notion of forcing in \( \mathcal{M} \), it is coherent and continuous in \( \mathcal{M}(\mathcal{G}_1) \), and hence it satisfies the set-\( c \)-o-\( m \) in \( \mathcal{M}(\mathcal{G}_1) \). Thus \( S = \mathcal{M}(\mathcal{G}_1) \), \( S = K_\mathcal{G}(a) \in \mathcal{G} \).

Let \( \Phi \) be a sentence of the forcing language saying that \( a \) is a maximal antichain in \( Q \). Note that the absolute definition of \( Q \) implies that \( \Phi \) means the same in every generic extension of the model \( \mathcal{M} \).

We define \( T = \{\langle p, q, r \rangle: p \in \mathcal{P}_1 \land q \in \mathcal{P}_2 \land \Phi \land (p, q, r) \in a\} \) and prove that \( T \) is predense in \( \mathcal{R} \). Of course \( T \) is a class of \( \mathcal{M} \), because forcing is always definable in the ground model.

Take \( \langle p, q, r \rangle \in T \). We have to find \( \langle p_1, q_1, r_1 \rangle \in T \) compatible with \( \langle p, q, r \rangle \). Let us take \( G_1 \subseteq P_1 \), \( P \)-generic over \( \mathcal{M} \) such that \( p_1 \in G_1 \). Consider two cases:

1. \( \mathcal{M}(\mathcal{G}_1) \models \Phi \). Then \( K_\mathcal{G}(a) \) is a maximal antichain in \( Q \), and so we can find \( q_1 \in K_\mathcal{G}(a) \) compatible with \( q \).

2. \( \mathcal{M}(\mathcal{G}_1) \models \neg \Phi \). We take \( q_1 = q \).

In both cases \( \mathcal{M}(\mathcal{G}_1) \models \Phi \land (p_1 \ni q_1 \land (p, q) \in a) \), and so there is a \( p_1 \ni q_1 \) such that \( p_1 \ni q \land (p, q) \in a \). This means that \( p_1 \) and \( p_1 \) are compatible since both are in \( G_1 \).
g and q, are compatible by construction. Thus \( \langle p, q \rangle \) and \( \langle q, p \rangle \) are compatible and \( \langle q, g \rangle, \langle p, q \rangle \in T \), which proves the predensity of \( T \).

By Lemma 2, \( G \cap T \neq 0 \). Let \( \langle p, q \rangle \in G \cap T \). Then \( p \in G_1 \), \( q \in G_2 \), \( \phi \in G_1 \), and \( \psi \in G_2 \). By \( \mathfrak{M}[G_1] \models \Box \phi \rightarrow \Box \psi \in \mathfrak{M}[G_2] \), but \( \mathfrak{M}[G_1] \models \Box \phi \), so \( \phi \in G_1 \) and \( \psi \in G_2 \). Simultaneously \( q \in G_1 \), and so \( G_2 \cap S \neq 0 \).

By Lemma 2 \( G_2 \) is \( Q \)-generic over \( \mathfrak{M}[G_1] \). The equality now follows from the fact that \( \mathfrak{M}[G] \) is the least model of ZF\(^C\) containing \( \mathfrak{M} \) and \( \mathfrak{M}[G_1] \). Then \( G = G_1 \times G_2 \) is \( R \)-generic over \( \mathfrak{M} \), and \( \mathfrak{M}[G] = [\mathfrak{M}[G_1], \mathfrak{M}[G_2]] \).

**Lemma 6.** Assume that \( P, Q \) and \( R \) are as in Lemma 5. Let \( G_1 \subseteq P \) be \( P\)-generic over \( \mathfrak{M} \) and let \( G_2 \subseteq Q \) be \( Q\)-generic over \( \mathfrak{M}[G_1] \). Then \( G = G_1 \times G_2 \) is \( R\)-generic over \( \mathfrak{M} \) and \( \mathfrak{M}[G] = \mathfrak{M}[G_1][G_2] \).

