Generalizing Vaught sentences from ω to strong cofinality ω

by

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Abstract. We prove a strengthening of C. C. Chang's interpolation theorem involving infinitary formulas with conjunctions of the size a strong limit cardinal with cofinality ω (Corollary 3.3 below). This yields a strengthening of Chang's Scott type isomorphism theorem involving the same type of formulas (Corollary 3.5). In the last section, using game formulas we reformulate J. Green's Σ_1-compactness proof.

Introduction. In this paper we generalize a part of Vaught's theory [V] from $L_{\omega_1\omega}$ to a situation involving both $L_{\varkappa^+\omega}$ and $L_{\varkappa^+\varkappa}$ with $\varkappa$ a strong limit cardinal with cofinality ω. As a result, we obtain strengthenings of Chang's interpolation theorem in [C2] (3.3 below) and of a part of Chang's generalization of Scott's isomorphism theorem in [C1] (3.5 below).

Although the generalization from [V] is rather straightforward, it has perhaps an interesting technical aspect in the use of Carol Karp's notion of truth in an ascending chain of models (c.f. [Ka]). Using this notion, we obtain results such as 3.3 and 3.5 which are formulated without this notion (and without the game sentences of Vaught).

Most of the proofs are only sketched since they are modelled very closely after existing proofs.

Copying Vaught's proof of the Barwise Σ_1-compactness theorem, in § 5 we give another proof of Judy Green's Σ_1-compactness theorem [G].

Games like those underlying the notion of Vaught sentences were used in [Mo], and in [C-M] they are used in connection with cardinals of strong cofinality ω.

§ 1. $\varkappa$-Vaught sentences. Throughout §§ 1-4, $\varkappa$ is a strong limit cardinal with cofinality ω and $\langle\varkappa_n\colon n<\omega\rangle$ is a fixed sequence of cardinals such that $2^{\varkappa_n}\leqslant\varkappa_{n+1}$ and $\varkappa=\bigcup_{n<\omega}\varkappa_n$. Though everything goes through

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for $\varkappa = \aleph_0$ (as it is already known), it is convenient to assume that $\varkappa \geqslant \aleph_0$ and also $\varkappa_0 \geqslant \aleph_0$.

$L_{\lambda\mu}$ is the first order logic using conjunctions and disjunctions of sets of cardinality $<\lambda$ and existential and universal quantifications over a sequence of length $<\mu$ of variables. Note that $L_{\varkappa\varkappa}$ and $L_{\varkappa\omega}$ are equivalent in expressive power to $L_{\varkappa^+\varkappa}$ and $L_{\varkappa^+\omega}$, respectively, since $\varkappa$ is singular. However in most cases it will be important for us that formulas are written as required in $L_{\varkappa\varkappa}$ and $L_{\varkappa\omega}$.

If one wants to emphasize which relation (and operation and individual constant) symbols are used, one writes e.g. $L_{\varkappa\omega}(\underset{\sim}{\vec{R}})$ to denote a logic using basic symbols in L and the additional symbols in the sequence $\underset{\sim}{\vec{R}}$.

In what follows, $\vec{x}_n, \vec{y}_n$ denote sequences of distinct variables, the length of $\vec{x}_n, \vec{y}_n$ is $\varkappa_n$. A *$\varkappa$-Vaught sentence* Γ is a sentence of the form

$$\forall \vec{x}_0 \exists \vec{y}_0 \bigvee_{l_0 \in \varkappa} \ldots \forall \vec{x}_n \exists \vec{y}_n \bigvee_{l_n \in \varkappa} \ldots \bigwedge_{n \in \omega} N^{l_0, \ldots, l_n}(\vec{x}_0, \ldots, \vec{y}_n)$$

where the $N^{\vec{l}}$ are (for simplicity) quantifier-free formulas of $L_{\varkappa\omega}$ with the (free) variables indicated. $\varkappa$-Vaught sentences are a direct generalization of the game sentences considered in [V].

It is fairly obvious how to interpret a $\varkappa$-Vaught sentence in a structure. Similarly as in [V], [Ma 1], [Ma 2] etc. we can talk about a game in this connection. The first move of e.g. the $\forall$-player consists of a sequence $\vec{a}_0$ of elements of the given structure, corresponding to $\vec{x}_0$. For later use we write down a formal definition of truth for a $\varkappa$-Vaught sentences.

For this purpose, we need to introduce infinitary predicate symbols with "arities" $<\varkappa$. Let, for each $n<\omega$ and each $\vec{l} = \langle l_0, \ldots, l_n\rangle \in {}^n\varkappa$, $\underset{\sim}{T}{}_n^{\vec{l}}$ denote a predicate symbol of arity $\varkappa_0 + \varkappa_0 + \varkappa_1 + \varkappa_1 + \ldots + \varkappa_{n-1} + \varkappa_{n-1}$ (ordinal sum; it is equal to $\varkappa_{n-1} + \varkappa_{n-1}$ if $n>0$ and to 0 if $n=0$). Then, with Γ above and arbitrary L-structure $\mathfrak{A}$, $\mathfrak{A} \models \Gamma$ iff there are relations $T_n^{\vec{l}}$ on A such that $(\mathfrak{A}, T_n^{\vec{l}})_{n<\omega, \vec{l} \in {}^n\varkappa}$ satisfies the formula

$$\text{(1)} \qquad T_0^{\varnothing} \wedge \bigwedge_{n<\omega} \forall \vec{x}_0 \ldots \vec{x}_{n-1} \vec{y}_0 \ldots \vec{y}_{n-1} \bigwedge_{l_0 \in \varkappa} \ldots \bigwedge_{l_{n-1} \in \varkappa} \Big[T_n^{\vec{l}}(\vec{x}_0, \vec{y}_0, \ldots, \vec{x}_{n-1}, \vec{y}_{n-1}) \rightarrow \forall \vec{x}_n \exists \vec{y}_n \bigvee_{l_n \in \varkappa} T_n^{\vec{l}, l_n}(\vec{x}_0, \ldots, \vec{y}_n) \wedge \bigwedge_{i<n} N_i^{l_0, \ldots, l_i}(\vec{x}_0, \ldots, \vec{y}_i)\Big]$$

$$(\text{here } \vec{l} = \langle l_0, \ldots, l_{n-1}\rangle).$$

Next we describe Karp's notion of truth in an ascending chain of models (see [Ka]; the notion was introduced and used as early as 1959 in Karp's thesis). Though Karp did not consider infinitary predicates, they are essential for us; however, they do not present any difficulty. Let $\overline{\mathfrak{A}} = \langle \mathfrak{A}_n : n<\omega\rangle$ be an ascending chain of models, called hereafter

a *chain-model*. A sequence of elements of $A = \bigcup_{n<\omega} A_n$ is *bounded* if all its elements come from one of the sets A_n. For a formula $\varphi \in L_{\infty\infty}$ and for a bounded sequence $\vec{a}$ of elements of A interpreting $\vec{x}$, we define

$$\overline{\mathfrak{A}} \models^K \varphi[\vec{a}/\vec{x}]$$

by the usual inductive clauses for atomic φ and for Boolean combinations and by the following modified quantifier clauses:

