Uncountable β-models with countable height

by

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Abstract. A model of $ZFC^- + V = HC$ is constructed so that its Cohen generic extensions are elementary. This construction is used to prove that there exist uncountable β-models of second order arithmetic with countable height. In presence of Martin's Axiom they can have power $2^{\aleph_0}$.

1. Introduction. In the present paper we prove the existence of uncountable standard models with countable height of the theory ZFC^- (ZF-set theory without the power set axiom and with the axiom scheme of choice:

$$(x)_y(Ez)\Phi(x, z) \to (Ef)\big[\mathrm{Func}(f) \;\&\; (x)_y[x \in \mathrm{dom}(f) \;\&\; \Phi(x, f(x))]\big]).$$

Hence we conclude that there exist uncountable β-models for second order arithmetics A_2 with only countably many non-similar well-orderings. Assuming Martin's Axiom, we can strengthen these results by replacing the words "uncountable" by the words "of the power of the continuum".

A. Mostowski in [1] raised the following problem: are there two non-isomorphic ω-models of A_2 of the same power $> \omega_1$?

Our paper gives a positive answer for every power $\leqslant 2^{\aleph_0}$, but with the aid of Martin's Axiom. The result in the case of $2^{\aleph_0} = \aleph_1$ is well known.

Gerald E. Sacks has kindly informed me that he proved the theorem on the existence of β-models of power $2^{\aleph_0}$ without any additional assumptions. His proof, which he sketched during the Logical Semester in Warsaw, was much more involved than the proof given below.

In the proof we use the method of forcing.

2. Forcing with proper classes. In the present section we quote some facts about forcing with proper classes in ZFC^-, introduced by A. Zarach in [3]. We omit the proofs of facts which are proved there.

Let $\mathfrak{M}$ be a countable standard model of ZFC^-. We call a class $\boldsymbol{P} \subseteq \mathfrak{M}$ a *notion of forcing* iff there is a partial ordering $\leqslant_{\boldsymbol{P}}$ of $\boldsymbol{P}$ (denoted further as $\leqslant$) with the greatest element $\mathbf{1}_{\boldsymbol{P}}$, definable in $\mathfrak{M}$ (possibly with parameters). The elements of $\boldsymbol{P}$ will then be called *conditions*. A condition p is stronger than q (or p is an extension of q) iff $p \leqslant q$.

DEFINITION 1. A set $G \subseteq \boldsymbol{P}$ is *$\boldsymbol{P}$-generic over $\mathfrak{M}$* iff G satisfies the following three conditions:

(1) $p \in G \;\&\; p \leqslant q \rightarrow q \in G$.

(2) $p \in G \;\&\; q \in G \rightarrow (Er)[r \in G \;\&\; r \leqslant p \;\&\; r \leqslant q]$.

(3) If $S \subseteq \boldsymbol{P}$ is a class of $\mathfrak{M}$ and S is dense in $\boldsymbol{P}$, then $G \cap S \neq 0$.

LEMMA 2. *The condition of density in* (3) *can be replaced by one of the following conditions*:

(a) *S is predense, i.e.*

$$(p)(Eq)(Er)[q \in S \;\&\; r \leqslant p \;\&\; r \leqslant q]\,,$$

(b) *S is a maximal antichain, i.e. S is a maximal subclass of $\boldsymbol{P}$ with the property*

$$(p)(q)(r)[p \in S \;\&\; q \in S \;\&\; p \neq q \;\&\; r \leqslant p \rightarrow \neg\, r \leqslant q]\,.$$

An easy proof is left to the reader. Note that an antichain is maximal iff it is predense.

DEFINITION 3. Let us suppose that for every $\alpha \in On \cap \mathfrak{M}$ we can define (uniformly in α) the set $\boldsymbol{P}_\alpha \subseteq \boldsymbol{P}$, $\boldsymbol{P}_\alpha \in \mathfrak{M}$, the class (of $\mathfrak{M}$) $\boldsymbol{P}^\alpha \subseteq \boldsymbol{P}$ and the mapping $F^P_\alpha\colon \boldsymbol{P}_\alpha \times \boldsymbol{P}^\alpha \rightarrow \boldsymbol{P}$ such that $\boldsymbol{1}_P \in \boldsymbol{P}_\alpha$, $\boldsymbol{P}_\alpha \subseteq \boldsymbol{P}_\beta$ for $\alpha < \beta$, $\boldsymbol{1}_P \in \boldsymbol{P}^\alpha$, $\bigcup_{\alpha \in On \cap \mathfrak{M}} \boldsymbol{P}_\alpha = \boldsymbol{P}$, F^P_α is one-to-one, onto and order-preserving (in $\boldsymbol{P}_\alpha \times \boldsymbol{P}^\alpha$ we consider the standard product ordering) and moreover, for $p, q, r \in \boldsymbol{P}$, if $p = F^P_\alpha(\langle q, r\rangle)$ then p is the weakest common extension of q and r. Then we call $\boldsymbol{P}$ a *coherent notion of forcing*.

DEFINITION 4. A coherent notion of forcing is *continuous* iff for every limit ordinal α, $\boldsymbol{P}_\alpha = \bigcup_{\beta<\alpha} \boldsymbol{P}_\beta$. A notion of forcing $\boldsymbol{P}$ satisfies in $\mathfrak{M}$ the *set chain condition* (set-c-c) iff every antichain of $\boldsymbol{P}$ definable in $\mathfrak{M}$ is an element of $\mathfrak{M}$.