Proof: Conditions (1) and (2) of Definition 1 are easy to check. By Lemma 2 it will be sufficient to prove that \( G \) intersects every maximal antichain in \( \mathfrak{M} \), parametrically definable in \( \mathfrak{M} \).

Let \( S \subseteq R \) be such an antichain. Since \( R \) is coherent and continuous, it satisfies the \( c \)-condition, and so \( S \subseteq \mathfrak{M} \). Thus \( S \subseteq R_\alpha \) for some \( \alpha \in \text{On} \cap \mathfrak{M} \).

Let \( \langle q, r \rangle \in Q \). Define \( S(q) \subseteq P \) as follows:

\[
S(q) = \{ p \in P : \langle p, q \rangle, \langle q, r \rangle \in S, \langle p, q, r \rangle \in S \}.
\]

Of course \( S(q) \subseteq P \), so \( S \subseteq \mathfrak{M} \). We are going to show that \( S(q) \) is predense in \( P \).

Take \( p \in P \). By the maximality of \( S \), \( (p, q) \) is compatible with \( \langle p, q \rangle \). Let \( \langle r_1, r_2 \rangle \) be a common extension of \( \langle p, q \rangle \) and \( \langle p, q \rangle \). Then \( r_1 = r_2 \) to obtain a \( p \in S(q) \). By the compatibility of \( p \) and \( p_1 \), \( q \) is predense.

Next define \( S' = \{ q \in Q : \forall p (p \in S(q) \land q \in S(q)) \} \). Now \( S' \subseteq S \subseteq Q \). We show that \( S' \) is predense in \( P \).

Take \( q \in Q \). Then \( S(q) \subseteq \mathfrak{M} \) and \( S(q) \subseteq Q \). Hence by Lemma 2, \( G_1 \cap S(q) \neq 0 \). \( S(q) \subseteq P \), and so there exists a \( p_1 \in P \) such that \( p_1 \in G_1 \) and \( S(q) \). Then we can find a \( q \in Q \) with \( \langle p_1, q \rangle \in S \). Thus \( p_2 \in S(q) \), and \( \langle p_2, q \rangle \in S \). Since \( p_1 \in G_1 \), and \( \langle p_2, q \rangle \in S \), we have \( q \in S' \). \( q \) is compatible with \( q_1 \) and \( q_2 \) is arbitrary in \( G_2 \), and so \( S' \in Q \).

Thus by Lemma 2 we can take \( q \in S' \cap G_1 \). Hence there is a \( p \in G_1 \) such that \( \langle p, q \rangle \in S \). But then \( p \in G_1 \), \( q \in G_2 \), and \( \langle p, q \rangle \in S \) and so \( S \cap G_2 \neq 0 \), which proves the genericity of \( G \).

The equality \( \mathfrak{M}[G] = \mathfrak{M}[G_1][G_2] \) can now be proved exactly as in Lemma 5. Q.E.D.

The importance of Lemmas 5 and 6 is that we can consider product forcing whose factors are proper classes. The classical proof (see Shoenfield [2]) could be applied only in the case where \( P \) is a set in \( \mathfrak{M} \). Our Lemmas allow us to prove the existence of a pair of models \( \mathfrak{M} \) and \( \mathfrak{N} \) of ZFC\(^C\), \( \mathfrak{M} \) being a generic extension of \( \mathfrak{M} \) and such that \( \mathfrak{M} \preceq \mathfrak{N} \). The idea of this proof is as follows: add a class of generic subsets of a given model \( \mathfrak{M} \) to obtain a model \( \mathfrak{M} \) and next add one generic subset more to obtain \( \mathfrak{N} \). The product lemmas allow us to prove that the second extension is elementary. This fact is the key tool in our considerations.