$\overline{\mathfrak{A}} \models^K \forall \vec{y}\, \varphi[\vec{a}/\vec{x}]$

$\Leftrightarrow$ for all bounded sequences $\vec{b}$ (of appropriate length), $\overline{\mathfrak{A}} \models^K \varphi[\vec{a}/\vec{x}, \vec{b}/\vec{y}]$,

$\overline{\mathfrak{A}} \models^K \exists \vec{y}\, \varphi[\vec{a}/\vec{x}] \Leftrightarrow$ for some bounded sequence $\vec{b}$, $\overline{\mathfrak{A}} \models^K \varphi[\vec{a}/\vec{x}, \vec{b}/\vec{y}]$.

Any chain model $\overline{\mathfrak{A}}$ gives rise to its union, $\bigcup_{n<\omega} \mathfrak{A}_n$; however in case we have infinitary predicates, these will not be defined but for the bounded combinations of arguments. Whenever we use the symbols $\overline{\mathfrak{A}}$ (or $\overline{\mathfrak{B}}$), it denotes a chain-model $\langle \mathfrak{A}_n : n<\omega\rangle$ (or $\langle \mathfrak{B}_n : n<\omega\rangle$) and if $\mathfrak{A}$ (or $\mathfrak{B}$) is used in the same context then it is the union $\bigcup_{n<\omega} \mathfrak{A}_n$ (or $\bigcup_{n<\omega} \mathfrak{B}_n$). Notice that for a sentence φ in $L_{\infty\omega}$, $\overline{\mathfrak{A}} \models^K \varphi \Leftrightarrow \mathfrak{A} \models \varphi$.

For relations $\vec{T}$ on $A = \bigcup_{n<\omega} A_n$, $(\overline{\mathfrak{A}}, \vec{T})$ denotes the chain-model $\langle (\mathfrak{A}_n, \overrightarrow{T \restriction A_n}) : n<\omega\rangle$.

For a $\varkappa$-Vaught sentence Γ, we write

$$\overline{\mathfrak{A}} \models^K \Gamma$$

if for some (infinitary) relations $T_n^{\vec{l}}$, we have

$$(\overline{\mathfrak{A}}, T_n^{\vec{l}})_{n\in\omega, \vec{l} \in {}^n\varkappa} \models^K \sigma$$

where σ is the sentence in (1).

Intuitively, $\overline{\mathfrak{A}} \models^K \Gamma$ involves the restriction of the range of both the universally and existentially quantified sequences of variables to bounded sequences of element. Actually, for our purposes only the restriction of the universal quantifiers is of importance.

The chain-model $\langle \mathfrak{A}_n : n<\omega\rangle$ is called a *chain-$\varkappa$-model* if $|A_n| \leqslant \varkappa_n$ $(n<\omega)$.

§ 2. The game form theorem. A *Σ_1^1-over-$L_{\varkappa\omega}$ sentence* is of the form $\exists \underset{\sim}{\vec{R}} \sigma$ where $\underset{\sim}{\vec{R}}$ is an arbitrary list of length at most $\varkappa$ of "new" relation (and possibly operation and individual constant) symbols and σ is a sentence of $L_{\varkappa\omega}(\underset{\sim}{\vec{R}})$. One of the basic tools in Vaught's work is the game form theorem for $L_{\omega_1\omega}$: any Σ_1^1-over-$L_{\omega_1\omega}$ sentence is equivalent for countable structures to an (ω-) Vaught sentence (Corollary 3.3 in [V]). The corresponding result that we need is

THEOREM 2.1. *For any Σ_1^1-over-$L_{\varkappa\omega}$ sentence $\exists\vec{\underset{\sim}{R}}\sigma$ there is a $\varkappa$-Vaught sentence Γ such that $\exists\vec{\underset{\sim}{R}}\sigma \models \Gamma$ and for any structure $\mathfrak{A}$ of power $\leqslant \varkappa$,*

$\mathfrak{A} \models \exists\vec{\underset{\sim}{R}}\sigma$ if and only if $\mathfrak{A} \models \Gamma$ if and only if there is a chain-$\varkappa$-model $\overline{\mathfrak{A}} = \langle \mathfrak{A}_n \colon n < \omega\rangle$ such that

$$\overline{\mathfrak{A}} \models^K \Gamma \quad \textit{and} \quad \bigcup_{n<\omega} \mathfrak{A}_n = \mathfrak{A}.$$

Proof. The proof is very similar to Svenonius' proof of his theorem from which Vaught deduces his own (see Theorem 3.2 in [V]). The details also will resemble those in [Ma 1], [Ma 2]. We will limit ourselves to giving the main constructions.

(a) To construct Γ, we first take a set V of variables such that $|V| = \varkappa$. Let $V = \bigcup_{n<\omega} V_n$, $V_n \subset V_{n+1}$ and $|V_n| = \varkappa_n$. We assume σ as well as any other formula is written with using only $\exists$, $\bigvee$ and finitary connectives. Consider all subformulas $\theta(\vec{u})$ of σ; substitute arbitrary variables $\vec{v}$ in V for *all* free variables $\vec{u}$ in θ to obtain $\theta(\vec{v})$; let Θ be the closure of the set of all these $\theta(\vec{v})$ under finitary connectives. Notice that $|\Theta| \leqslant \varkappa$.

(b) We next take Θ_n for each $n < \omega$ such that $\bigcup_{n<\omega}\Theta_n = \Theta$, $\Theta_n \subseteq \Theta_{n+1}$, $|\Theta_n| \leqslant \varkappa_n$, every formula in Θ_n has all its free variables in V_n, and for every disjunction $\bigvee\Sigma$ in Θ_n, $|\Sigma| \leqslant \varkappa_n$.