In [3] it is proved that a coherent and continuous notion of forcing satisfies the set-c-c, and if $G \subseteq \boldsymbol{P}$ is $\boldsymbol{P}$-generic over $\mathfrak{M}$ then $\mathfrak{M}[G] \models \mathrm{ZFC}^-$ and $\mathfrak{M}[G]$ is the smallest model of ZFC^- containing $\mathfrak{M}$ (as subset) and all $G_\alpha = G \cap \boldsymbol{P}_\alpha$ (as elements). In this case we have $\mathfrak{M}[G] = \bigcup_{\alpha \in On \cap \mathfrak{M}} \mathfrak{M}[G_\alpha]$ and G_α is $\boldsymbol{P}_\alpha$-generic over $\mathfrak{M}$. In [3] A. Zarach used Shoenfield's definition of $\mathfrak{M}[G]$:

$$\mathfrak{M}[G] = \{K_G(a)\colon\; a \in \mathfrak{M}\}\,,$$

where

$$K_G(a) = \{K_G(b)\colon\; (Ep)[p \in G \;\&\; \langle b, p\rangle \in a]\}\,.$$

3. The product lemma. Let us consider in $\mathfrak{M}$ two coherent and continuous notions of forcing $\boldsymbol{P}$ and $\boldsymbol{Q}$. Next assume that the formulae defining in $\mathfrak{M}$ the sets $\boldsymbol{P}$, $\boldsymbol{Q}$, their decompositions and isomorphisms F^P_α and F^Q_α, define the same notions in each standard model of ZFC^- of the same height as $\mathfrak{M}$ and containing $\mathfrak{M}$. For instance, we can satisfy this condition in the case where $\mathfrak{M} \models V = L$ (restrict the definitions to the class L; absolutness of L in the models of the same height gives uniform definitions of these notions). The second case is where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are classes of finite subsets of $On \cap \mathfrak{M}$, satisfying some absolute conditions. E.g. $\boldsymbol{P}$ and $\boldsymbol{Q}$ may be equal to the class of functions from finite subsets of $(On \cap \mathfrak{M}) \times \omega$ to 2 or one of them is this class and the second is a set of $\mathfrak{M}$. This second case will be of special importance in our construction, though product lemmas are proved under the general assumptions stated above.

Let us define the notion of forcing $\boldsymbol{R} = \boldsymbol{P} \times \boldsymbol{Q}$ with the product ordering. It can easily be shown that $\boldsymbol{R}$ is a coherent notion of forcing:

$$\boldsymbol{R}_\alpha = (\boldsymbol{P}_\alpha) \times (\boldsymbol{Q}_\alpha)\,, \qquad \boldsymbol{R}^\alpha = (\boldsymbol{P}^\alpha) \times (\boldsymbol{Q}^\alpha)\,,$$

$$F^R_\alpha(\langle\langle p_1, q_1\rangle, \langle p_2, q_2\rangle\rangle) = \langle F^P_\alpha(\langle p_1, p_2\rangle), F^Q_\alpha(\langle q_1, q_2\rangle)\rangle\,.$$

If $\boldsymbol{P}$ and $\boldsymbol{Q}$ are continuous, then so is $\boldsymbol{R}$. Next observe that if $\boldsymbol{P}$ and $\boldsymbol{Q}$ have absolute definitions, then so does $\boldsymbol{R}$.

LEMMA 5. *Let $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ be as above. Let $G \subseteq \boldsymbol{R}$ be $\boldsymbol{R}$-generic over $\mathfrak{M}$. Then there exist $G_1 \subseteq \boldsymbol{P}$ and $G_2 \subseteq \boldsymbol{Q}$ such that $G = G_1 \times G_2$, G_1 is $\boldsymbol{P}$-generic over $\mathfrak{M}$ and G_2 is $\boldsymbol{Q}$-generic over $\mathfrak{M}[G_1]$. Moreover $\mathfrak{M}[G] = \mathfrak{M}[G_1][G_2]$.*

Proof. Define G_1 and G_2 as follows:

$$G_1 = \{p \in \boldsymbol{P}\colon\; (Eq)[\langle p, q\rangle \in G]\} \quad \text{and} \quad G_2 = \{q \in \boldsymbol{Q}\colon\; (Ep)[\langle p, q\rangle \in G]\}\,.$$

The proofs that $G = G_1 \times G_2$, G_1 is $\boldsymbol{P}$-generic over $\mathfrak{M}$ and G_2 satisfies conditions (1) and (2) of Definition 1 are trivial.

Let $S \subseteq \boldsymbol{Q}$ be a maximal antichain parametrically definable in $\mathfrak{M}[G_1]$. Since $\boldsymbol{Q}$ is a coherent and continuous notion of forcing in $\mathfrak{M}$, it is coherent and continuous in $\mathfrak{M}[G_1]$, and hence it satisfies the set-c-c in $\mathfrak{M}[G_1]$. Thus $S \in \mathfrak{M}[G_1]$, $S = K_{G_1}(a)$, $a \in \mathfrak{M}$.

Let Φ be a sentence of the forcing language saying that a is a maximal antichain in $\boldsymbol{Q}$. Note that the absolute definition of $\boldsymbol{Q}$ implies that Φ means the same in every generic extension of the model $\mathfrak{M}$.

We define $T = \{\langle p, q\rangle \in \boldsymbol{R}\colon\; p \Vdash_P \text{“}\Phi \rightarrow \hat{q} \in a\text{”}\}$ and prove that T is predense in $\boldsymbol{R}$. Of course T is a class of $\mathfrak{M}$, because forcing is always definable in the ground model.

Take $\langle p, q\rangle \in \boldsymbol{R}$. We have to find $\langle p_1, q_1\rangle \in T$ compatible with $\langle p, q\rangle$. Let us take $G_1' \subseteq \boldsymbol{P}$, $\boldsymbol{P}$-generic over $\mathfrak{M}$ such that $p \in G_1'$. Consider two cases:

1. $\mathfrak{M}[G_1'] \models \Phi$. Then $K_{G_1'}(a)$ is a maximal antichain in $\boldsymbol{Q}$, and so we can find $q_1 \in K_{G_1'}(a)$ compatible with q.