4. Isomorphisms and automorphisms of notions of forcing. Let \( P \) and \( Q \) be two coherent and continuous notions of forcing in \( \mathfrak{M} \) and let \( F : P \to Q \) be an isomorphism of \( P \) and \( Q \), definable with parameters in \( \mathfrak{M} \). We extend \( F \) to a mapping \( F' : \mathfrak{M} \to \mathfrak{M} \) as follows: for \( a \in \mathfrak{M} \) let \( F'(a) = \langle (F'(b), F(p)) : (p \in P \land \lambda \in \mathfrak{M}) \rangle \).

Proof: The \( Q \)-genericity of \( F'(a) \) can be proved exactly as in the case of \( P \) and \( Q \) being sets.

Assume inductively that for \( b \in \mathfrak{M} \) less rank than \( \text{rank}(a) \) we have \( K_{F}(b) = K_{F'}(b) \). Then \( x \in K_{F}(a) \). Then \( x \in K_{F}(b) \) where \( \langle p, q \rangle \in F \) for some \( p \in P \). But then \( \text{rank}(b) \) \( \text{rank}(a) \), and so \( x \in K_{F'}(b) \).

At the same time \( F(p) \in F(G) \), and so \( x \in K_{F'}(a) \). For the second inclusion, run the argument backward, Q.E.D.

The reader should observe that if \( F^{-1} \) is the isomorphism inverse to \( F \), then \( (F^{-1})' \) is not an inverse mapping to \( F' \). But an easy proof shows that for every \( a \in \mathfrak{M} \), \( K_{F}(a) = K_{F'}(F^{-1}(a)) \); hence every condition \( p \in P \) forces \( "a = F^{-1}(F'(a))" \).

**Lemma 7.** Let \( p \in P \) and \( a_1, \ldots, a_n \in \mathfrak{M} \). Then

\[
\exists p \cdot \phi(a_1, \ldots, a_n) = F(p) \models \phi(F(a_1), \ldots, F(a_n)).
\]

Proof: Assume the left-hand side and take \( G \subseteq Q \). \( Q \)-generic over \( \mathfrak{M} \) such that \( F'(p) \in G \). Then \( p \in F^{-1}(G) \) and by Lemma 7, \( F^{-1}(G) \) is \( F \)-generic over \( \mathfrak{M} \). Thus \( \mathfrak{M}[F^{-1}(G)] = [\mathfrak{M}[F^{-1}(G)] \models \phi(a_1, \ldots, a_n) \}, \)

From Lemma 7 we see that \( \mathfrak{M}[G] = \mathfrak{M}[F^{-1}(G)] \) so

\[
\mathfrak{M}[G] = \mathfrak{M}[F^{-1}(G)].
\]

Thus

\[
\mathfrak{M}[G] = \mathfrak{M}[F^{-1}(G)].
\]

and so

\[
\mathfrak{M}[G] = \mathfrak{M}[F^{-1}(G)].
\]
Since $\mathcal{G}$ is arbitrary, we have proved the right-hand side. For the second implication use $P^{-1}$ and the remark before the Lemma. Q.E.D.

**Lemma 9.** $P^{a}(\mathcal{G}) = \mathcal{G}$ for $a \in \mathcal{M}$.  

An easy proof by induction is left to the reader.

**Remark.** The sign $\ast$ on the left-hand side of the equation in Lemma 9 is used in the sense of $P$ and the one on the right-hand side in the sense of $\mathcal{G}$.

**Definition 10.** A coherent and continuous notion of forcing $\mathcal{G}$ is homogeneous in $\mathcal{M}$ if for every pair of conditions $p, q \in \mathcal{M}$ there exists an $a \in 2^{\omega} \cap \mathcal{M}$ and an automorphism $f: P_{a} \to P_{a}$, $f \in \mathcal{M}$ such that $p, q \in P_{a}$, and $f(p)$ is compatible with $q$.