(c) Next we exhibit a list of "Henkin formulas". By induction on n, we define the disjoint sequences $\vec{x}_n, \vec{y}_n$ of variables in V and the set H_n of formulas as follows. Let $\langle \exists x_\alpha \psi_\alpha \colon \alpha < \varkappa_n\rangle$ be a sequence enumerating all members of Θ_n of the form $\exists x\psi$. Let y_α for $\alpha < \varkappa_n$ be arbitrary distinct variables in $V_{n+1} - V_n$. Let $\vec{x}_n$ be a sequence of variables containing precisely the variables in $V_n - (\mathrm{rn}\,\vec{x}_0 \cup \ldots \cup \mathrm{rn}\,\vec{y}_{n-1})$, and let

$$\vec{y}_n = \langle y_\alpha \colon \alpha < \varkappa_n\rangle .$$

Finally, put

$$H_n = H_{n-1} \cup \{\exists x\psi_\alpha \to \psi_\alpha(y_\alpha \text{ for } x_\alpha) \colon \alpha < \varkappa_n\}.$$

(d) Next we let $\langle \bigvee \Sigma_\alpha \colon \alpha < \varkappa_n\rangle$ be a sequence enumerating all non-empty infinitary disjunctions in Θ_n (if it so happens that Θ_n does not contain any such, we simply add one to it). Recall that $|\Sigma_\alpha| \leqslant \varkappa_n$ for all $\alpha < \varkappa_n$. There are $\varkappa_n^{\varkappa_n} < \varkappa$ choice functions $h \in \prod_{\alpha<\varkappa_n} \Sigma_\alpha$, let us index them as h_l $(l < \varkappa)$. Define $H_n^{l_0,\ldots,l_n}$ to be

$$H_n \cup \bigcup_{i\leqslant n}\{\bigvee \Sigma_\alpha \to h_{l_i}(\alpha) \colon \alpha < \varkappa_n\}.$$

Notice that $H_n^{l_0,\ldots,l_n} \subseteq H_{n+1}^{l_0,\ldots,l_n,l_{n+1}}$.

(e) For any $n < \omega$, put $N_n^{\vec{l}}(\vec{x}_0, \vec{y}_0, \ldots, \vec{x}_n, \vec{y}_n)$ $(\vec{l} = \langle l_0, \ldots, l_n\rangle)$ to be the conjunction of all finite disjunctions δ of atomic and negated atomic L-formulas (i.e., no symbols in $\vec{\underset{\sim}{R}}$ are allowed!) which contain the variables in $\vec{x}_0, \ldots, \vec{y}_n$ at most and such that δ is a *consequence of* $H_n^{l_0,\ldots,l_n} \cup \{\sigma\}$ with all free variables $\vec{x}_0, \ldots, \vec{y}_n$ *held constant* (i.e., regarded individual constants). Thus, $N_n^{l_0,\ldots,l_n}$ is a quantifier-free formula of $L_{\varkappa^+\omega}$("="$L_{\varkappa\omega}$). Finally, define Γ as

$$\forall\vec{x}_0\exists\vec{y}_0 \bigvee l_0 < \varkappa \ldots \forall\vec{x}_n\exists\vec{y}_n \bigvee l_n < \varkappa \ldots \bigwedge n < \omega N_n^{l_0,\ldots,l_n}(\vec{x}_0, \ldots, \vec{y}_n).$$

(f) We submit that it is obvious that $\exists\vec{\underset{\sim}{R}}\sigma \models \Gamma$. In fact, if $\mathfrak{A} \models \exists\vec{\underset{\sim}{R}}\sigma$, i.e. $(\mathfrak{A}, \vec{R}) \models \sigma$ for some $\vec{R}$ on A, then for arbitrary (interpretation of) $\vec{x}_0, \vec{x}_1, \ldots$ in A we can define elements (interpreting) $\vec{y}_0, \vec{y}_1, \ldots$ in A and ordinals $l_0, l_1, \ldots < \varkappa$ such that (i) y_n and l_n depend only on $\vec{x}_0, \ldots, \vec{x}_n$ and (ii) the $\vec{x}_0, \ldots, \vec{y}_n$ satisfy every element of $H_n^{l_0,\ldots,l_n}$ (with the chosen $l_0, \ldots, l_n$), thus also they satisfy $N_n^{l_0,\ldots,l_n}(\vec{x}_0, \ldots, \vec{y}_n)$.

(g) It is easy to see that the second condition of the theorem implies the third one. To complete the proof suppose that $\overline{\mathfrak{A}}$ is a chain-$\varkappa$-model, (hence $|A_n| \leqslant \varkappa_n$) and $\overline{\mathfrak{A}} \models \Gamma$. We want to see that $\mathfrak{A} \models \exists\vec{\underset{\sim}{R}}\sigma$. Let $\vec{a}_n$ be an enumeration of A_n such that length $(\vec{a}_n) = \varkappa_n$. Apply $\overline{\mathfrak{A}} \models^K \Gamma$ to obtain $\vec{b}_n$ corresponding to $\vec{y}_n$ and $l_n < \varkappa$ $(n < \omega)$ such that

$$\mathfrak{A} \models N_n^{l_0,\ldots,l_n}[\vec{a}_0/\vec{x}_0, \ldots, \vec{b}_n/\vec{y}_n]. \tag{3}$$

(Recall that the $N_n^{\vec{l}}$ are quantifier free, hence $\overline{\mathfrak{A}} \models^K N_n^{\vec{l}}[\vec{a}_0/\vec{x}_0, \ldots]$ implies (3).)

Now, let Δ be the diagram of $\mathfrak{A}$, i.e. the set of atomic and negated atomic formulas θ with variables in V such that $\mathfrak{A} \models \theta[\vec{a}_0/\vec{x}_0, \vec{b}_0/\vec{y}_0, \ldots]$. Let Ψ be the set of all valid formulas. Consider the following set T of formulas:

$$\bigcup_{n<\omega} H_n^{l_0,\ldots,l_n} \cup \{\sigma\} \cup \Delta \cup \Psi .$$

Just as in [Ma 2], we see that T is finitely satisfiable; this fact is a consequence of (3) and the definition of the $N_n^{\vec{l}}$. Next we apply the compactness theorem for finitary propositional logic to obtain an assignment a of truth-values to elements of Θ which makes every element of T "true" and which "respects" ("commutes with") finitary connectives. Finally, define for any relation symbol $\underset{\sim}{R}$ in $\vec{\underset{\sim}{R}}$, the corresponding relation R on A by

$$Rc_1 \ldots c_m \text{ is true} \underset{\mathrm{df}}{\Leftrightarrow} a(\underset{\sim}{R}\; z_1 \ldots z_m) \text{ is "true"}$$

where $z_1, \ldots, z_m$ are variables in V and c_i corresponds to z_i under the interpretations $\vec{a}_n/\vec{x}_n$, $\vec{b}_n/\vec{y}_n$ $(n < \omega)$. On the basis of the definition of $H_n^{\vec{l}}$.

we can verify that this definition is legitimate and by induction on φ in Θ we can prove that

$$(\mathfrak{A}, \vec{R}) \models \varphi[\vec{a}_0/\vec{x}_0, \vec{b}_0/\vec{y}_0, \ldots] \Leftrightarrow \alpha(\varphi) \text{ is "true"}$$

(see the corresponding part of the proof of 2.9 in [Ma 2]). So, in fact $(\mathfrak{A}, R) \models \sigma$ as required. Q.E.D.