2. $\mathfrak{M}[G_1'] \models \neg\Phi$. We take $q_1 = q$.

In both cases $\mathfrak{M}[G_1'] \models \text{“}\Phi \rightarrow \hat{q}_1 \in a\text{”}$, and so there is a $p_1 \in G_1'$ such that $p_1 \Vdash_P \text{“}\Phi \rightarrow \hat{q}_1 \in a\text{”}$. p and p_1 are compatible since both are in G_1'.

q and q_1 are compatible by construction. Thus $\langle p, q\rangle$ and $\langle p_1, q_1\rangle$ are compatible and $\langle p_1, q_1\rangle \in T$, which proves the predensity of T.

By Lemma 2, $G \cap T \neq 0$. Let $\langle p, q\rangle \in G \cap T$. Then $p \in G_1$, $p \Vdash_P$ "$\Phi \to \hat{q} \in a$", and so $\mathfrak{M}[G_1] \models$ "$\Phi \to \hat{q} \in a$". But $\mathfrak{M}[G_1] \models \Phi$, so $q \in S$. Simultaneously $q \in G_2$, and so $G_2 \cap S \neq 0$.

By Lemma 2 G_2 is $\boldsymbol{Q}$-generic over $\mathfrak{M}[G_1]$. The equality now follows from the fact that $\mathfrak{M}[G]$ is the least model of ZFC$^-$ containing $\mathfrak{M}$ and all $G_\alpha = (G_1)_\alpha \times (G_2)_\alpha$ and $\mathfrak{M}[G_1][G_2]$ is the least model of ZFC$^-$ containing $\mathfrak{M}$, all $(G_1)_\alpha$ and all $(G_2)_\alpha$. Both conditions say the same because $\mathfrak{M}[G]$ and $\mathfrak{M}[G_1][G_2]$ are models of ZF$^-$. Q.E.D.

LEMMA 6. *Assume that $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ are as in Lemma 5. Let $G_1 \subseteq \boldsymbol{P}$ be $\boldsymbol{P}$-generic over $\mathfrak{M}$ and let $G_2 \subseteq \boldsymbol{Q}$ be $\boldsymbol{Q}$-generic over $\mathfrak{M}[G_1]$. Then $G = G_1 \times G_2$ is $\boldsymbol{R}$-generic over $\mathfrak{M}$ and $\mathfrak{M}[G] = \mathfrak{M}[G_1][G_2]$.*

Proof. Conditions (1) and (2) of Definition 1 are easy to check. By Lemma 2 it will be sufficient to prove that G intersects every maximal antichain in $\boldsymbol{R}$, parametrically definable in $\mathfrak{M}$.

Let $S \subseteq \boldsymbol{R}$ be such an antichain. Since $\boldsymbol{R}$ is coherent and continuous, it satisfies the set-c-c, and so $S \in \mathfrak{M}$. Thus $S \subseteq \boldsymbol{R}_\alpha$ for some $\alpha \in On \cap \mathfrak{M}$.

Let $q_1 \in \boldsymbol{Q}$. Define $S(q_1) \subseteq \boldsymbol{P}$ as follows:

$$S(q_1) = \{p \in \boldsymbol{P}\colon (Eq)(Er)[r \leqslant q \ \&\ r \leqslant q_1 \ \&\ \langle p, q\rangle \in S]\}.$$

Of course $S(q_1) \subseteq \boldsymbol{P}_\alpha$, so $S(q_1) \in \mathfrak{M}$. We are going to show that $S(q_1)$ is predense in $\boldsymbol{P}$.

Take $p_1 \in \boldsymbol{P}$. By the maximality of S, $\langle p_1, q_1\rangle$ is compatible with some $\langle p, q\rangle \in S$. Let $\langle r_1, r_2\rangle$ be a common extension of $\langle p, q\rangle$ and $\langle p_1, q_1\rangle$. Put $r = r_2$ to obtain $p \in S(q_1)$. By the compatibility of p and p_1, $S(q_1)$ is predense.

Next define $S' = \{q \in \boldsymbol{Q}\colon (Ep)[p \in (G_1)_\alpha \ \&\ \langle p, q\rangle \in S]\}$. S' is a class of $\mathfrak{M}[G_1]$, since $(G_1)_\alpha \in \mathfrak{M}[G_1]$ and $S' \subseteq \boldsymbol{Q}_\alpha$. We show that S' is predense in $\boldsymbol{Q}$.

Take $q_1 \in \boldsymbol{Q}$. Then $S(q_1) \in \mathfrak{M}$ and $S(q_1)$ is predense in $\boldsymbol{P}$, hence by Lemma 2, $G_1 \cap S(q_1) \neq 0$. $S(q_1) \subseteq \boldsymbol{P}_\alpha$, and so there exists a $p_1 \in \boldsymbol{P}_\alpha$ such that $p_1 \in G_1 \cap S(q_1)$. Then we can find a q compatible with q_1 such that $\langle p_1, q\rangle \in S$. Since $p_1 \in (G_1)_\alpha$ and $\langle p_1, q\rangle \in S$, we have $q \in S'$. q is compatible with q_1 and q_1 is arbitrary in $\boldsymbol{Q}$, and so S' is predense in $\boldsymbol{Q}$.

Thus by Lemma 2 we can take $q \in S' \cap G_2$. Hence there is a $p \in (G_1)_\alpha$ such that $\langle p, q\rangle \in S$. But $p \in G_1$, $q \in G_2$ and $\langle p, q\rangle \in S$, and so $S \cap (G_1 \times G_2) \neq 0$, which proves the genericity of G.