**Lemma 11.** If $\mathcal{G}$ is homogeneous, $x_{1}, \ldots, x_{n} \in \mathcal{M}$, $G$ is $P$-generic over $\mathcal{M}$ and $\mathcal{M}(G)$, $\Phi(x_{1}, \ldots, x_{n})$ is $P$-generic over $\mathcal{M}(G)$, then $1_{P} \vDash \Phi(x_{1}, \ldots, x_{n})$. Extend $f$ (taken from Definition 10) to an automorphism $F$ of $P$. Then by Lemma 8 and next by Lemma 7 we have $F(p) \vDash \Phi(x_{1}, \ldots, x_{n})$. But $F(p) = f(p)$ is compatible with $q$, which is a contradiction. Q.E.D.

Note that if $P$ is defined absolutely and is homogeneous in $\mathcal{M}$, then it is homogeneous in every extension of $\mathcal{M}$ of the same height.

5. Construction of an elementary generic extension. Let $\mathcal{M}$ be a countable standard model for ZFC. We fix the following notion of forcing $\mathcal{P}$:

$$\mathcal{P} = \{ f: \text{Fun}(f) \& \text{dom}(f) \subseteq \text{On} \cap \omega \& \text{rg}(f) \subseteq 2 \& \text{Fin}(f) \},$$

where $\text{Fin}(f)$ is a formula "$f$ is a finite set".

$$P_{n} = \{ f \in \mathcal{P} : \text{dom}(f) \subseteq a \cap \omega \},$$

$$P^{a} = \{ f \in \mathcal{P} : f \cap (a \cap \omega) = 0 \},$$

$$F_{P}^{a}(p, q) = p \cup q \quad \text{for} \quad p \in P_{a} \quad \text{and} \quad q \in P^{a}.$$  

The ordering of $P$ is the inverse inclusion.

One can easily check that $\mathcal{P}$ is a coherent, continuous and homogeneous notion of forcing. Next observe that every finite subset of $\mathcal{M}$ belongs to $\mathcal{M}$, and so the definition of $P$ is absolute, because we can define $P$ outside the model $\mathcal{M}$ as a subset of $(\text{On} \cap \mathcal{M}) \times a \cap \omega$. Thus $\mathcal{P}$ is a class of every model of the same height as $\mathcal{M}$. Observe also that, for every $a \in \text{On} \cap \mathcal{M}$, $P^{a}$ is also a coherent, continuous and homogeneous notion of forcing, defined absolutely. The decomposition of $P^{a}$ can be defined as follows:

$$(P^{a})_{a} = \{ f \in P^{a} : \text{dom}(f) \subseteq (\beta-a) \cap \omega \},$$

$$(P^{a})^{a} = \{ f \in P^{a} : f \cap ((\beta-a) \cap \omega) = 0 \},$$

$$F_{P}^{a}(p, q) = p \cup q \quad \text{for} \quad p \in (P^{a})_{a} \quad \text{and} \quad q \in (P^{a})^{a}.$$  

The ordering of $P^{a}$ is also the inverse inclusion. The homogeneity of these notions of forcing is a consequence of the finiteness of the conditions.

Finally we define the notion of forcing $Q$:

$$Q = \{ f: \text{Fun}(f) \& \text{dom}(f) \subseteq a \& \text{rg}(f) \subseteq 2 \& \text{Fin}(f) \}.$$  

We can easily see that $Q \in \mathcal{M}$, and so $Q$ belongs to every extension of $\mathcal{M}$. We order $Q$ by the inverse inclusion. Of course $Q$ can be considered as a coherent and continuous notion of forcing, and thus product lemma will be applicable to $P \times Q$ and $P^{a} \times Q$. Observe that $Q, P \times Q$ and $P^{a} \times Q$ are homogeneous notions of forcing.

**Lemma 12.** In the model $\mathcal{M}$ (and hence in every extension of it) the notions of forcing $P \times Q$ and $P$ are isomorphic.  