Very similarly we could prove (compare [Ma1] and [Ma 2])

THEOREM 2.2. *For any Σ_1^1-over-$L_{\varkappa\omega}$ sentence $\exists \vec{R}\ \sigma$ there is a positive $\varkappa$-Vaught sentence Γ* (i.e., *the $N_n^{\vec{l}}$ are positive*) *such that $\exists \vec{R}\ \sigma \models \Gamma$ and for any $\mathfrak{B}$ of power $\leqslant \varkappa$*

there in $\mathfrak{A} \models \exists \vec{R}\ \sigma$ such that $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$ if and only if $\mathfrak{B} \models \Gamma$ if and only if for some chain-$\varkappa$-model $\overline{\mathfrak{B}}$ with union $\mathfrak{B}$ we have $\overline{\mathfrak{B}} \models^K \Gamma$.

We could in fact develop a generalization of Lindström games and establish generalizations of results in [Ma 2].

§ 3. Approximations of $\varkappa$-Vaught sentences. Here we follow § 4 of [V] very closely.

Let Γ be a $\varkappa$-Vaught sentence in the notation used in § 1. Define for any ordinal α and any sequence $\vec{l} = l_0, \ldots, l_{n-1}$ of indices $< \varkappa$ the formula $\delta_\alpha^{\vec{l}}(\vec{x}_0, \ldots, \vec{y}_{n-1})$ in $L_{\infty\varkappa}$ as follows:

$\delta_\alpha^{\vec{l}}$ is $\bigwedge_{i<n} N_i^{l_0,\ldots,l_i}(\vec{x}_0, \ldots, \vec{y}_i)$

(in particular, $\delta_0^{\varnothing}$ is identically true),

$\delta_{\alpha+1}^{\vec{l}}$ is $\forall \vec{x}_n \exists \vec{y}_n \bigvee l_n < \varkappa \delta_\alpha^{\vec{l}, l_n}(\vec{x}_0, \ldots, \vec{x}_n, \vec{y}_n)$,

$\delta_\lambda^{\vec{l}}$ is $\bigwedge \beta < \lambda \delta_\beta^{\vec{l}}$ if λ is a limit ordinal.

We write δ_α for $\delta_\alpha^{\varnothing}$. Note that for $\alpha < \varkappa^+$, $\delta_\alpha \in L_{\varkappa^+\varkappa}$ ("$=$"$L_{\varkappa\varkappa}$).
We can easily prove the next

THEOREM 3.1. *$\delta_\alpha \models \delta_\beta$ if $\beta < \alpha$, $\Gamma \models \delta_\alpha$ for any ordinal α.*

THEOREM 3.2. *Let ψ be a sentence in some logic $L_{\varkappa\omega}(\vec{\underset{\sim}{S}})$. Suppose that for every chain-$\varkappa$-model $(\overline{\mathfrak{A}}, \vec{S})$ such that $\overline{\mathfrak{A}} \models^K \Gamma$ we have $(\mathfrak{A}, \vec{S}) \models \psi$ (in intuitive notation, (Γ in $\varkappa$-sense) $\models \psi$). Then there is $\alpha < \varkappa^+$ such that $\delta_\alpha \models \psi$.*

Proof. The proof will follow Vaught's proof of his 4.11 in [V] very closely.

(a) First we formalize the definition of $\delta_\alpha^{\vec{l}}$ as in Definition 4.5 in [V]. We introduce the unary predicates $\underset{\sim}{U}$, $\underset{\sim}{A}$, the binary predicate $<$, the individual constant c and for each $n < \omega$, and $l_0, \ldots, l_{n-1} < \varkappa$ the $1 + \varkappa_0 +$ $+ \varkappa_0 + \ldots + \varkappa_{n-1} + \varkappa_{n-1} = \varkappa_{n-1} + \varkappa_{n-1}$-ary (for $n = 0$, the unary) predicate symbol $D^{l_0,\ldots,l_{n-1}}$. Let ξ^+ be the following sentence in $L'_{\varkappa^+\varkappa}$ ($L_{\varkappa^+\varkappa}$ extended by the new symbols):

$$\text{"}(U, <) \text{ is a linear ordering with last element } c\text{"}$$
$$\wedge \bigwedge n < \omega \forall \vec{x}_0 \ldots \vec{x}_{n-1} \vec{y}_0 \ldots \vec{y}_{n-1} \bigwedge l_0, \ldots, l_{n-1} < \varkappa$$
$$\{D^{l_0,\ldots,l_{n-1}}(u;\ \vec{x}_0, \vec{y}_0, \ldots, \vec{x}_{n-1}, \vec{y}_{n-1})$$
$$\rightarrow [(\forall v < u) \forall \vec{x}_n \exists \vec{y}_n \bigvee l_n < \varkappa D^{l_0,\ldots,l_{n-1},l_n}(v;\ \vec{x}_0, \ldots, \vec{y}_n) \wedge$$
$$\wedge \bigwedge_{i<n} N_i^{l_0,\ldots,l_i}(\vec{x}_0, \ldots, \vec{y}_i)]\} \wedge D^{\varnothing}(c).$$

Here we suppressed $\underset{\sim}{A}$; the variables in $\vec{x}_n, \vec{y}_n$ are understood to be relativized to $\underset{\sim}{A}$.

(b) Given any α and a model $\mathfrak{A}$ of δ_α, we can "make" $\mathfrak{A}$ into a model $\mathfrak{B}$ of ξ^+ such that $(\underset{\sim}{U}^{\mathfrak{B}}, <^{\mathfrak{B}})$ is a well-ordering of type $\alpha+1$ and $\mathfrak{A}$ is the reduct of the substructure of $\mathfrak{B}$ with domain $\underset{\sim}{A}^{\mathfrak{B}}$. In $\mathfrak{B}$, $D^{\vec{l}}(\beta, \ldots)$ "denotes" $\delta_\beta^{\vec{l}}$ as interpreted in $\mathfrak{A}$.

Now assume that $\delta_\alpha \wedge \neg\psi$ has a model for each $\alpha < \varkappa^+$ (the negation of the assertion of 3.2). Hence $\xi^+ \wedge \neg(\psi)^{\underset{\sim}{A}} \wedge$"$\underset{\sim}{A} \neq 0$" has a model for each $\alpha < \varkappa^+$ such that the order type of $(U, <)$ is $\alpha+1$ ($\psi^{(\underset{\sim}{A})}$ is ψ relativized to $\underset{\sim}{A}$).

(c) Now, apply the following lemma discussed in the next section.