The equality $\mathfrak{M}[G] = \mathfrak{M}[G_1][G_2]$ can now be proved exactly as in Lemma 5. Q.E.D.

The importance of Lemmas 5 and 6 is that we can consider product forcing whose factors are proper classes. The classical proof (see Shoenfield [2]) could be applied only in the case where $\boldsymbol{P}$ is a set in $\mathfrak{M}$. Our Lemmas allow us to prove the existence of a pair of models $\mathfrak{M}$ and $\mathfrak{N}$ of ZFC$^-$, $\mathfrak{N}$ being a generic extension of $\mathfrak{M}$ and such that $\mathfrak{M} \precneqq \mathfrak{N}$. The idea of this proof is as follows: add a class of generic subsets of a given model $\mathfrak{M}_0$ to obtain a model $\mathfrak{M}$ and next add one generic subset more to obtain $\mathfrak{N}$. The product lemmas allow us to prove that the second extension is elementary. This fact is the key tool in our considerations.

4. Isomorphisms and automorphisms of notions of forcing. Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be two coherent and continuous notions of forcing in $\mathfrak{M}$ and let $F\colon \boldsymbol{P} \to \boldsymbol{Q}$ be an isomorphism of $\boldsymbol{P}$ and $\boldsymbol{Q}$, definable with parameters in $\mathfrak{M}$. We extend F to a mapping $F^*\colon \mathfrak{M} \to \mathfrak{M}$ as follows: for $a \in \mathfrak{M}$ let $F^*(a) = \{\langle F^*(b), F(p)\rangle\colon p \in \boldsymbol{P} \ \&\ \langle b, p\rangle \in a\}$. Of course the precise definition of F^* should be carried out by induction on $\operatorname{rank}(a)$. We prove some lemmas on F^*.

LEMMA 7. *If $G \subseteq \boldsymbol{P}$ is $\boldsymbol{P}$-generic over $\mathfrak{M}$, then $F(G) \subseteq \boldsymbol{Q}$ is $\boldsymbol{Q}$-generic over $\mathfrak{M}$ and for every $a \in \mathfrak{M}$ we have $K_G(a) = K_{F(G)}(F^*(a))$.*

Proof. The $\boldsymbol{Q}$-genericity of $F(G)$ can be proved exactly as in the case of $\boldsymbol{P}$ and $\boldsymbol{Q}$ being sets.

Assume inductively that for b of rank less than $\operatorname{rank}(a)$ we have $K_G(b) = K_{F(G)}(F^*(b))$. Let $x \in K_G(a)$. Then $x = K_G(b)$ where $\langle b, p\rangle \in a$ for some $p \in G$. But then $\operatorname{rank}(b) < \operatorname{rank}(a)$, and so $x = K_{F(G)}(F^*(b))$. Since $\langle b, p\rangle \in a$, $\langle F^*(b), F(p)\rangle \in F^*(a)$. At the same time $F(p) \in F(G)$, and so $x \in K_{F(G)}(F^*(a))$. For the second inclusion, run the argument backward. Q.E.D.

The reader should observe that if F^{-1} is the isomorphism inverse to F, then $(F^{-1})^*$ is not an inverse mapping to F^*. But an easy proof shows that for every $a \in \mathfrak{M}$, $K_G(a) = K_G\big((F^{-1})^*(F^*(a))\big)$; hence every condition $p \in \boldsymbol{P}$ forces "$a = (F^{-1})^*(F^*(a))$".

LEMMA 8. *Let $p \in \boldsymbol{P}$ and $x_1, \ldots, x_n \in \mathfrak{M}$. Then*

$$p \Vdash_{\boldsymbol{P}} \Phi(x_1, \ldots, x_n) \equiv F(p) \Vdash_{\boldsymbol{Q}} \Phi\big(F^*(x_1), \ldots, F^*(x_n)\big).$$

Proof. Assume the left-hand side and take $G \subseteq \boldsymbol{Q}$, $\boldsymbol{Q}$-generic over $\mathfrak{M}$ such that $F(p) \in G$. Then $p \in F^{-1}(G)$ and by Lemma 7, $F^{-1}(G)$ is $\boldsymbol{P}$-generic over $\mathfrak{M}$. Thus $\mathfrak{M}[F^{-1}(G)] \models$ "$\Phi(x_1, \ldots, x_n)$", i.e.

$$\mathfrak{M}[F^{-1}(G)] \models \Phi[K_{F^{-1}(G)}(x_1), \ldots, K_{F^{-1}(G)}(x_n)].$$

From Lemma 7 we see that $\mathfrak{M}[G] = \mathfrak{M}[F^{-1}(G)]$; so

$$\mathfrak{M}[G] \models \Phi[K_{F^{-1}(G)}(x_1), \ldots, K_{F^{-1}(G)}(x_n)].$$

Thus

$$\mathfrak{M}[G] \models \Phi\big[K_G\big(F^*(x_1)\big), \ldots, K_G\big(F^*(x_n)\big)\big],$$

and so

$$\mathfrak{M}[G] \models \text{"}\Phi\big(F^*(x_1), \ldots, F^*(x_n)\big)\text{"}.$$

Since G is arbitrary, we have proved the right-hand side. For the second implication use F^{-1} and the remark before the Lemma. Q.E.D.

LEMMA 9. *$F^*(\hat{x}) = \hat{x}$ for $x \in \mathfrak{M}$.*

An easy proof by induction is left to the reader.

Remark. The sign $\hat{}$ on the left-hand side of the equality in Lemma 9 is used in the sense of $\boldsymbol{P}$ and the one on the right-hand side in the sense of $\boldsymbol{Q}$.