**Proof.** We define an isomorphism $F: P^{a} \times Q \to P^{a}$ as follows:

$$F((f_{1}, f_{2})) = \langle \langle \beta, \alpha \rangle, \delta \rangle: \beta > a \cap \omega \& \langle \beta, \alpha \rangle \in \text{dom}(f_{1}) \& f_{2}(\beta, \alpha) = \delta \cup \cup \langle \langle \beta, \alpha \rangle, \delta \rangle: \beta < \beta \cap \omega \& \delta \in \text{dom}(f_{2}) \& f_{2}(\beta, \alpha) = \delta \cup \cup \langle \langle \beta, \alpha \rangle, \delta \rangle: \beta \cap \omega < \beta \cap \omega \& \delta \in \text{dom}(f_{2}) \& f_{2}(\beta, \alpha) = \delta \cup \cup \langle \langle \beta, \alpha \rangle, \delta \rangle: \beta \cap \omega < \beta \cap \omega \& \delta \in \text{dom}(f_{2}) \& f_{2}(\beta, \alpha) = \delta.$$  

Of course $F$ is a class of every extension of $\mathcal{M}$. We leave it to the reader to check that $F$ is an isomorphism. Q.E.D.

**Theorem 13.** There exists a model $\mathcal{M}$ of the same height as $\mathcal{M}$ such that if $H \subseteq Q$ is $\mathcal{M}$-generic over $\mathcal{M}$, then $\mathcal{M} \models \text{ZFC}$.  

**Proof.** Let $\mathcal{M} \subseteq \mathcal{N}$ be $P$-generic over $\mathcal{M}$, and put $\mathcal{M} = \mathcal{M}(G)$. By Lemma 6, $G \cap H$ is $P \times Q$-generic over $\mathcal{M}$ and $\mathcal{M}(G[H]) = \mathcal{M}(G \times H)$. We must show that $\mathcal{M} \models \text{ZFC}$.  

Let $\Phi$ be a formula of the language of the set theory and $x_{1}, \ldots, x_{n} \in \mathcal{M}$. It is sufficient to show that $\mathcal{M} \models \Phi(x_{1}, \ldots, x_{n}) \models \mathcal{M}(G[H])$. But in the model $\mathcal{M}(G[H])$ the notions of forcing $P^{a}$ and $P^{a} \times Q$ are isomorphic. Let $P$ be this isomorphism. Then by Lemma 8

$$1_{P \times Q} \vDash \Phi(\beta_{1}, \ldots, \beta_{n}).$$

But in the model $\mathcal{M}(G[H])$ the notions of forcing $P^{a}$ and $P^{a} \times Q$ are isomorphic. Let $P$ be this isomorphism.

Let us consider the model $\mathcal{M}(G[H])$ by Lemma 6

$$\mathcal{M}(G[H]) = \mathcal{M}(G[H]) \equiv \mathcal{M}(G[H]) \models \Phi(\beta_{1}, \ldots, \beta_{n}),$$

i.e. $\mathcal{M}(H) \models \Phi(x_{1}, \ldots, x_{n})$. Q.E.D.
We can extend this result to the following:

**Theorem 14.** Let $M_a$ be a countable standard model of $ZFC^-$ and let $P \in M_a$ be a notion of forcing in $M_a$. Then there exists a model $M \supseteq M_a$ of the same height such that every $P$-generic extension of $M$ is an elementary extension.

The idea of the proof of Theorem 14 is the same: add a proper class of $P$-generic subsets of $P$ and next add one more subset. We then use the following notions of forcing:

- $R = \{f : \text{Fun}(f) \cap \text{dom}(f) \subseteq ON \cap \text{rg}(f) \subseteq P - \{1_P\} \& \text{Fin}(f)\}$,
- $f_1 \subseteq_R f_2 = (\alpha \in \text{dom}(f_2) \land \exists \beta \in \text{dom}(f_1) \land f_1(\beta) = f_2(\alpha))$,
- $R_a = \{f \in R : \text{dom}(f) \subseteq a\}$,
- $R_a = \{f \in R : \text{dom}(f) \cap a = \emptyset\}$,
- $P_a^R(p, q) = p \lor q$.

The details of the proof are left to the reader. Q.E.D.