LEMMA. *If an $L'_{\varkappa\varkappa}$ sentence φ has, for any $\alpha < \varkappa^+$, a model $\mathfrak{B}_\alpha$ such that $(\underset{\sim}{U}^{\mathfrak{B}_\alpha}, <^{\mathfrak{B}_\alpha})$ is well-ordered with order-type $> \alpha$, then ψ has a chain-$\varkappa$-model $\overline{\mathfrak{A}}$ such that $(\underset{\sim}{U}^{\mathfrak{A}}, <^{\mathfrak{A}})$ is not well-ordered.*

Hence, by (b), there is a chain-$\varkappa$-model $\overline{\mathfrak{B}}$ such that $\overline{\mathfrak{B}} \models^K \xi^+ \wedge \neg(\psi)^{\underset{\sim}{A}} \wedge$ $\wedge$"$\underset{\sim}{A} \neq 0$" and $(\underset{\sim}{U}^{\mathfrak{B}}, <^{\mathfrak{B}})$ is not well-ordered.

(d) Hence we conclude, as Vaught does, that $\overline{\mathfrak{A}} \models^K \Gamma$ where $\overline{\mathfrak{A}}$ is the obvious L-chain-$\varkappa$-model derived from $\overline{\mathfrak{B}}$. In fact, let $c^{\mathfrak{B}} = u_0 >^{\mathfrak{B}} u_1 >^{\mathfrak{B}} u_2 >^{\mathfrak{B}} \ldots$ be a descending sequence of elements and define the relations $T_n^{\vec{l}}$ on A by

$$T_n^{\vec{l}}(\vec{x}_0, \ldots, \vec{y}_{n-1}) \text{ is true} \Leftrightarrow D^{\vec{l}}(u_n;\ \vec{x}_0, \ldots, \vec{y}_{n-1}) \text{ is true in } \mathfrak{B}.$$

Then it is clear by (c) that

$$(\overline{\mathfrak{A}}, T_n^{\vec{l}}) \models^K \varphi$$

where φ is the sentence in (1) of § 1, hence $\overline{\mathfrak{A}} \models^K \Gamma$. Since we also have $(\mathfrak{A}, \vec{S}) \models \neg\psi$, we have arrived at a contradiction to the hypothesis of the theorem. Q.E.D.

Theorems 2.1 and 3.2 at once yield

Corollary 3.3. (Main result). *For any Σ_1^1-over-$L_{\varkappa\omega}$ sentence $\exists \underset{\sim}{\vec{R}}\,\sigma$ there are $L_{\varkappa\varkappa}$ sentences δ_a for $a < \varkappa^+$ such that*

$$\exists \underset{\sim}{\vec{R}}\,\sigma \models \delta_a$$

and for any $\psi \in L_{\varkappa\omega}(\underset{\sim}{\vec{S}})$, if

$$\exists \underset{\sim}{\vec{R}}\,\sigma \models \psi$$

then for some $a < \varkappa^+$, $\delta_a \models \psi$.

Corollary 3.4. *In the notation of 3.3, $\exists \underset{\sim}{\vec{R}}\,\sigma$ is equivalent to $\bigwedge_{a<\varkappa^+}\delta_a$ for structures of power $\leqslant \varkappa$.*

Proof. Assume $\mathfrak{A}$ has domain A, $|A| \leqslant \varkappa$. Let $\Delta_{\mathfrak{A}}$ be the diagram of $\mathfrak{A}$ written with new constants $\underset{\sim}{a}$ for $a \in A$. Let ψ be

$$\neg[\forall x \bigvee\{x = \underset{\sim}{a}\colon a \in A\} \wedge \bigwedge \Delta_{\mathfrak{A}}].$$

Suppose $\mathfrak{A} \not\models \exists \underset{\sim}{\vec{R}}\,\sigma$. Then clearly, $\exists \underset{\sim}{\vec{R}}\,\sigma \models \psi$. Hence by 3.3, for some $a < \varkappa^+$ we have $\delta_a \models \psi$, hence $\mathfrak{A} \not\models \delta_a$. Q.E.D.

Corollary 3.5. *Every structure $\mathfrak{A}$ of power $\leqslant \varkappa$ can be characterized up to isomorphism among structures of power $\leqslant \varkappa$ by a conjunction $\bigwedge_{a<\varkappa^+}\delta_a$ of $L_{\varkappa\varkappa}$ sentences δ_a.*

Proof. Apply 3.4 to $\exists \underset{\sim}{\vec{R}}\,\sigma$ where σ is

$$\forall x \bigvee\{x = \underset{\sim}{a}\colon a \in A\} \wedge \bigwedge \Delta_{\mathfrak{A}}$$

and the $\underset{\sim}{\vec{R}}$ are the additional individual constants $\underset{\sim}{a}$ $(a \in A)$.

Remarks. 3.3 is an analogue (a generalization of a part of) 4.11 in [V]. It is a strengthening of Chang's interpolation theorem connecting $L_{\varkappa\omega}$ and $L_{\varkappa\varkappa}$ [C2]. 3.5 is a strengthening of a part of Chang's generalization of Scott's isomorphism theorem [C1] (see the third paragraph on page 45 in [C1]; now $\varkappa^{\underset{\sim}{\varkappa}} = 2^\varkappa$).

Using 2.2 instead of 2.1, we obtain

Corollary 3.6. *The class of homomorphic images of power $\leqslant \varkappa$ of models of $\exists \underset{\sim}{\vec{R}}\,\sigma$ is axiomatized among structures of power $\leqslant \varkappa$ by a conjunction $\bigwedge_{a<\varkappa^+}\delta_a$ of positive $L_{\varkappa\varkappa}$ sentences δ_a $(a < \varkappa^+)$.*

Corollary 3.7. (Chang [C2]). *Any sentence of $L_{\varkappa^+\omega}$ preserved under homomorphic images is equivalent to a positive sentence in $L_{\varkappa\varkappa}$.*

§ 4. Non-characterizability of well-order.

Theorem 4.1 below was very likely known to Karp (who wrote in [Ka] that "the proofs in [Ke] of a number of important results for $L_{\omega_1\omega}$ generalize ... by replacing the ordinary consistency properties by $\varkappa$-consistency properties"). In [Ka], Karp proves a model existence theorem concerning chain-$\varkappa$-models and $\varkappa$-consistency properties; this is used in the proof of 4.1. For the rather long definition of $\varkappa$-consistency properties, we refer to [Ka].

Theorem 4.1. (i) *Suppose $\varphi \in L'_{\varkappa\varkappa}$ where $L'_{\varkappa\varkappa}$ contains the unary predicate $\underset{\sim}{U}$ and the binary predicate $<$ and possibly some infinitary $<\varkappa$-ary predicates. Suppose that for each $a < \varkappa^+$ there is a model $\mathfrak{B}_a$ of φ such that $(\underset{\sim}{U}^{\mathfrak{B}_a}, <^{\mathfrak{B}_a})$ is a well-ordering of type $\geqslant a$. Then φ has a chain-$\varkappa$-model $\overline{\mathfrak{A}}$ such that $(\underset{\sim}{U}^{\mathfrak{A}}, <^{\mathfrak{A}})$ contains a copy of the ordered set of the rationals.*

(ii) *The same conclusion holds if we change $\mathfrak{B}_a$ to a chain-model $\overline{\mathfrak{B}}_a$ of φ.*

The proofs use the model existence theorem in [Ka] and are exact replicas of the proofs of Theorem 12 in [Ke]. We only define the consistency properties used in the proofs.