DEFINITION 10. A coherent and continuous notion of forcing $\boldsymbol{P}$ is *homogeneous* in $\mathfrak{M}$ iff for every pair of conditions $p, q \in \boldsymbol{P}$ there exists an $\alpha \in On \cap \mathfrak{M}$ and an automorphism $f\colon \boldsymbol{P}_\alpha \to \boldsymbol{P}_\alpha$, $f \in \mathfrak{M}$ such that $p, q \in \boldsymbol{P}_\alpha$ and $f(p)$ is compatible with q.

LEMMA 11. *If $\boldsymbol{P}$ is homogeneous, $x_1, \ldots, x_n \in \mathfrak{M}$, G is $\boldsymbol{P}$-generic over $\mathfrak{M}$ and $\mathfrak{M}[G] \models \Phi[x_1, \ldots, x_n]$, then $\mathbf{1}_{\boldsymbol{P}} \Vdash_{\boldsymbol{P}} \Phi(\hat{x}_1, \ldots, \hat{x}_n)$.*

Proof. Let $p \Vdash_{\boldsymbol{P}} \Phi(\hat{x}_1, \ldots, \hat{x}_n)$ and $q \Vdash_{\boldsymbol{P}} \neg\Phi(\hat{x}_1, \ldots, \hat{x}_n)$. Extend f (taken from Definition 10) to an automorphism F of $\boldsymbol{P}$. Then by Lemma 8 and next by Lemma 7 we have $F(p) \Vdash_{\boldsymbol{P}} \Phi(\hat{x}_1, \ldots, \hat{x}_n)$. But $F(p) = f(p)$ is compatible with q, which is a contradiction. Q.E.D.

Note that if $\boldsymbol{P}$ is defined absolutely and is homogeneous in $\mathfrak{M}$, then it is homogeneous in every extension of $\mathfrak{M}$ of the same height.

5. Construction of an elementary generic extension. Let $\mathfrak{M}_0$ be a countable standard model for ZFC$^-$. We fix the following notion of forcing $\boldsymbol{P}$:

$$\boldsymbol{P} = \{f\colon \operatorname{Func}(f) \,\&\, \operatorname{dom}(f) \subseteq On \times \omega \,\&\, \operatorname{rg}(f) \subseteq 2 \,\&\, \operatorname{Fin}(f)\},$$

where $\operatorname{Fin}(f)$ is a formula "f is a finite set".

$$\boldsymbol{P}_\alpha = \{f \in \boldsymbol{P}\colon \operatorname{dom}(f) \subseteq \alpha \times \omega\},$$

$$\boldsymbol{P}^\alpha = \{f \in \boldsymbol{P}\colon \operatorname{dom}(f) \cap (\alpha \times \omega) = 0\},$$

$$F_\alpha^{\boldsymbol{P}}(\langle p, q\rangle) = p \cup q \quad \text{for} \quad p \in \boldsymbol{P}_\alpha \text{ and } q \in \boldsymbol{P}^\alpha.$$

The ordering of $\boldsymbol{P}$ is the inverse inclusion.

One can easily check that $\boldsymbol{P}$ is a coherent, continuous and homogeneous notion of forcing. Next observe that every finite subset of $\mathfrak{M}_0$ belongs to $\mathfrak{M}_0$, and so the definition of $\boldsymbol{P}$ is absolute, because we can define $\boldsymbol{P}$ outside the model $\mathfrak{M}_0$ as a subset of $(On \cap \mathfrak{M}) \times \omega \times 2$. Thus $\boldsymbol{P}$ is a class of every model of the same height as $\mathfrak{M}_0$. Observe also that, for every $\alpha \in On \cap \mathfrak{M}_0$, $\boldsymbol{P}^\alpha$ is also a coherent, continuous and homogeneous notion of forcing, defined absolutely. The decomposition of $\boldsymbol{P}^\alpha$ can be defined as follows:

$$(\boldsymbol{P}^\alpha)_\beta = \{f \in \boldsymbol{P}^\alpha\colon \operatorname{dom}(f) \subseteq (\beta - \alpha) \times \omega\},$$

$$(\boldsymbol{P}^\alpha)^\beta = \{f \in \boldsymbol{P}^\alpha\colon \operatorname{dom}(f) \cap ((\beta - \alpha) \times \omega) = 0\},$$

$$F_\beta^{\boldsymbol{P}^\alpha}(\langle p, q\rangle) = p \cup q \quad \text{for} \quad p \in (\boldsymbol{P}^\alpha)_\beta \text{ and } q \in (\boldsymbol{P}^\alpha)^\beta.$$

The ordering of $\boldsymbol{P}^\alpha$ is also the inverse inclusion. The homogeneity of these notions of forcing is a consequence of the finiteness of the conditions.

Finally we define the notion of forcing $\boldsymbol{Q}$:

$$\boldsymbol{Q} = \{f\colon \operatorname{Func}(f) \,\&\, \operatorname{dom}(f) \subseteq \omega \,\&\, \operatorname{rg}(f) \subseteq 2 \,\&\, \operatorname{Fin}(f)\}.$$

We can easily see that $\boldsymbol{Q} \in \mathfrak{M}$, and so $\boldsymbol{Q}$ belongs to every extension of $\mathfrak{M}$. We order $\boldsymbol{Q}$ by the inverse inclusion. Of course $\boldsymbol{Q}$ can be considered as a coherent and continuous notion of forcing, and thus product lemmas will be applicable to $\boldsymbol{P} \times \boldsymbol{Q}$ and $\boldsymbol{P}^\alpha \times \boldsymbol{Q}$. Observe that $\boldsymbol{Q}$, $\boldsymbol{P} \times \boldsymbol{Q}$ and $\boldsymbol{P}^\alpha \times \boldsymbol{Q}$ are homogeneous notions of forcing.