The phenomenon described in Theorems 13 and 14 cannot take place in the case of models of $ZF$, because we have the following well-known

**Lemma 15.** If $M \preceq M$ and both are standard models of $ZF$, then $M = M \cap M$, where $a$ is the height of $M$.

**Proof.** Assume that $M \supseteq M$ and $M$. Let $x$ be an element of $M - M$ of minimal rank. Then $x \subseteq M$. Assume that rank $(x)$ is less than $\alpha$. Then for some $y \in M$, $x \subseteq y$. Take $x = p(y) \cap M$. Then $x$ is the power set of $y$ in $M$; since $M \preceq M$, $x$ is the power set of $y$ in $M$. But $x \subseteq y$ and $x \subseteq \alpha$, which contradicts the definition of power set. Thus rank $(x) \geq \alpha$. Q.E.D.

The following lemma is the "key lemma" of the paper. It makes possible to describe inside the model $M$ that $M \preceq M[G]$.

**Lemma 16.** Let $M$ be a standard model of $ZFC^-, Q$ a notion of forcing defined at the beginning of this section and $G \subseteq Q$ a $Q$-generic set over $M$. Then $M \preceq M[G]$ iff

$$M \models \langle x_1, \ldots, x_n \rangle \models \exists \Phi(x_1, \ldots, x_n) = 1_g \models \Phi(\bar{a}_1, \ldots, \bar{a}_n).$$

for every formula $\Phi$ of the language of the set theory with free variables $x_1, \ldots, x_n$.

**Proof.** Assume $M \preceq M[G]$. Then by Lemma 11

$$M \models \Phi(x_1, \ldots, x_n) = M[G] \models \Phi(x_1, \ldots, x_n) = 1_g \models \Phi(\bar{a}_1, \ldots, \bar{a}_n).$$

On the other hand, assume that $M$ satisfies the schema. Then

$$M \models \Phi(x_1, \ldots, x_n) = 1_g \models \Phi(\bar{a}_1, \ldots, \bar{a}_n) = M[G] \models \Phi(\bar{a}_1, \ldots, \bar{a}_n).$$

Q.E.D.

Observe that the formula $1_g \models \Phi(\bar{a}_1, \ldots, \bar{a}_n)$ depends only on the formula $\Phi$, not on the model $M$, i.e. there is a formula For$_g(y, z_1, \ldots, z_n)$ such that

$$M \models \text{For}_g[\langle y, z_1, \ldots, z_n \rangle = M \models 1_g \models \Phi(\bar{a}_1, \ldots, \bar{a}_n)].$$

**6. Uncountable models of $ZFC^-$ and $A_\alpha$.**

**Definition 17.** An ordinal $\alpha$ is a model number iff there is a standard model $M$ of $ZFC^-$ such that $\alpha = \text{On} \cap M$. $HC$ is the class of all hereditarily countable sets.

By Zbierski's theorem (cf. [4]) model numbers are exactly the heights of $\beta$-models of $A_\alpha$.

**Theorem 18.** Assume Martin's Axiom. Then for every model number $\alpha$ there exists a model $M$ of $ZFC^- \cup V = HC$, of power $2^a$ and height $\alpha$.

**Proof.** Let $M_\alpha$ be a countable standard model of $ZFC^- \cup V = HC$ of height $\alpha$. By Theorem 13 there exists a model $M$ of the same height, such that $0 \subseteq M$ is $\mathcal{Q}$-generic over $M$, then $M \preceq M[G]$.

We define a transfinite sequence of standard models for $ZFC^- \cup V = HC$. Assume that $0 < \eta < 2^\alpha$ we have defined models $M_\eta$ satisfying the following conditions:

1. $M_0 = M_\eta$ (note that if $M \cup V = HC$, then so does every generic extension of $M$).
2. $M_{\eta+1} \preceq M_\eta$ for $\eta < \eta$.
3. $M_\eta \models HC$ for $\eta < \eta$.
4. $M_\eta \preceq \frac{1}{2} + M_\eta$.
5. $M_\eta$ is a standard model of $ZFC^- \cup V = HC$ of height $\eta$.