Using the sets C_n of new constants such that $|C_n| = \varkappa_n$ as in [Ka] and also the further new constants d_r for r a rational number, we define S_1 to be the set of all sets s of sentences of $L'_{\varkappa\varkappa}$ enlarged by adding $\bigcup_{n<\omega} C_n$ and $\{d_r\colon r \text{ is a rational}\}$ such that

(iii) for some $n < \omega$, all new constants in s belong to $D \underset{\text{df}}{=} C_n \cup \{d_{r_1}, \ldots, d_{r_n}\}$ for some rationals $r_1 < \ldots < r_n$,

(iv) for every $a < \varkappa^+$, there is a model $\mathfrak{B}_a$ of s such that $(U^{\mathfrak{B}_a}, <^{\mathfrak{B}_a})$ is a well-ordered set of type $\geqslant a$ and in fact, (the interpretations of) $d_{r_1}, \ldots, d_{r_n}$ satisfy

$$a \leqslant d_{r_1} \leqslant d_{r_1} + a \leqslant d_{r_2} \leqslant \ldots \leqslant d_{r_{n-1}} + a \leqslant d_{r_n}.$$

Then S_1 can be shown to be a $\varkappa$-consistency property with respect to $\langle C_n\colon n < \omega\rangle$ (see [Ka]) exactly as the corresponding proof in [Ke].

Since by the hypothesis in (i) we have $\{\varphi\} \in S_1$, the model existence theorem in [Ka] establishes (i).

For proving (ii), we use S_2 defined as S_1 with (iv)′ replacing (iv):

(iv)′ for every $a < \varkappa^+$, there is a chain-model $\langle \mathfrak{B}_{an}\colon n < \omega\rangle$ and there are some $m < \omega$ and interpretations $\bar{d}$ of all constants d in D (D is as in (iii)) such that $\bar{d} \in B_{am}$ $(d \in D)$ and $\langle(\mathfrak{B}_{a,m+i}, \bar{d})_{d\in D}\colon i < \omega\rangle$ is a model of s with the rest of (iv) holding for $(U^{\mathfrak{B}_a}, <^{\mathfrak{B}_a})$.

Alternatively, it is not difficult to give a direct proof of 4.1 avoiding consistency properties in a way similar to the proof of the interpolation theorem for $L_{\varkappa\omega}$ in [C2].

§ 5. The Σ_1-compactness theorem of Green.

A beautiful application of his theory is given by Vaught for proving the Σ_1-compactness theorem of Barwise [B1], see § 5 of [V]. Barwise shows in [B2] that Σ_1-compactness for admissible (in fact for transitive prim. closed) sets A is equivalent to the strict-$\underset{\sim}{\Pi}_1^1$ (s-$\underset{\sim}{\Pi}_1^1$) reflection principle holding for A. The essential part of Vaught's proof, in fact, establishes the s-$\underset{\sim}{\Pi}_1^1$ reflection principle for countable admissible sets. We give some definitions.

An ordinary first order formula in an extended language of set theory (containing $\in$ and $\approx$ and some additional predicate variables) is called *essentially universal* (e.u.) if it is in negation normal form and all existential quantifiers in the formula appear in contexts like $\exists x \in y$ (i.e., $\exists x[x \in y \to \ldots]$). The notion of essentially existential (e.e.) is defined similarly. A *strict*-Σ_1^1 (s-Σ_1^1) formula is a second order formula of the form $\exists \underset{\sim}{S}_1 \ldots \underset{\sim}{S}_k \theta$ where θ is e.u. (we also assume that no free predicate variable other than $\in$ and $\approx$ appears in $\exists \vec{\underset{\sim}{S}}\, \theta$). Let A be a non-empty transitive set. We write $\underset{\frown}{A}$ for the structure $(A, \in \restriction A)$. The *strict*-$\underset{\sim}{\Pi_1^1}$ *reflection principle* (s-$\underset{\sim}{\Pi_1^1}$ R.P.) for A says, in the contrapositive form, that for any s-Σ_1^1 formula $\exists \vec{\underset{\sim}{S}}\, \theta(\vec{x})$ and elements $\vec{a}$ of A, we have:

$$\forall w \in A[w \text{ transitive and } \vec{a} \in w \Rightarrow \underset{\frown}{w} \models \exists \vec{\underset{\sim}{S}}\, \theta[\vec{a}]] \Rightarrow \underset{\frown}{A} \models \exists \vec{\underset{\sim}{S}}\, \theta[\vec{a}].$$

For the reader unfamiliar with the subject, the best way to get a feeling of these notions is to see, first, that the converse of the s-$\underset{\sim}{\Pi_1^1}$ R.P. is trivially true, and second, that the set $R(\omega)$ of hereditarily finite sets satisfies the s-$\underset{\sim}{\Pi_1^1}$ R.P., and in fact this last fact is more or less a reformulation of the König lemma on finitary trees.

Now, the nice thing about Vaught's proof of the s-$\underset{\sim}{\Pi_1^1}$ R.P. for countable admissible sets is that it breaks the argument into two theorems, the first talking about arbitrary *countable* structures, the second about arbitrary *admissible* sets.

The first of these theorems is that a s-Σ_1^1 formula is equivalent for countable structures to a *game*-s-Σ_1^1 formula of the form

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots \bigwedge n < \omega N_n(\vec{u};\, x_0, \ldots, y_n)$$

where $\langle N_n;\, n < \omega\rangle$ is recursive, each N_n is quantifier-free and

$$\models N_n \to y_n \in x_0 \cup \ldots \cup y_{n-1} \cup \{x_0, \ldots, y_{n-1}\} \cup u_0 \cup \ldots \cup u_l .$$

(Actually, it follows from a result of [Ma 1] that the same conclusion is true for any Σ_1^1 statement preserved for transitive substructures.)

The second part of Vaught's argument is the theorem that the statement of the s-$\underset{\sim}{\Pi_1^1}$ R.P. with a game-s-Σ_1^1 formula holds for arbitrary admissible sets.

We will use a slightly modified form of this second theorem of Vaught, involving set-primitive recursiveness, see [J-Ka]. A set closed under set-primitive recursive functions is called prim. closed. The use of prim. closed sets would not be absolutely necessary but it is convenient because many set-theoretic predicates are absolute with respect to transitive prim. closed sets.