LEMMA 12. *In the model $\mathfrak{M}_0$ (and hence in every extension of it the notions of forcing $\boldsymbol{P}^\alpha \times \boldsymbol{Q}$ and $\boldsymbol{P}^\alpha$ are isomorphic.*

Proof. We define an isomorphism $F\colon \boldsymbol{P}^\alpha \times \boldsymbol{Q} \to \boldsymbol{P}^\alpha$ as follows:

$$\begin{aligned} F(\langle f_1, f_2\rangle) = \{\langle\langle\beta, n\rangle, i\rangle\colon \beta \geqslant \alpha + \omega \,\&\, \langle\beta, n\rangle \in \operatorname{dom}(f_1) \,\&\, f_1(\langle\beta, n\rangle) = i\} \cup \\ \cup \{\langle\langle\beta, n\rangle, i\rangle\colon \alpha < \beta < \alpha + \omega \,\&\, \langle\beta - 1, n\rangle \in \operatorname{dom}(f_1) \,\&\, \\ \& f_1(\langle\beta - 1, n\rangle) = i\} \cup \{\langle\langle\alpha, n\rangle, i\rangle\colon n \in \operatorname{dom}(f_2) \,\&\, f_2(n) = i\}. \end{aligned}$$

Of course F is a class of every extension of $\mathfrak{M}_0$. We leave it to the reader to check that F is an isomorphism. Q.E.D.

THEOREM 13. *There exists a model $\mathfrak{M}$ of the same height as $\mathfrak{M}_0$ such that if $H \subseteq \boldsymbol{Q}$ is $\boldsymbol{Q}$-generic over $\mathfrak{M}$, then $\mathfrak{M} \precsim \mathfrak{M}[H]$.*

Proof. Let $G \subseteq \boldsymbol{P}$ be $\boldsymbol{P}$-generic over $\mathfrak{M}_0$ and put $\mathfrak{M} = \mathfrak{M}_0[G]$. By Lemma 6, $G \times H$ is $\boldsymbol{P} \times \boldsymbol{Q}$-generic over $\mathfrak{M}_0$ and $\mathfrak{M}[H] = \mathfrak{M}_0[G][H] = \mathfrak{M}_0[G \times H]$. We must show that $\mathfrak{M} \precsim \mathfrak{M}[H]$.

Let Φ be a formula of the language of the set theory and $x_1, \ldots, x_n \in \mathfrak{M}$. It is sufficient to show that $\mathfrak{M} \models \Phi[x_1, \ldots, x_n] \to \mathfrak{M}[H] \models \Phi[x_1, \ldots, x_n]$. For the converse implication take $\neg\Phi$.

From section 2, $x_1, \ldots, x_n \in \mathfrak{M}_0[G_\alpha]$ for some α, $\alpha \in On \cap \mathfrak{M}_0$. Thus we have $\mathfrak{M}_0[G_\alpha][G^\alpha] \models \Phi[x_1, \ldots, x_n]$, and so $\mathfrak{M}_0[G_\alpha][G^\alpha] \models \Phi(\hat{x}_1, \ldots, \hat{x}_n)$, where the sign $\hat{}$ is used in $\mathfrak{M}_0[G_\alpha]$, in the sense of forcing $\boldsymbol{P}^\alpha$. We have seen that $\boldsymbol{P}^\alpha$ is a homogeneous notion of forcing in $\mathfrak{M}_0[G_\alpha]$, so by Lemma 11, $\mathbf{1}_{\boldsymbol{P}^\alpha} \Vdash_{\boldsymbol{P}^\alpha} \Phi(\hat{x}_1, \ldots, \hat{x}_n)$. But in the model $\mathfrak{M}_0[G_\alpha]$ the notions of forcing $\boldsymbol{P}^\alpha$ and $\boldsymbol{P}^\alpha \times \boldsymbol{Q}$ are isomorphic. Let F be this isomorphism. Then by Lemma 8 $\mathbf{1}_{\boldsymbol{P}^\alpha \times \boldsymbol{Q}} \Vdash_{\boldsymbol{P}^\alpha \times \boldsymbol{Q}} \Phi(F^*(\hat{x}_1), \ldots, F^*(\hat{x}_n))$ and by Lemma 9, $\mathbf{1}_{\boldsymbol{P}^\alpha \times \boldsymbol{Q}} \Vdash_{\boldsymbol{P}^\alpha \times \boldsymbol{Q}} \Phi(\hat{x}_1, \ldots, \hat{x}_n)$, where the sign $\hat{}$ is now used in the sense of $\boldsymbol{P}^\alpha \times \boldsymbol{Q}$.

Let us consider the model $\mathfrak{M}_0[G_\alpha][G^\alpha][H]$. By Lemma 6

$$\mathfrak{M}_0[G_\alpha][G^\alpha][H] = \mathfrak{M}_0[G_\alpha][G^\alpha \times H],$$

where $G^\alpha \times H$ is $\boldsymbol{P}^\alpha \times \boldsymbol{Q}$-generic over $\mathfrak{M}_0[G_\alpha]$. So by the truth lemma,

$$\mathfrak{M}_0[G_\alpha][G^\alpha \times H] \models \Phi(\hat{x}_1, \ldots, \hat{x}_n),$$

i.e. $\mathfrak{M}[H] \models \Phi[x_1, \ldots, x_n]$. Q.E.D.