We define $M$. There are two cases:

Case $1. \eta$ is a limit ordinal. Then we define $M_\eta = \bigcup M_\eta$. Of course $M_\eta$ satisfies all the conditions 1-5.

Case $2. \eta = \eta + 1$. Then $M_\eta = M_\eta \preceq M_\eta$; thus, by Lemma 16, $M_\eta$ satisfies the schema

$$\langle x_1, \ldots, x_n \rangle \models \exists \Phi(x_1, \ldots, x_n) = 1_g \models \Phi(\bar{a}_1, \ldots, \bar{a}_n).$$

By Martin's Axiom take $M_\eta \models \exists \Phi(x_1, \ldots, x_n)$. Put $M_\eta = M[G]$. Then $M_\eta$ again satisfies all the conditions 1-5.

Finally put $M = \bigcup M_\eta$. Then $M$ is the required model. Q.E.D.

**Corollary 19.** Under Martin's Axiom we can prove the existence of a model of $ZFC^-$ of power $\omega_1$ and height $\omega_1$.

**Corollary 20.** Assume Martin's Axiom. Then for every model number $\alpha$ there exists a $\beta$-model of $A_\alpha$ of height $\alpha$ and power $2^{\omega_1}$.

For a proof take the continuum of a model constructed in Theorem 18. The Corollary is then a consequence of Zbierski's Theorem. Q.E.D.
Two notes on abstract model theory

I. Properties invariant on the range of definable relations between structures

by

Solomon Feferman (1) (Stanford, Cal.)

Abstract. Suppose \( L \) is any model-theoretic language satisfying the many-sorted interpolation property and that \( \mathcal{K} \) is an \( L \)-definable or even \( L \)-projective relation between \( L \)-structures. It is shown that if (1) an \( L \)-sentence \( \varphi \) holds in \( \mathcal{K} \), just in case it holds in \( \mathcal{K} \) whenever \( \mathcal{K}(\mathcal{M}, \mathcal{N}) \) and \( \mathcal{K}(\mathcal{N}, \mathcal{M}) \) then (2) there is an \( L \)-sentence \( \varphi \) such that \( \varphi \) holds in \( \mathcal{K} \) if and only if \( \varphi \) holds in \( \mathcal{K} \) whenever \( \mathcal{K}(\mathcal{M}, \mathcal{N}) \). This has various results of Beth, Robinson, Gaifman, Barwise and Rosenthal for familiar languages as immediate corollaries.

Introduction. Abstract (or general) model-theory deals with notions that are applicable to all model-theoretic languages \( L \). Each such \( L \) is determined by a relation \( \mathcal{R}^{*}_{\varphi} \), called its satisfaction relation, in which \( \mathcal{R} \) ranges over a collection \( \text{Str}_{\varphi} \) of structures for \( L \) and \( \varphi \) ranges over a collection of objects \( \text{St}_{\varphi} \), called the sentences of \( L \). The notions of general model theory are just those which can be expressed in terms of those basic ones (using ordinary set-theoretical concepts). Examples of such are: elementary class, projective class, L"o"wenheim-Skolem properties, Hanf number, interpolation property, compactness property, categoricity. Typically, the results apply to all \( L \) satisfying some simple conditions or characterize some given \( L \) by means of such conditions.

Lindström (J.2) provided the first work clearly of this character. Its point of departure (via [L.1]) was Mostowski’s characterization of \( L_{n,x} \) among certain languages with generalized quantifiers [Mo]. Since then, the subject of general model theory has been especially developed by Barwise [B2]-[B4]. The present two notes are a sequel to my own contribution in § 3 of [F2], making essential use, as there, of many-sorted structures.

(1) Guggenheim Fellow 1972–73. The author is indebted to the Guggenheim Foundation and to the U.E.R. de Mathématiques, Université Paris VII, for their generous assistance during the period in which these notes were prepared for publication.