Let us call a game formula of the form

$$\forall x_0 \exists y_0 \ldots \forall x_n \exists y_n \ldots \forall n < \omega N(n;\, \vec{u}, \langle x_0, \ldots, y_n\rangle)$$

a *special game* formula if $N(v, \vec{u}, z)$ is an e.e. formula containing only $\in$ and $\approx$, and

$$\models N(n, \vec{u}, \langle x_0, \ldots, y_n\rangle) \to y_n \in x_0 \cup \ldots y_{n-1} \cup u_0 \cup \ldots \cup u_l .$$

THEOREM 5.1 (Vaught [V]). *Let A be an admissible set, $\omega \in A$. Let $\Gamma(\vec{u})$ be a special game formula, $\vec{a} \in A$. Suppose that for every transitive prim. closed set $w \in A$ such that $a_1, \ldots, a_l \in w$, we have $\underset{\frown}{w} \models \Gamma[\vec{a}]$. Then $\underset{\frown}{A} \models \Gamma[\vec{a}]$.*

Inspection of the proof of 5.3 in [V] reveals that it establishes 5.1. (We use that $\omega \in A$ implies that any $a \in A$ is an element of a transitive prim. closed $w \in A$. The assumption on N that it is e.e. is used to make sure that $\underset{\frown}{w} \models N[\ldots] \Rightarrow \underset{\frown}{A} \models N[\ldots]$.)

We are going to show that 5.1 can be used to prove Judy Green's compactness theorem [G]. In order to do this, we will "bring s-Σ_1^1 formulas to a special game form".

Let A be an admissible set. Let $B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots$ and let

$$k \underset{\mathrm{df}}{=} \langle B_n : n < \omega\rangle \in A.$$

Assume that for some $C \in A$ we have ${}^{B_n}B_n, {}^{B_n}2, {}^{B_n}\omega \subseteq C$ for all $n < \omega$. Assume moreover that

(i) for every element $a \in A$ there is a function $f \in A$ such that $\operatorname{dom} f \subseteq B = \bigcup_{n<\omega} B_n$ and $\operatorname{rn} f = a$,

(ii) A is of cofinality ω, i.e. $A = \bigcup_{n<\omega} A_n$ for some $A_n \in A$ $(n < \omega)$.

EXAMPLE 1. Let $B_n = n$. Let A be a countable admissible set such that $\omega \in A$ and every element of A is countable "within" A. Then the hypotheses are satisfied.

EXAMPLE 2. Let $\langle B_n |\ n < \omega\rangle$ be a sequence of transitive sets such that $\omega \in B_0$, and for each $n < \omega$, B_n is closed under pairs and the power set of B_n is an element of B_{n+1}. Let A be the smallest admissible set such that $\langle B_n :\, n < \omega\rangle \in A$. Then as Green shows in [G], A satisfies the above hypotheses.

THEOREM 5.2 (Green [G]). *Any admissible set A satisfying the above hypotheses satisfies the s-$\underset{\sim}{\Pi_1^1}$ R.P., or equivalently, is Σ_1-compact.*

Let $\exists \vec{\underset{\sim}{S}}\, \sigma(u)$ be a s-Σ_1^1 formula, $a \in A$. We will construct some special game formula $\Gamma(u, u_1, u_2)$ such that the following two statements hold:

(iii) For every transitive prim. closed $w \in A$ such that $a, k, C \in w$, we have

$$\underset{\frown}{w} \models \exists \vec{\underset{\sim}{S}}\, \sigma[a] \Rightarrow \underset{\frown}{w} \models \Gamma[a, k, C].$$

(iv) $\underset{\frown}{A} \models \Gamma[a, k, C] \Rightarrow \underset{\frown}{A} \models \exists \vec{\underset{\sim}{S}}\, \sigma[a]$.

Together with 5.1, (iii) and (iv) give the s-$\underset{\sim}{\Pi_1^1}$ R.P. for A.

We note that (iii) and (iv) do not require that A be admissible but require (i) and (ii) above. It would be nicer to have a single equivalence instead of (iii) and (iv) similarly to the first theorem mentioned above in Vaught's argument; however, the converse of (iii) does not seem to hold.

(a) To begin the description of Γ and to give an intuitive idea of the proof, we consider an auxiliary language suggested by Green's "indexed languages" in [G]. Define D to be the smallest set of elements of A such that if $f \in A$ is a function with $\mathrm{rn} f \subset D$ and *either* $\mathrm{dom} f = B$ *or* $\mathrm{dom} f = 0$ *or* $0 \in \mathrm{dom} f \subseteq B_n$ for some $n \in \omega$ (we assume w.l.o.g. that $0 \in B_0$), then $f \in D$. We will consider the language $L(\underset{\sim}{\vec{S}}, D)$ of set theory ($\in$ and $\approx$) augmented with the relation symbols $\underset{\sim}{\vec{S}}$ and with all individual constants $\underline{f}$ for $f \in D$. $\underline{f}$ will not denote f, rather we have the following inductive definition:

$$\mathit{denote}\ (f) = \{\mathit{denote}\ (g) \colon g \in \mathrm{rn} f\}\,.$$

So, every formula of $L(\underset{\sim}{\vec{S}}, D)$ has a *standard* interpretation in any transitive set- (or class-) model with $\underline{f}$ denoting *denote* (f). It is easy to see that assumption (i) implies that for every $a \in A$ there is $f \in D$ such that *denote* $(f) = a$. This fact will be used in the final part of the argument, (f) below. The reason for the appearance of the two kinds of domains of elements of D is the following. Very roughly speaking, the main feature of the proof (as well as of Judy Green's) is a "two stage instantiation" of restricted existence statements. In other words, having been forced to make $\exists x \in \underline{f}\varphi(x)$ true, we first force some $\exists x \in \underline{f \restriction B_n}\varphi(x)$ true, then secondly we force some $\varphi(\underline{g})$ to be true for some $g \in \overline{\mathrm{rn}(f \restriction B_n)}$. We have to do this because we have to deal with many existence statements simultaneously but we can handle them only if there are not too many *possibilities* for the simultaneous instantiations. Made in one step, for the above simultaneous instantiations we have too many possibilities, roughly as "many" as the set ${}^{\chi}(\mathit{denote}\ (f))$ (where χ is the set of statements considered) that can even be of power greater than A. In the two stage version however, the first stage entails ${}^{\chi}\omega$, the second ${}^{\chi}B_n$ "many" possibilities and both sets will be elements of A. This explanation will become clearer upon seeing the proof that follows.