We can extend this result to the following:

THEOREM 14. *Let $\mathfrak{M}_0$ be a countable standard model of* ZFC$^-$ *and let $\boldsymbol{P} \in \mathfrak{M}_0$ be a notion of forcing in $\mathfrak{M}_0$. Then there exists a model $\mathfrak{M} \supseteq \mathfrak{M}_0$, of the same height such that every $\boldsymbol{P}$-generic extension of $\mathfrak{M}$ is an elementary extension.*

The idea of the proof of Theorem 14 is the same: add a proper class of $\boldsymbol{P}$-generic subsets of $\boldsymbol{P}$ and next add one more subset. We then use the following notions of forcing:

$$\boldsymbol{R} = \{f\colon \mathrm{Func}(f) \;\&\; \mathrm{dom}(f) \subseteq On \;\&\; \mathrm{rg}(f) \subseteq \boldsymbol{P} - \{\mathbf{1}_P\} \;\&\; \mathrm{Fin}(f)\},$$

$$f_1 \leqslant_R f_2 \equiv (\alpha)[\alpha \in \mathrm{dom}(f_2) \rightarrow \alpha \in \mathrm{dom}(f_1) \;\&\; f_1(\alpha) \leqslant_P f_2(\alpha)],$$

$$\boldsymbol{R}_\alpha = \{f \in \boldsymbol{R}\colon \mathrm{dom}(f) \subseteq \alpha\},$$

$$\boldsymbol{R}_\alpha = \{f \in \boldsymbol{R}\colon \mathrm{dom}(f) \cap \alpha = 0\},$$

$$F^R_\alpha(\langle p, q\rangle) = p \cup q.$$

The details of the proof are left to the reader. Q.E.D.

The phenomenon described in Theorems 13 and 14 cannot take place in the case of models of ZF, because we have the following well-known

LEMMA 15. *If $\mathfrak{M} \prec \mathfrak{N}$ and both are standard models of* ZF, *then $\mathfrak{M} = \boldsymbol{R}_\alpha \cap \mathfrak{N}$, where α is the height of $\mathfrak{M}$.*

Proof. Assume that $\mathfrak{N} - \mathfrak{M} \neq 0$. Let x be an element of $\mathfrak{N} - \mathfrak{M}$ of minimal rank. Then $x \subseteq \mathfrak{M}$. Assume that $\mathrm{rank}(x)$ is less than α. Then for some $y \in \mathfrak{M}$, $x \subseteq y$. Take $z = \wp(y) \cap \mathfrak{M}$. Then z is the power set of y in $\mathfrak{M}$; since $\mathfrak{M} \prec \mathfrak{N}$, z is the power set of y in $\mathfrak{N}$. But $x \subseteq y$ and $x \notin z$, which contradicts the definition of power set. Thus $\mathrm{rank}(x) \geqslant \alpha$. Q.E.D.

The following lemma is the "key lemma" of the paper. It makes possible to describe inside the model $\mathfrak{M}$ that $\mathfrak{M} \prec \mathfrak{M}[G]$.

LEMMA 16. *Let $\mathfrak{M}$ be a standard model of* ZFC$^-$, *$\boldsymbol{Q}$ a notion of forcing defined at the beginning of this section and $G \subseteq \boldsymbol{Q}$ a $\boldsymbol{Q}$-generic set over $\mathfrak{M}$. Then $\mathfrak{M} \prec \mathfrak{M}[G]$ iff*

$$\mathfrak{M} \models (x_1) \ldots (x_n)[\Phi(x_1, \ldots, x_n) \equiv \mathbf{1}_Q \Vdash_Q \Phi(\hat{x}_1, \ldots, \hat{x}_n)].$$

for every formula Φ of the language of the set theory with free variables $x_1, \ldots, x_n$.

Proof. Assume $\mathfrak{M} \prec \mathfrak{M}[G]$. Then by Lemma 11

$$\mathfrak{M} \models \Phi[x_1, \ldots, x_n] \equiv \mathfrak{M}[G] \models \Phi[x_1, \ldots, x_n] \equiv \mathbf{1}_Q \Vdash_Q \Phi(\hat{x}_1, \ldots, \hat{x}_n).$$

On the other hand, assume that $\mathfrak{M}$ satisfies the schema. Then

$$\mathfrak{M} \models \Phi[x_1, \ldots, x_n] \equiv \mathbf{1}_Q \Vdash_Q \Phi(\hat{x}_1, \ldots, \hat{x}_n) \equiv \mathfrak{M}[G] \models \Phi[x_1, \ldots, x_n].$$ Q.E.D.

Observe that the formula $\mathbf{1}_Q \Vdash_Q \Phi(\hat{x}_1, \ldots, \hat{x}_n)$ depends only on the formula Φ, not on the model $\mathfrak{M}$, i.e. there is a formula $\mathrm{Forc}_\Phi(y, z, x_1, \ldots, x_n)$ such that

$$\mathfrak{M} \models \mathrm{Forc}_\Phi[\leqslant_Q, p, x_1, \ldots, x_n] \equiv \mathfrak{M} \models \mathbf{1}_Q \Vdash_Q \Phi(\hat{x}_1, \ldots, \hat{x}_n).$$

6. Uncountable models of ZFC$^-$ and A_2.

DEFINITION 17. An ordinal α is a *model number* iff there is a standard model $\mathfrak{M}$ of ZFC$^- +$ V $=$ HC such that $\alpha = On \cap \mathfrak{M}$. HC is the class of all hereditarily countable sets.

By Zbierski's theorem (cf. [4]) model numbers are exactly the heights of β-models of A_2.

THEOREM 18. *Assume Martin's Axiom. Then for every model number α there exists a model $\mathfrak{N}$ of* ZFC$^- +$ V $=$ HC, *of power $2^{\aleph_0}$ and height α.*

Proof. Let $\mathfrak{M}_0$ be a countable standard model of ZFC$^- +$ V $=$ HC of height α. By Theorem 13 there exists a model $\mathfrak{M}$ of the same height, such that if $G \subseteq \boldsymbol{Q}$ is $\boldsymbol{Q}$-generic over $\mathfrak{M}$, then $\mathfrak{M} \prec \mathfrak{M}[G]$.