(b) Let F be a function, $F \in A$, $\mathrm{dom} F = B_n$, and let h_0, h_1, h_2 be functions with domain B_n, $\mathrm{rn} h_0 \subset \omega$, $\mathrm{rn} h_1 \subset B_n$, $\mathrm{rn} h_2 \subset 2$. We describe a set $\Phi = \Phi(F, h_0, h_1, h_2)$ of sentences of the language $L(\underset{\sim}{\vec{S}}, D)$ constructed on the basis of F, h_0, h_1, h_2. Φ will be a set-primitive recursive function of $\omega, a, k, F, h_0, h_1, h_2$ and σ (the formula originally given) (ω here is redundant since $\omega = \mathrm{dom} k$). As a preliminary remark, we note that F is used here for enumerating some sentences of $L(\underset{\sim}{\vec{S}}, D)$ as well as some constants in D and h_0, h_1, h_2 are used to instantiate existence statements and disjunctions. Recall that σ is essentially universal.

Φ is defined as the smallest set such that (v)-(xi) below hold.

(v) $\sigma(\underline{f}$ for $u) \in \Phi$ if $f \in D$, $b \in B_n$, $f = F(b)$ and $a = \mathit{denote}\ (f)$.

(vi) If $b \in B_n$,

$$F(b) = \exists x \in \underline{f}\varphi(x)$$

and $F(b)$ belongs to Φ and $\mathrm{dom} f = B$, then

$$\exists x \in \underline{f \restriction B_m}\varphi(x) \in \Phi$$

where $m = h_0(b)$ (recall that $h_0 \colon B_n \to \omega$).

(vii) If $b \in B_n$,

$$F(b) = \exists x \in \underline{g}\varphi(x) \in \Phi\,,$$

$0 \neq \mathrm{dom} g \subseteq B_m$ then *either* $h_1(b) \notin \mathrm{dom} g$ and $\varphi(\underline{g(0)}) \in \Phi$ *or* $h_1(b) \in \mathrm{dom}(g)$ and $\varphi(\underline{f}) \in \Phi$ for $f = g(h_1(b))$ (recall that $h_1 \colon B_n \to B_n$).

(viii) If $b \in B_n$,

$$F(b) = \varphi_1 \vee \varphi_2 \in \Phi\,,$$

then $\varphi_{h_2(b)} \in \Phi$ (recall that $h_2 \colon B_n \to 2$).

(ix) If $\varphi_1 \wedge \varphi_2 \in \Phi$, then $\varphi_1, \varphi_2 \in \Phi$.

(x) If $\forall x\,\varphi(x) \in \Phi$,

$$f = F(b)\,, \quad \text{and} \quad f \in D\,,$$

then $\varphi(\underline{f}) \in \Phi$.

(xi) All equality axioms in $L(\underset{\sim}{\vec{S}}, D)$ with constants in $\mathrm{rn} F \cap D$ are in Φ.

(c) We claim that there is an essentially existential formula $N(v, u, u_1, u_2, z)$ with the following property. Given any $n \in \omega$, transitive prim. closed $w \subseteq A$ such that $a, k, C \in w$, and given any F_i and h_j^i in w ($i \leqslant n$, $j \leqslant 2$) then for $s = \langle F_0, h_0^0, h_1^0, h_2^0, \ldots, F_n, h_0^n, h_1^n, h_2^n\rangle$ we have

$$\widehat{w} \models N(n, a, k, C, s)$$

iff conditions (xii) and (xiii) below are satisfied.

(xii) For $i \leqslant n$ and $j \leqslant 2$, F_i and h_j^i are functions with domain B as above, $h_j^i \in C$.

(xiii) $\Phi \underset{\mathrm{df}}{=} \bigcup_{i \leqslant n} \Phi(F_i, h_0^i, h_1^i, h_2^i)$ is "*consistent*" in the sense that it does not contain π and $\neg\pi$ for any atomic π and whenever $\pi = \underline{f} \in \underline{g}$ or $\underline{f} \approx \underline{g}$ is in Φ, then π is true under the standard interpretation.

To see the existence of N, use the facts that *denote* is a set-primitive recursive function, Φ is a set-primitive recursive function of a, b, C and s, and that the graph of a set-primitive recursive function is definable by an e.e. formula which is absolute with respect to transitive prim. closed sets (see [J-Ka]).

We put $\Gamma(u, u_1, u_2)$ to be

$$\forall x_0 \exists y_0^0 \exists y_1^0 \exists y_2^0 \ldots \forall x_n \exists y_0^n \exists y_1^n \exists y_2^n \ldots \forall n < \omega$$
$$N(n;\ u, u_1, u_2, \langle x_0, y_0^0, y_1^0, y_2^0, \ldots, x_n, y_0^n, y_1^n, y_2^n \rangle).$$

(e) Suppose w is as above and $(w, \epsilon | w, S_1, \ldots, S_k) \models \sigma[a]$. To show (iii), we have to say how to find h_j^n once $F_0, \ldots, F_n$ are given. First we consider the set Φ' of sentences in $L(\vec{\underset{\sim}{S}}, D)$ which are substitution instances of subformulas of σ and which are true under the standard interpretation in $(w, \epsilon \upharpoonright w, S_1, \ldots, S_k)$. We "truly" instantiate these by functions h_j^n so that the above (vi), (vii), (viii) become true when we read Φ' for Φ. Now, since ${}^{B_n}\omega, {}^{B_n}B_0, {}^{B_n}2 \subseteq C$, we have $h_j^n \in C$. Clearly, $\Phi(F_n, h_0^n, h_1^n, h_2^n) \subseteq \Phi'$ and by induction, $\bigcup_{i \leq n} \Phi(F_i, h_0^i, h_1^i, h_2^i) \subseteq \Phi'$, and so, by (c), $N(n, a, k, C, s)$ is true in $\hat{\underset{\sim}{w}}$. This shows (iii).

(f) Assume $\hat{\underset{\sim}{A}} \models \Gamma[a, k, C]$.

Then it is sufficient to apply this fact for some $F_0, F_1, \ldots$ in A such that $\mathrm{dom} F_n = B_n$ and $\bigcup_{n<\omega} \mathrm{rn} F_n = A$. Taking the h_j^n given by the assumption, we see that

$$T = \bigcup_{n<\omega} \Phi(F_n, h_0^n, h_1^n, h_2^n)$$

is "consistent" in the above sense. It is easy to see that for some relations $S_1, \ldots, S_k$ on A, $(A, \epsilon \upharpoonright A, S_1, \ldots, S_k)$ will be a model of T under the standard interpretation of constants. The relations S_j are defined according to which of the atomic and negated atomic formulas containing S_j are in T. Then an induction will show that all elements of T become true in the model. The induction is based on the definition of T and for the case the formula is of the form $\forall x \varphi$, it uses the above-mentioned consequence of assumption (i) that every $b \in A$ is *denote* (f) for some $f \in D$.

This proves (iv) since clearly, $\sigma \in T$ (see (v)).

By what was said above, this finishes the proof of Green's theorem.

Finally we mention that the A-recursive enumerability of valid sentences of A for an A with the above assumptions (actually, (ii) is not needed) can be proved similarly, following again § 5 of [V]. This result is also due to Green [G].

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