We define a transfinite sequence of standard models for ZFC$^- +$ V $=$ HC. Assume that of $\xi < \eta < 2^{\aleph_0}$ we have defined models $\mathfrak{N}_\xi$ satisfying the following conditions:

1. $\mathfrak{N}_0 = \mathfrak{M}$ (note that if $\mathfrak{M}_0 \models$ V $=$ HC, then so does every generic extension of it).
2. $\mathfrak{N}_{\xi_1} \prec \mathfrak{N}_{\xi_2}$ for $\xi_1 < \xi_2$.
3. $\mathfrak{N}_{\xi_1} \notin \mathfrak{N}_{\xi_2}$ for $\xi_1 < \xi_2$.
4. $\overline{\overline{\mathfrak{N}_\xi}} \leqslant \bar{\bar{\xi}} + \aleph_0$.
5. $\mathfrak{N}_\xi$ is a standard model of ZFC$^- +$ V $=$ HC of height α.

We define $\mathfrak{N}_\eta$. There are two cases:

Case 1. η is a limit ordinal. Then we define $\mathfrak{N}_\eta = \bigcup_{\xi<\eta} \mathfrak{N}_\xi$. Of course $\mathfrak{N}_\eta$ satisfies all the conditions 1-5.

Case 2. $\eta = \xi + 1$. Then $\mathfrak{M} = \mathfrak{N}_0 \prec \mathfrak{N}_\xi$; thus, by Lemma 16, $\mathfrak{N}_\xi$ satisfies the schema

$$(x_1) \ldots (x_n)[\Phi(x_1, \ldots, x_n) \equiv \mathbf{1}_Q \Vdash_Q \Phi(\hat{x}_1, \ldots, \hat{x}_n)].$$

By Martin's Axiom take $G \subseteq \boldsymbol{Q}$, $\boldsymbol{Q}$-generic over $\mathfrak{N}_\xi$. Then, by Lemma 16, $\mathfrak{N}_\xi \prec \mathfrak{N}_\xi[G]$. Put $\mathfrak{N}_\eta = \mathfrak{N}_\xi[G]$. $\mathfrak{N}_\eta$ again satisfies all the conditions 1-5.

Finally put $\mathfrak{N} = \bigcup_{\xi<2^{\aleph_0}} \mathfrak{N}_\xi$. Then $\mathfrak{N}$ is the required model. Q.E.D.

COROLLARY 19. *Without Martin's Axiom we can prove the existence of a model of* ZFC$^-$ *of power ω_1 and height α.*

COROLLARY 20. *Assume Martin's Axiom. Then for every model number α there exists a β-model of A_2 of height α and power $2^{\aleph_0}$.*

For a proof take the continuum of a model constructed in Theorem 18. The Corollary is then a consequence of Zbierski's Theorem. Q.E.D.

The model constructed in Corollary 20 is not isomorphic to the principal model of A_2. If we let α be the smallest model number, we shall obtain a β-model for A_2 of power $2^{\aleph_0}$, which is not elementarily equivalent to the principal model of A_2. Namely, our model will not contain any other model of A_2 as an element.

COROLLARY 21. *Without Martin's Axiom we can prove the existence of uncountable β-models of A_2 of height α for every model number α.*

An analogous result concerning the Kelley-Morse set theory was proved in the same way in the author's doctoral thesis.

Added in proof. A proof of a much stronger theorem was recently given by H. Friedman in his unpublished paper *Uncountable models of set theory*.

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Two notes on abstract model theory
I. Properties invariant on the range of definable relations between structures

by

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Abstract. Suppose L is any model-theoretic language satisfying the many-sorted interpolation property and that $\mathfrak{R}$ is an L-definable or even L-projective relation between L-structures. It is shown that if (1) an L-sentence φ holds in $\mathfrak{N}_1$ just in case it holds in $\mathfrak{N}_2$ whenever $\mathfrak{R}(\mathfrak{M}, \mathfrak{N}_1)$ and $\mathfrak{R}(\mathfrak{M}, \mathfrak{N}_2)$ then (2) there is an L-sentence ψ such that φ holds in $\mathfrak{N}$ if and only if ψ holds in $\mathfrak{M}$ whenever $\mathfrak{R}(\mathfrak{M}, \mathfrak{N})$. This has various results of Beth, Robinson, Gaifman, Barwise and Rosenthal for familiar languages as immediate corollaries.

Introduction. *Abstract* (or *general*) *model-theory* deals with notions that are applicable to all *model-theoretic languages* L. Each such L is determined by a relation $\mathfrak{M} \models_L \varphi$, called its *satisfaction relation*, in which $\mathfrak{M}$ ranges over a collection Str_L of *structures* for L and φ ranges over a collection of objects Stc_L called the *sentences* of L. The notions of general model theory are just those which can be expressed in terms of these basic ones (using ordinary set-theoretical concepts). Examples of such are: *elementary class*, *projective class*, *Löwenheim-Skolem properties*, *Hanf number*, *interpolation property*, *compactness properties*, *categoricity*. Typically, the results apply to all L satisfying some simple conditions or characterize some given L_0 by means of such conditions.

Lindström [L2] provided the first work clearly of this character. Its point of departure (via [L1]) was Mostowski's characterization of $L_{\omega,\omega}$ among certain languages with generalized quantifiers [Mo]. Since then, the subject of general model theory has been especially developed by Barwise [B2]–[B4]. The present two notes are a sequel to my own contribution in § 3 of [F2], making essential use, as there, of *many-sorted* structures.

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